Higher-order Uniformity and Applications

Lectured by Julia Wolf

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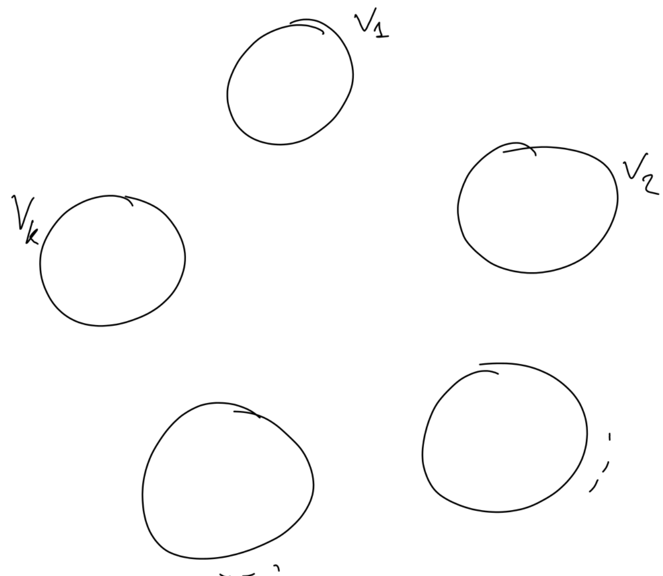
Lecture 1

1 The regularity lemma and applications

Theorem 1.1 (Szemerédi)**.** For all $\varepsilon > 0$, there exists $k = k(\varepsilon)$ such that the vertex set of any sufficiently large graph $G = (V, E)$ can be partitioned into $V_1, \ldots, V_s$, $s \leq k$ such that for all but an ε-proportion of pairs (V_i, V_j), $G(V_i, V_j)$ is *ε-regular.*



Remark.

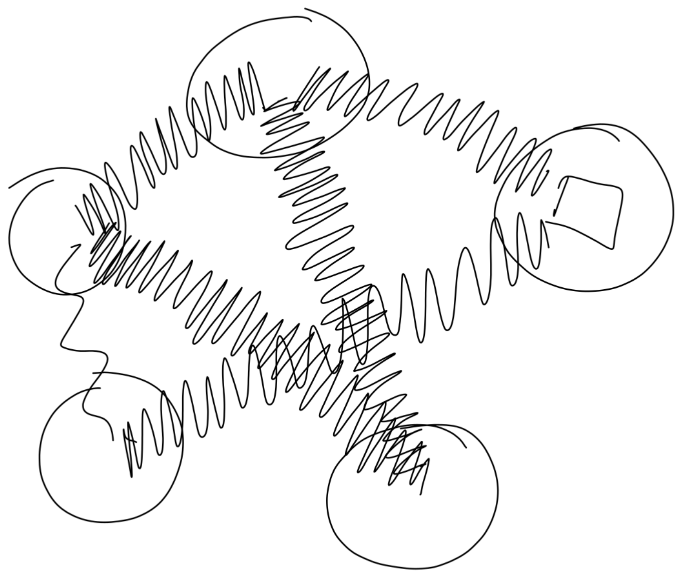
- ε-regular means: looks like a "random" graph. We will define it more thoroughly at the end of the lecture.
- In Theorem 1.1, it is also possible to ensure that all the V_i are almost the same size.
- k is known to have a bad dependence on ε: we have

$$2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}},$$

  where the tower of exponentials is of size ε^{-C}.

Theorem 1.2 (Removal)**.** For all $\delta > 0$, there exists $\eta > \eta(\delta)$ such that the following holds. If G is a graph on n vertices with at most ηn^3 triangles, then it is possible to remove $\leq \delta n^2$ edges to make it triangle-free.

Sketch proof. Apply Theorem 1.1 with $\varepsilon = \frac{\delta}{4}$ to obtain $V_1, \ldots, V_s$, $s \leq k = k(\delta)$ of almost equal size.

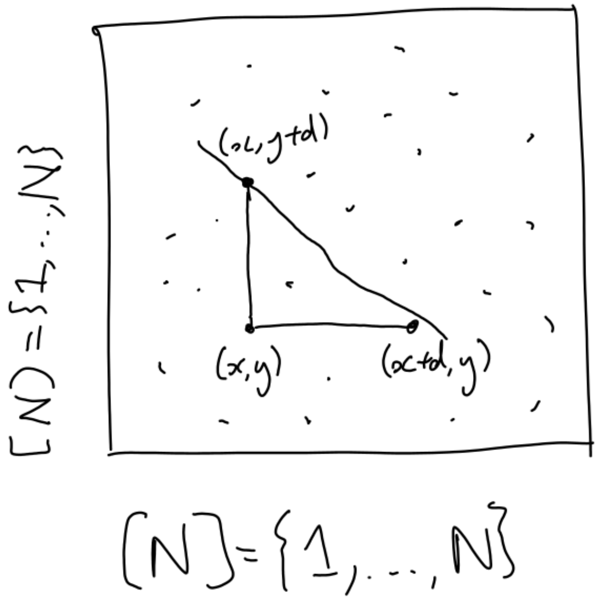


Remove all edges between V_i and V_j where $G(V_i, V_j)$ fails to be ε-regular, or where the density of $G(V_i, V_j) \leq 2\varepsilon$. The number of edges removed in this way is $\leq \varepsilon n^2 + 2\varepsilon n^2 < \delta n^2$.

