Probabilistic Combinatorics

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Lecture 1

1 Introduction

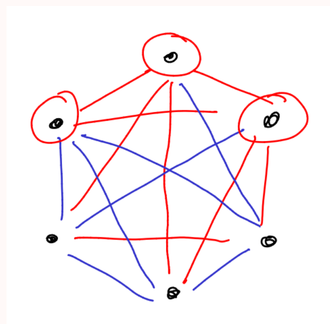
The course is Probabilistic Combinatorics, being lectured by

$$\text{Julian} \qquad \underbrace{\text{Sa}}\underbrace{\text{has}}\underbrace{\text{ra}}\underbrace{\text{bu}}\underbrace{\text{dhe}}_{\text{“}day''}$$

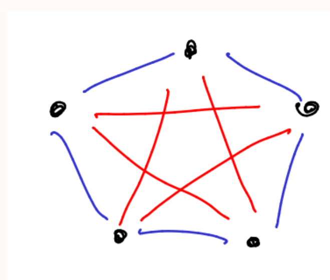
We will be studying in particular the Ramsey numbers. Starting with the simplest to define:

$$R(k) = \min\{n : \text{every red/blue colouring of } E(K_n) \text{ contains a monochromatic } K_n\}.$$

Example. $R(3) \le 6$: pick a vertex. Without loss of generality say it has at least 3 red neighbours. If any of these are connected by a red edge, then we get a red triangle by using the original vertex. If none of them are connected by a red edge, then we get a blue triangle between them.



$R(3) > 5$: by considering the following picture.



Thus $R(3) = 6$.

Main question: how fast does $R(k) \to \infty$ as $k \to \infty$?

We will also study:

$$R(l, k) = \min\{n : \text{every red/blue colouring of } E(K_n) \text{ contains either a blue } K_l \text{ or a red } K_k\}.$$

For a fixed graph H, we define

$$R(H) = \min\{n : \text{every red/blue colouring of } E(K_n) \text{ contains a monochromatic } H\}.$$

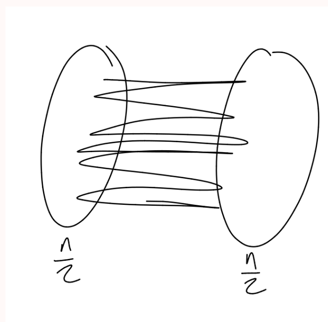
Example. $R(k) = R(K_k)$.

For H a fixed graph, we define

$$\mathrm{ex}(n, H) = \max\{e(G) : G \text{ is a graph on } n \text{ vertices and } G \not\supset H\}.$$

Example. Mantel's Theorem says that

$$\mathrm{ex}(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$



1.1 Binomial Random Graph

Definition 1.1 (Binomial random graph)**.** Given $n \in \mathbb{N}$, $p \in (0, 1)$, the *binomial random graph* $G(n, p)$ is the probability space defined on all graphs on n vertices, where each edge is included independently with probability p.

1.2 Topics in this course

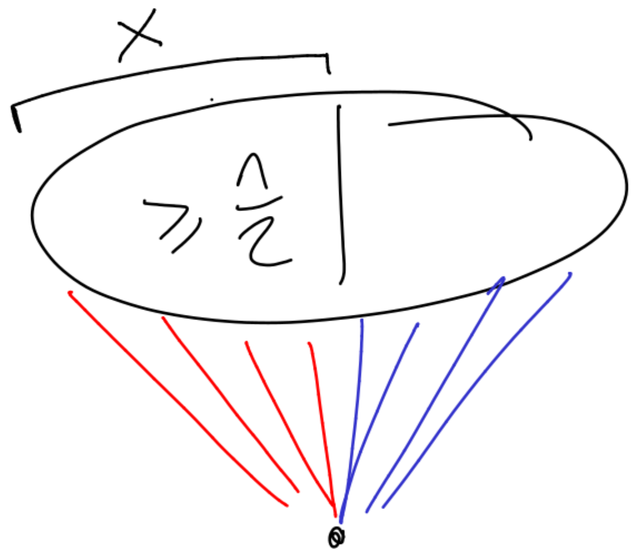
- First examples: "first moment method".
- $R(3, k)$:
  - deletion method
  - Lovász local lemma
  - Semi-random method
  - Hard core model
  - The triangle free process

- Dependent random choice:
  - Ramsey numbers $R(H)$ with H sparse
  - Sidorenko conjecture
  - Extremal numbers of bipartite graphs
- Pseudo-randomness
  - $R(H)$ of bounded-degree graphs
  - size-ramsey numbers
  - $R(3, 3, k)$
- Szemeredi-Regularity lemma
  - Roth's Theorem on 3-term arithmetic progressions in dense sets
  - Ramsey-Turán
- Method of graph containers:
  - Counting graphs with no C_4
  - $R(3, 3, k)$
  - $R(4, k)$

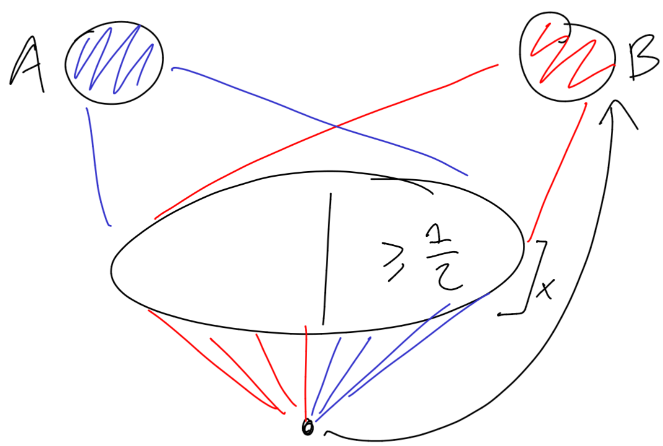
1.3 Brief introduction to $R(k)$

Theorem (Erdos-Szekeres, '35)**.** $R(k) \leq 4^k$.