Claim: The "reduced" graph is triangle-free.

Indeed, if x, y, z form a triangle in G', then (x, y, z) must lie in $V_i \times V_j \times V_k$ with $G(V_i, V_j)$, $G(V_i, V_k)$, $G(V_j, V_k)$ all regular and dense (density $\geq 2\varepsilon$). Hence we in fact have $\geq \varepsilon^3 \left(\frac{n}{k}\right)^3 = \frac{\delta^3}{2^6} \frac{n^3}{k(\delta)}$ triangles in G. This is a contradiction if $\eta < \frac{\delta^3}{2^6 k(\delta)}$. □

Theorem 1.3 (Corners)**.** For all $\alpha > 0$, there exists $N_0 = N_0(\alpha)$ such that for all $N \geq N_0$, the following holds. Let $A \subseteq [N]^2$ of density α ($|A|/N^2 = \alpha$). Then A contains a triple of the form (x, y), $(x + d, y)$, $(x, y + d)$ with $d > 0$.



Remark. The theorem as stated can be fairly easily deduced from a version where we only ask that $d \neq 0$. To do this: note that if A is symmetric, then existence of a triangle with $d < 0$ implies existence of one with $d > 0$. Then one can "make A symmetric" (in exchange for a loss in density) by intersecting A with a reflection of A through a suitably chosen point.

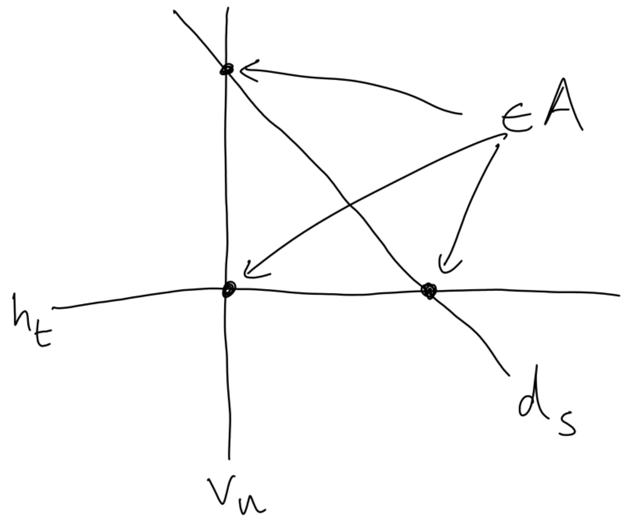
Proof. Let

$$X = \{v_u = \{(x, y) : x = u\} : u \in [N]\}$$
$$Y = \{h_t = \{(x, y) : y = t\} : t \in [N]\}$$
$$Z = \{d_s = \{(x, y) : x + y = s\}\}$$

We define a tripartite graph on parts X, Y, Z, where vertices are joined by an edge if and only if the intersection of the corresponding lines lies in A. A triangle in this graph corresponds to three points

$$(u, t), (u, s - u), (s - t, t) \in A.$$

Setting $d = s - u - t$, these points are (u, t), $(u, t + d)$, $(u + d, t)$.



If A contains no corner with $d \neq 0$, then the only triangles in this graph are the degenerate ones ($d = 0$). There are $\alpha N^2 = |A|$ many of these, and they are edge disjoint. Pick $\delta = \frac{\alpha}{2}$, then by triangle-removal, there exists $\eta = \eta(\alpha)$ such that we can destroy ηN^3 triangles by removing at most δN^2 edges.

If $\alpha N^2 < \eta N^3$, then should be able to remove all triangles. But this is a contradiction since all the triangles are edge disjoint. □

Theorem 1.4 (Roth)**.** For all $\alpha > 0$, there exists $N_0 = N_0(\alpha)$ such that for all $N \geq N_0$, every $A \subseteq [N]$ of density α $(= |A|/N)$ contains a non-trivial 3-AP (a triple $x, x+d, x+2d$).

Sketch proof. Let $B = \{(x, y) \in [N]^2 : x - y \in A\}$. By Theorem 1.3, B contains $(x, y), (x, y+d), (x+d, y)$ with $d \neq 0$. Then $x - y, x - (y+d) = x - y + d, x + d - y = x - y + d \in A$. □

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