Proof sketch. Let $n \geq 4^k$ and let χ be a colouring of K_n. Pick a vertex v. It must have $\geq \frac{1}{2}n$ neighbours connected to it by the same colour.



Now ignore everything that was connected using the other colour. Pick a new vertex from what remains, and apply the process again:

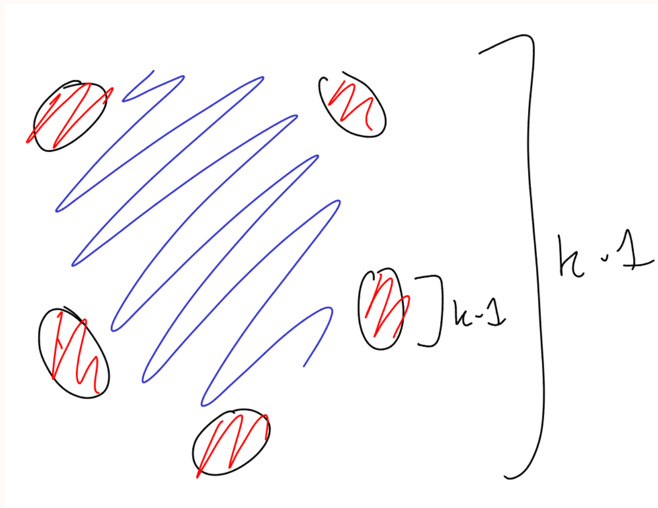


We continue until either A or B gets to size k. We basically just want

$$n\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^k \geq 1,$$

i.e. $n \geq 4^k$. $\square$

How about a lower bound?



Example.
This gives $R(k) \geq (k-1)^2 + 1$. Quite a long way from 4^k!

Theorem (Erdos, '47)**.** $R(k) \geq (1 - o(1)) \frac{k}{\sqrt{2}e} 2^{k/2}$.

Notation. $o(1)$ denotes a quantity that $\to 0$ as $k \to \infty$.

Proof. Let $n = (1 - o(1)) \frac{k}{\sqrt{2}e} e^{k/2}$ we choose χ to be a random colouring of $E(K_n)$ where the colour of each edge is chosen red / blue uniformly and independently. We have

$$\begin{aligned}
\mathbb{P}(\chi \text{ has a monochromatic } K_k) &= \mathbb{P}\left(\bigcup_{K \in [n]^{(k)}} \{K \text{ is a monochromatic clique}\} \right) \\
&\leq 2 \binom{n}{k} 2^{-\binom{k}{2}} \\
&\leq 2 \left(\frac{en}{k} \right)^k 2^{-\binom{k}{2}} \\
&= \left(\frac{en}{k} 2^{\left(\frac{k-1}{2}\right)} \right) \\
&< 1
\end{aligned}$$

by the choice of n. $\square$

Lecture 2 Big question: Is there an "explicit" construction that gives $R(k) \geq (1 + \varepsilon)^k$, for some fixed $\varepsilon > 0$?

2 The Ramsey numbers $R(3,k)$

Theorem (Erdos-Szekeres, 1935)**.** $R(l,k) \le \binom{k+l-2}{l-1}$.

Proof. Induction. See e.g. GT. $\square$

Note. $R(k) = R(k,k) \le \binom{2(k-1)}{k-1} \sim c\frac{4^k}{\sqrt{k}}$.

Corollary. $R(3,k) \le k^2$.

Idea 1: If randomly colour K_n, we have $\sim \frac{1}{8}\binom{n}{3}$ triangles $\rightsquigarrow$:(

Idea 2: We want to bias our distribution in favour of K_k – colour "red". Say $G \sim G(n,p)$. Sad news: if $p \gg \frac{1}{n}$ then G contains a K_3 with high probability. $\rightsquigarrow$:(

Theorem (Erdos, 1960s)**.** $R(3,k) \ge c\left(\frac{k}{\log k}\right)^{3/2}$ for some $c > 0$.

Proof sketch. Sample $G \sim G(n,p)$, then define $\tilde{G} = G -$ (a vertex from each triangle). For this to work we need $\frac{\#\text{vertices}}{2} > \#\text{triangles}$. So $\frac{n}{2} > p^3\binom{n}{3}$. So take $p \sim n^{-2/3}$. $\square$

Remark. This is called the "deletion method".

Fact (Markov)**.** Let X be a non-negative random variable. Let $t > 0$. Then

$$\mathbb{P}(X \ge t) \le \frac{\mathbb{E}X}{t}.$$

That is

$$\mathbb{P}(X \ge s\mathbb{E}X) \le \frac{1}{s},$$

for any $s > 0$.

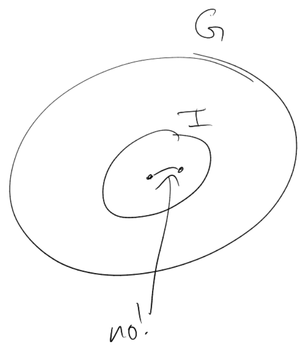
Fact. For $1 \le k \le n$ we have

$$\binom{n}{k} \le \left(\frac{en}{k}\right)^k$$

and

$$\binom{n}{k} \ge \left(\frac{n}{k}\right)^k.$$

Definition 2.1 (Independence number)**.** Let G be a graph. Then $I \subset V(G)$ is *independent* if $x \not\sim y$ in G for all $x, y \in I$.
We use $\alpha(G) =$ size of the largest independent set.



Note. We are trying to construct a triangle free graph on $k^{3/2-o(1)}$ vertices with $\alpha(G) < k$.

Lemma 2.2. Assuming that:

- $\frac{1}{n} \ll p \leq \frac{1}{2}$
- $G \sim G(n, p)$

Then
$$\alpha(G) \leq (2 + o(1)) \frac{\log np}{p},$$
with probability tending to 1 as $n \to \infty$ (known as "with high probability (w.h.p.)").

Remark. We actually have $=$ in this lemma.

Proof. Let $k = (2 + \delta) \frac{\log np}{p}$, for some $\delta > 0$. Let X be the random variable that counts the number of independent k sets in G.

$$\begin{aligned}
\mathbb{E}X &= \mathbb{E} \sum_{I \in [n]^{(k)}} \mathbb{1}_{I \text{ is independent}} \\
&= \sum_{I \in [n]^{(k)}} \mathbb{P}(I \text{ is independent}) \\
&= \binom{n}{k} (1-p)^{\binom{k}{2}} \\
&\le \left(\frac{en}{k}\right)^k e^{-p\binom{k}{2}} \\
&\le \left(\frac{en}{k} \cdot e^{-\left(\frac{k-1}{2}\right)p}\right)^k \\
&\to 0
\end{aligned}$$

by plugging in k. So by Markov

$$\mathbb{P}(X \ge 1) \le \mathbb{E}X \to 0. \qquad \square$$

We have a method for bounding the probability that X is large: Markov says that $\mathbb{P}(X > t) \le \frac{\mathbb{E}X}{t}$. What about bounding the probability that it is small?

Fact: Let X be a random variable with $\mathbb{E}X$, $\operatorname{Var} X$ finite. Then for all $t > 0$,

$$\mathbb{P}(|X - \mathbb{E}X| \ge t) \le \frac{\operatorname{Var} X}{t^2}.$$

Reminder:

$$\operatorname{Var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \mathbb{E}(X - \mathbb{E}X)^2.$$

Lemma 2.3. Assuming that:

- $\frac{1}{n} \ll p \le 1$
- $G \sim G(n,p)$

Then

$$\#\text{triangles in } G = (1 + o(1))p^3 \binom{n}{3}$$

with high probability.

Proof. Let X be the random variable counting the number of triangles in G. Note

$$X = \sum_{T \in [n]^{(3)}} \mathbb{1}_{T^{(2)} \subset G}.$$

Mean is straightforward:

$$\mathbb{E}X = p^3 \binom{n}{3}.$$

Variance is a little more tricky. First:

$$X^2 = \sum_{S,T \in [n]^{(3)}} \mathbb{1}_{S^{(2)}, T^{(2)} \subset G}.$$

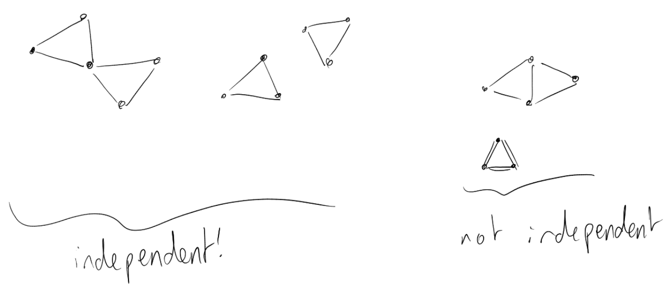
Lecture 3

We can write

$$\mathbb{E}X^2 = \sum_{S,T \in [n]^{(2)}} \mathbb{P}(T^{(2)}, S^{(2)} \subset G)$$

$$(\mathbb{E}X)^2 = \sum_{S,T \in [n]^{(2)}} \mathbb{P}(T^{(2)} \subset G)\mathbb{P}(S^{(2)} \subset G)$$

Since the pairs of events are independent whenever $|S^{(2)} \cap T^{(2)}| = 0$, we have



$$\begin{aligned} \operatorname{Var} X &\leq \sum_{\substack{S,T \in [n]^{(2)} \\ |S^{(2)} \cap T^{(2)}| = 1}} \\ &\leq p^5 n^4 + \mathbb{E}X \end{aligned}$$

Now note

$$
\begin{aligned}
\mathbb{P}(|X - \mathbb{E}X| \geq \delta \mathbb{E}X) &\leq \frac{\operatorname{Var} X}{\delta^2 (\mathbb{E}X)^2} \\
&\leq \frac{p^5 n^4}{\delta^2 (\mathbb{E}X)^2} + \frac{1}{\delta^2 \mathbb{E}X} \\
&\leq C \frac{p^5 n^4}{(n^3 p^3)^2} + \frac{1}{\delta^2 \mathbb{E}X} \\
&\leq C \frac{1}{n^2 p} + \frac{1}{\delta^2 \mathbb{E}X} \\
&\to 0
\end{aligned}
$$

□

Recall that before we showed that for $G \sim G(n,p)$, $\frac{1}{n} \ll p \leq \frac{1}{2}$, we have

$$\alpha(G) \leq (2 + o(1)) \frac{\log(np)}{p}$$

with high probability.

Proof of lower bound on $R(3,k)$. Set $n = \left(\frac{k}{\log k}\right)^{3/2}$, $p = n^{-2/3}$. Let $G \sim G(n,p)$ and let $\tilde{G}$ with a vertex deleted from each triangle. Then with high probability we have

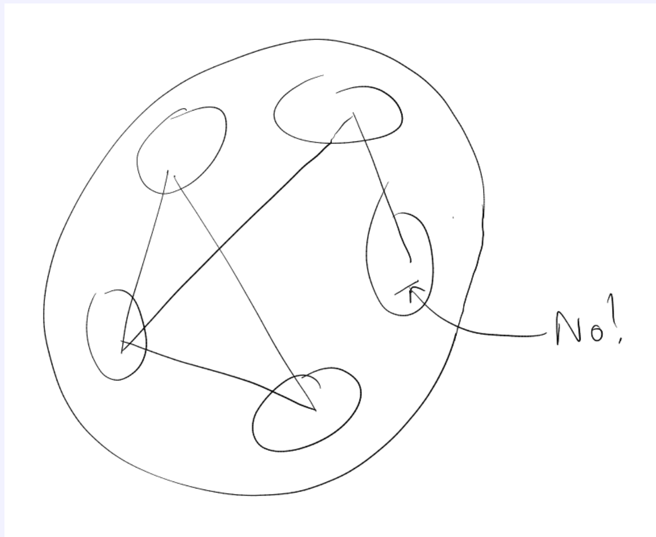
$$
\begin{aligned}
\alpha(\tilde{G}) &\leq \alpha(G) \\
&\leq (2 + o(1)) \frac{\log np}{p} \\
&\leq \left(\frac{2}{3} + o(1)\right) n^{2/3} \log n \\
&\leq \left(\frac{2}{3} + o(1)\right) \frac{k}{\log k} (\log k^{3/2}) \\
&\leq (1 + o(1)) k \qquad (*)
\end{aligned}
$$

We also have with high probability:

$$
\begin{aligned}
|V(\tilde{G})| &\geq n - (\#\text{of triangles in } G) \\
&\geq n - (1 + o(1)) \frac{p^3 n^3}{6} \\
&= n - (1 + o(1)) \frac{n}{6} \\
&\geq \frac{n}{2} \qquad (**)
\end{aligned}
$$

So with high probability $(*)$ and $(**)$ hold. So there must exist a graph $\tilde{G}$ on $\frac{n}{2} = \frac{1}{2}\left(\frac{k}{\log k}\right)^{3/2}$ with no triangles and independence number $< k$ (actually not quite, but would have worked if we multiply one of the parameters by a constant or something). □

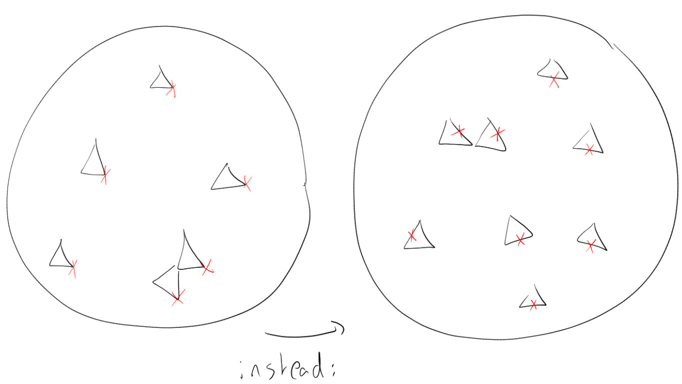
Definition (Chromatic number)**.** Let G be a graph. The chromatic number of G is the minimum k such that there exists $\chi : V(G) \to [k]$such that no edge has both ends in the same colour.



$\chi(G)$ denotes the chromatic number.

Similar to the above lower bound on $R(3, k)$, Erdős proved:

Theorem (Erdős)**.** There exist graphs G with arbitrarily large chromatic number and arbitrarily large girth, i.e. for all $g, k \in \mathbb{N}$ there exists G with girth $> g$ and $\chi(G) > k$.



Idea: let's delete an *edge* from each triangle, instead of a vertex (we have a lot more edges to spend than we have vertices!).

For this we need

$$p^3n^3 \ll pn^2$$

so $p = \frac{\gamma}{\sqrt{n}}$. For some small $\gamma > 0$.

Danger: when we delete an edge from each triangle, we need to ensure that we don't hurt $\alpha(G)$.

Theorem 2.4 (Erdos, 1960s)**.** $R(3,k) \geq c\frac{k^2}{(\log k)^2}$ for some $c > 0$.

To prove this, it is enough to prove for sufficiently large n that there exists a triangle free graph on n vertices with

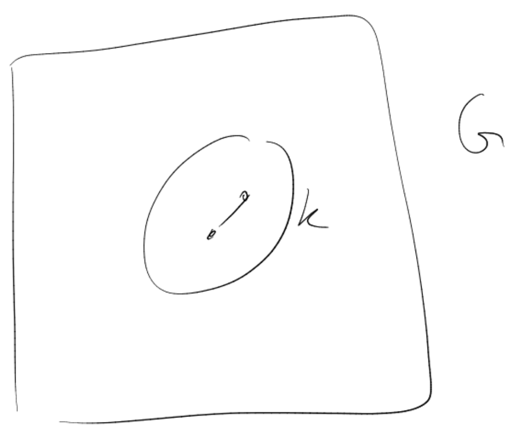
$$\alpha(G) \leq Cn^{\frac{1}{2}} \log n$$

for some $C > 0$.

This is because we can set $k = C\sqrt{n}\log n$, and then get $n = c\frac{k^2}{(\log k)^2}$.

More sketching of the idea: we know $\alpha(G) \leq 2\frac{\log n}{p}$.

What if $k > 2\frac{\log np}{p}$? Then there must be an edge (because $\alpha(G) < k$.



Not very impressive.

What if $k > 100\frac{\log np}{p}$? Then we actually get a lot of edges: expect pk^2, and can guarantee $\frac{pk^2}{16}$.

Lemma 2.5. Assuming that:

- $\frac{1}{n} \ll p \leq \frac{1}{2}$
- $G \sim G(n,p)$
- $k > C\frac{\log n}{p}$

Then with high probability every set of size $\geq k$ in G induces $\frac{pk^2}{16}$ edges.

Lecture 4

Proof. Fix $t = \frac{pk^2}{16}$ and fix a set S with size $|S| = k$.

$$\begin{aligned}
\mathbb{P}(e(G[S]) < t) &\leq \sum_{i=0}^{t} \binom{\binom{k}{2}}{i} p^i (1-p)^{\binom{k}{2}-i} \\
&\leq t\binom{\binom{k}{2}}{t} p^t (1-p)^{\binom{k}{2}-t} \\
&\leq t\left[\left(\frac{ek^2}{2t}\right)^t p^t\right] e^{-p\left[\binom{k}{2}-t\right]} \\
&\leq t(e8)^{\frac{pk^2}{16}} e^{-p\binom{k}{2}/2} \\
&\leq k^2\left[(e8)^1 e^{-4+o(1)}\right]^{\frac{pk^2}{16}} \\
&\leq e^{-\alpha pk^2}
\end{aligned}$$

for some $\alpha > 0$.

We now union bound on all $S \in [n]^{(k)}$.

$$\begin{aligned}
\mathbb{P}(\exists S \in [n]^{(k)} \text{ such that } e([S]) < t) &\leq \binom{n}{k}\mathbb{P}(e(G[S]) < t) \\
&\leq \left(\frac{en}{k}\right)^k e^{-\alpha k^2 p} \\
&= \left(\frac{en}{k} \cdot e^{-\alpha kp}\right)^k \\
&\leq \left(enp\left(\frac{1}{pn}\right)^{\alpha C}\right)^k
\end{aligned}$$

(using $kp \geq C\log n$) which $\to 0$ for $C > \frac{1}{\alpha}$. $\square$

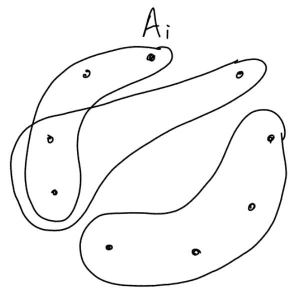
Lemma 2.6. Assuming that:

- $\mathcal{F} = \{A_i\}_i$ be a collection of events

- $\mathcal{E}_t$, for $t \in \mathbb{N}$, be the event that t independent events of $\mathcal{F}$ occur

Then

$$\mathbb{P}(\mathcal{E}_t) \leq \frac{1}{t!} \left(\sum_i \mathbb{P}(A_i) \right)^t .$$



Remark. $t! = t^t e^{-t} \sqrt{2\pi t}(1 + o(1))$.
We will use $t! \geq t^t e^{-t}/2$ for large t in the above lemma to get:

$$\mathbb{P}(\mathcal{E}_t) \leq 2 \left(\frac{e\mathbb{E}(\#\text{ of } A_i \text{ that hold}}{t} \right)^t .$$

Proof. Let $\mathcal{I} = \{(A_{i_1}, \dots, A_{i_k}) : A_{i_1}, \dots, A_{i_k} \text{ are independent}\}$.

$$\begin{aligned} \mathbb{1}_{\mathcal{E}_t} &\leq \frac{1}{t!} \sum_{(A_{i_1}, \dots, A_{i_k} \in \mathcal{I}} \mathbb{1}_{A_{i_1} \cap \dots \cap A_{i_k}} \\ \mathbb{P}(\mathcal{E}_t) &\leq \sum_{(A_{i_1}, \dots, A_{i_k}) \in \mathcal{I}} \mathbb{P}(A_{i_1} \cap \dots \cap A_{i_k}) \\ &\leq \frac{1}{t!} \sum_{i_1, \dots, i_t} \mathbb{P}(A_{i_1}) \cdots \mathbb{P}(A_{i_k}) \\ &= \frac{1}{t!} \left(\sum_i \mathbb{P}(A_i) \right)^t \end{aligned}$$

□

Theorem. Assuming that:

- $n \geq 2$

Then there exists a triangle-free graph G on n vertices with

$$\alpha(G) \leq C\sqrt{n}\log n.$$

Proof. Let n be large. Let $p = \frac{\gamma}{\sqrt{n}}$ for some $\gamma > 0$. Let $G \sim G(n,p)$, and let $\tilde{G}$ be G with a maximal collection of edge disjoint triangles T removed, i.e. $\tilde{G} = G - T$.

Clearly $\tilde{G}$ is triangle-free. We let $\mathcal{Q}$ be the event that every set of size k, where $k = C\sqrt{n}\log n$ contains at least $\frac{pk^2}{16}$ edges. We now consider

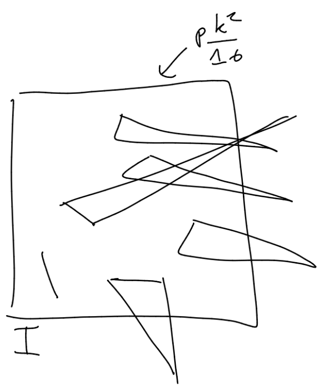
$$\mathbb{P}(\alpha(\tilde{G}) \geq k) = \mathbb{P}(\alpha(\tilde{G}) \geq k \cap \mathcal{Q}) + \mathbb{P}(\alpha(\tilde{G}) \geq k \cap \mathcal{Q}^c).$$

So

$$\mathbb{P}(\alpha(\tilde{G}) \geq k) \leq \underbrace{\mathbb{P}(\alpha(\tilde{G}) \geq k \cap \mathcal{Q})}_{(*)} + o(1).$$

We now union-bound

$$\begin{aligned}(*) &\leq \binom{n}{k}\mathbb{P}(I \text{ independent in } \tilde{G} \cap \mathcal{Q}) \\ &= \binom{n}{k}\mathbb{P}\left(T \text{ intersects } I \text{ in } \geq \frac{pk^2}{16} \text{ edges}\right)\end{aligned}$$



Define $\{T_i\}$ to be the collection of all triangles that meet I in at least one edge, and define the event $A_i = \{T_i \subset G\}$.

Note that the event $\mathcal{E}_t$, where $t = \frac{pk^2}{16}$, from the lemma holds on the event that T meets I in $\frac{pk^2}{16}$ edges.

This is where we use the fact that we only choose edge-disjoint triangles! By Lemma 2.6, we have

$$\begin{aligned}\mathbb{P}(\mathcal{E}_t) &\le 2\left(\frac{ek^2np^3}{t}\right)^t \\ &\le \left(\frac{16ek^2np^3}{pk^2}\right)^{\frac{pk^2}{16}} \\ &= (16e\gamma^2)^{\frac{pk^2}{16}}\end{aligned}$$

So

$$(*) \le (16e\gamma^2)^{\frac{pk^2}{16}}\left(\frac{en}{k}\right)^k \to 0,$$

if γ is chosen to be small and C large. □

Lecture 5

2.1 Lovasz-Erdős Local Lemma

Lemma 2.7 (Erdos-Lovasz, 70s)**.** Assuming that:

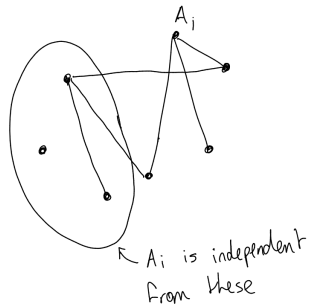
- $\mathcal{F} = \{A_i\}$ is a family of events in a probability space
- $\mathcal{G}$ a dependency graph of $\mathcal{F}$
- $\mathbb{P}(A_i) \le \frac{1}{e(\Delta+1)}$, where $\Delta = \Delta(\mathcal{G})$ is the max degree of $\mathcal{G}$

Then

$$\mathbb{P}\left(\bigcap_i A_i^c\right) > 0.$$

Definition 2.8 (Dependency graph)**.** If $\mathcal{F} = \{A_i\}$ is a family of events, a *dependency graph* $\mathcal{G}$ is a graph with vertex set $\mathcal{F}$, with the property that: for all i, the event A_i is independent from $\{A_j : A_j \not\sim A_i\}$.

Picture:



Reminder: A_i is *independent* from $\{A_j : A_j \not\sim A_i\}$ if E is formed by taking intersections of $A_j, A_j^c \not\sim A_i$, we have

$$\mathbb{P}(A_i \cap E) = \mathbb{P}(A_i)\mathbb{P}(E).$$

Theorem (Spencer)**.** $R(k) \geq (1 + o(1))\frac{\sqrt{2}k}{e}2^{k/2}$.

Proof. Let $\delta > 0$ and let $n = (1-\delta)\frac{\sqrt{2}k}{e}2^{k/2}$. Choose a colouring uniformly at random. For every $S \in [n]^{(k)}$, define the event

$$A_S = \text{“}S^{(2)} \text{ are monochromatic”}.$$

We clearly have $\mathbb{P}(A_S) = 2^{-\binom{k}{2}+1}$.

Let $\mathcal{G}$ be the dependency graph on $\{A_S\}_S$ where $A_S \sim A_T$ if $S^{(2)} \cap T^{(2)} \neq \emptyset$.

$$\Delta = \Delta(G) \leq \binom{k}{2} \cdot \binom{n-2}{k-2} \leq k^4 \cdot \left(\frac{1}{n^2}\right)\binom{n}{k}$$

We now check:

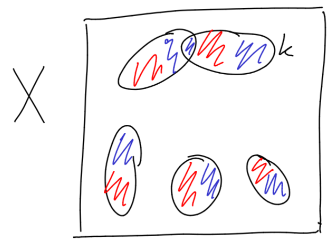
$$\Delta \cdot 2^{-\binom{k}{2}+1} \leq 2k^4\left(\frac{1}{n^2}\right)\left(\frac{en}{k}\right)^k 2^{-\binom{k}{2}} = \left[(1+o(1))\frac{1}{n^{2/k}}\left(\frac{en\sqrt{n}}{k} \cdot 2^{-k/2}\right)\right]^k.$$

This $\to 0$. So the condition in Local Lemma holds for sufficiently large k. $\square$

Remark. This is the best known lower bound for diagonal Ramsey.

Definition 2.9 (k-uniform hypergraph)**.** Say $\mathcal{H}$ is a *k-uniform hypergraph* on vertex set X if $\mathcal{H} \subset X^{(k)}$.

Definition 2.10 (Colourable k-uniform hypergraph)**.** Say a k-uniform hypergraph is 2-colourable if there exists $\chi : X \to \{\text{red}, \text{blue}\}$ such that there is no monochromatic $e \in \mathcal{H}$.



Theorem 2.11. Assuming that:

- $\mathcal{H}$ a k-uniform hypergraph with maximum degree d
- $d \le \frac{2^{k-1}}{ek} - 1$

Then $\mathcal{H}$ is 2-colourable.

Proof. We choose our colouring of the ground set X uniformly at random. For each $e \in \mathcal{H}$, define the event

$$A_e = \text{“}e \text{ monochromatic”}.$$

We have $\mathbb{P}(E_e) = 2^{-k+1}$. We want $2^{-k+1} \le \frac{1}{e(\Delta+1)}$. Note if we define our dependency graph $\mathcal{G}$ by $A_e \sim A_f$ if $e \cap f \neq \emptyset$, then

$$\Delta(\mathcal{G}) \le d \cdot k.$$

Now just use the bound on d in the hypothesis. □

Remark. Actually, our second application implies the first on $R(k)$: just let $\mathcal{H}$ be the $\binom{k}{2}$-uniform hypergraph on $X = [n]^{(2)}$, defined by

$$\mathcal{H} = \{S^{(2)} : S \in [n]^{(k)}\}.$$

Theorem 2.12 (Lopsided Local Lemma)**.** Assuming that:

- $\mathcal{F} = \{A_i\}$ a family of events
- $\mathcal{G}$ a dependency graph

- $x_1, \ldots, x_n \in (0,1)$ be such that $\forall i$,

$$\mathbb{P}(A_i) \le x_i \prod_{A_i \sim A_j} (1 - x_j)$$

(where $A_i \sim A_j$ denotes adjacency in the dependency graph).

Then

$$\mathbb{P}\left(\bigcap_{i=1}^n A_i^c\right) \ge \prod_{i=1}^n (1 - x_i).$$

Proof. We use

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\mathbb{P}(B).$$

Applying this many times, we have

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{i=1}^n A_i^c\right) &= \prod_{i=1}^n \mathbb{P}(A_i^c \mid A_1^c \cap \cdots \cap A_{i-1}^c) \\
&= \prod_{i=1}^n (1 - \underbrace{\mathbb{P}(A_i \mid A_1^c \cap \cdots \cap A_{i-1}^c))}_{\le x_i?}
\end{aligned}$$

We will prove that this term is $\le x_i$ by instead proving the more general statement:

$$(*) = \mathbb{P}\left(A_i \;\middle|\; \bigcap_{j \in S} A_j^c\right) \le x_i,$$

where $S \subset [n] \setminus \{j\}$. We prove by induction on $|S|$.

If $S = \emptyset$, then done by hypothesis.

If $S \ne \emptyset$, then define

$$\mathcal{D} = \bigcap_{\substack{j \in S \\ j \sim i}} A_j^c, \qquad \mathcal{I} = \bigcap_{j \in S j \not\sim i} A_j^c.$$

Note that

$$\begin{aligned}
(*) &= \mathbb{P}(A_i \mid D \cap I) \\
&= \frac{\mathbb{P}(A_i \cap D \cap I)}{\mathbb{P}(D \cap I)} \\
&\le \frac{\mathbb{P}(A_i \cap I)}{\mathbb{P}(D \cap I)} \\
&= \frac{\mathbb{P}(A_i)\mathbb{P}(I)}{\mathbb{P}(D \cap I)} \\
&= \frac{\mathbb{P}(A_i)}{\mathbb{P}(D \mid I)}
\end{aligned}$$

Lecture 6 Let $\{A_{i_1}, \dots, A_{i_s}\}$ be the events in the intersection D. Then

$$\begin{aligned}\mathbb{P}(D \mid I) &= \prod_{j=1}^{s} \mathbb{P}(A_{i_j}^c \mid I \cap A_{i_1}^c \cap \cdots \cap A_{i_{j-1}}^c) \\ &= \prod_{j=1}^{s} (1 - \mathbb{P}(A_{i_j} \mid I \cap A_{i_1}^c \cap \cdots \cap A_{i_{j-1}}^c))\end{aligned}$$

If no events in the intersection, then we are already done. Otherwise, we may apply the induction hypothesis to each term to get

$$\frac{\mathbb{P}(A_i)}{\mathbb{P}(D \mid I)} \leq \frac{x_i \prod_{j\sim i}(1-x_j)}{\prod_{j=1}^{s}(1-x_{i_j})} \leq x_i. \qquad \square$$

Proof of Local Lemma. Apply Lopsided Local Lemma with weights $x_i = \frac{1}{\Delta+1}$ for all i. We just need to check that the probability upper bounds in Lopsided Local Lemma hold.

By calculus, we have

$$\left(\frac{\Delta}{\Delta+1}\right)^{\Delta} \geq \frac{1}{e}.$$

Hence

$$\begin{aligned}\mathbb{P}(A_i) &\leq \frac{1}{e(\Delta+1)} \\ &\leq \frac{1}{\Delta+1}\left(\frac{\Delta}{\Delta+1}\right)^{\Delta} \\ &= \frac{1}{\Delta+1}\left(1-\frac{1}{\Delta+1}\right)^{\Delta} \\ &\leq \frac{1}{\Delta+1}\prod_{A_i\sim A_j}\left(1-\frac{1}{\Delta+1}\right)\end{aligned} \qquad \square$$

Now we will see another proof of $R(3,k) \geq c\frac{k^2}{(\log k)^2}$, in order to give some intuition as to what kind of situations are good for Local Lemma.

Proof of $R(3,k) \geq c\frac{k^2}{(\log k)^2}$ using Local Lemma. Recall that it is enough to show that for sufficiently large n, there exists a triangle -free graph on n vertices with $\alpha(G) \leq C\sqrt{n}\log n$.

Set $k = C\sqrt{n}\log n$, and sample $G \sim G(n,p)$ where $p = \frac{\gamma}{\sqrt{n}}$, with $\gamma > 0$ is chosen later.

Define the events:

- For each $T \in [n]^{(3)}$, define

$$A_T = \text{“}T^{(2)} \subset G\text{”}.$$

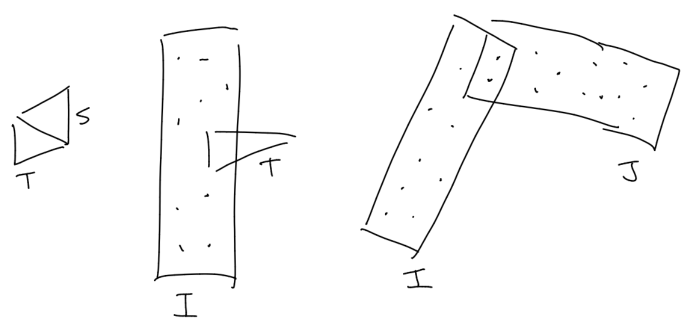
- For each $I \in [n]^{(k)}$, define
$$B_I = \text{“}I^{(k)} \text{ independent in } G\text{”}.$$

Define the dependency graph $\mathcal{G}$:

- $A_T \sim A_S$ for $S, T \in [n]^{(3)}$ if $S^{(2)} \cap T^{(2)} \neq \emptyset$.
- $A_T \sim B_I$ for $T \in [n]^{(3)}, I \in [n]^{(k)}$ if $S^{(2)} \cap I^{(2)} \neq \emptyset$.
- $B_I \sim B_J$ for $I, J \in [n]^{(k)}$ if $I^{(2)} \cap J^{(2)} \neq \emptyset$.

Now note:

- The event A_T is $\sim$ to $\leq 3n$ A_S, $S \in [n]^{(3)}$.
- The event A_T is $\sim$ to $\leq 3n^{k-2}$ B_I, $I \in [n]^{(k)}$.
- The event B_I is $\sim$ to $\leq k^2 n$ A_S, $S \in [n]^{(3)}$.
- The event B_I is $\sim$ to $\leq k^2 n^{k-2}$ B_J, $J \in [n]^{(k)}$.



We choose $x_T = 2p^3$ for all $T \in [n]^{(3)}$ (“we want to make it a bit bigger than the probability of it occuring so that we can afford the product stuff”) and $x_I = n^{-10k}$ for all $I \in [n]^{(k)}$.

Then

$$\begin{aligned} x_T \prod_{\substack{S \in [n]^{(3)} \\ T \sim S}} (1 - x_S) \prod_{\substack{I \in [n]^{(k)} \\ I \sim T}} (1 - x_I) &\geq 2p^3 (1 - 2p^3)^{3n} (1 - n^{-10k})^{3n^{k-2}} \\ &\geq 2p^3 \exp(-(1 + o(1))[2p^3(3n) + 3n^{-10k} n^{k-2}]) \\ &= 2p^3 (1 + o(1)) \\ &\geq p^3 \end{aligned}$$

for large enough n. We have $\mathbb{P}(B_I) = (1-p)^{\binom{k}{2}} \le e^{-p\binom{k}{2}}$, so

$$\begin{aligned} x_I(1-2p^3)^{k^2 n}(1-n^{-10k})^{k^2 n^{k-2}} &\ge n^{-10k}\exp(-(1+o(1))[2p^3k^2n + n^{-10k}k^2n^{k-2}]) \\ &= n^{-10k}\exp(-(1+o(1))(2\gamma^2)pk^2) \end{aligned}$$

Note

$$e^{-p\binom{k}{2}} = \exp\left(-\frac{\gamma}{\sqrt{n}}kC\sqrt{n}(\log n)\right) = n^{-\gamma C k}.$$

First choose γ small so $e^{-2\gamma^2 pk} \gg e^{-p\binom{k}{2}\frac{1}{2}}$, and then choose $C \gg \frac{1}{\gamma}$, then done. $\square$

2.2 Upper bounds on $R(3,k)$

Theorem 2.13 (Ajtai-Komlós-Szemerédi, 1980s + Shearer 80s)**.** $R(3,k) \le (1+o(1))\frac{k^2}{\log k}$.

Theorem 2.14 (Shearer, 1980s)**.** Assuming that:

- G a triangle-free graph on n vertices
- max degree d

Then

$$\alpha(G) \ge (1+o(1))\frac{n}{d}(\log d),$$

where $o(1) \to 0$ as $d \to \infty$.

Remark. If we take $G \sim G\left(n, \frac{d}{n}\right)$ and modify, for $d \ll \sqrt{n}$, one can show that there exist graphs with no triangles and

$$\alpha(G) \le (2+o(1))\frac{n}{d}\log d.$$

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