Analysis of Boolean Functions

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Lecture 1

Introduction

We will be analysing functions $f : \{0,1\}^n \to \{0,1\}$.

One reason to be interested in these is because of computers.

A more combinatorial reason is that a function $f : \{0,1\}^n \to \{0,1\}$ can be viewed as a function $f : \mathcal{P}([n]) \to \{0,1\}$, so can be viewed as a *set system* (a subset of $\mathcal{P}([n])$). Set systems are very much of interest in combinatorics (e.g. Sperner's Lemma, Kruskal-Katona, etc).

Remarks on differences between this course and additive combinatorics

In additive combinatorics, it is common to study $\mathbb{F}_2^n$ in a way that is basis-independent. When studying boolean functions, we *won't* be working in a basis-independent way.

Slogan: if you have a basis that you care about, then perhaps you are working in the boolean functions world, rather than the additive combinatorics world.

1 Discrete Fourier Analysis

Definition 1.1 (Character)**.** Let G be a finite Abelian group. A *character* on G is a homomorphism $\chi : G \to \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Remark. This definition doesn't change if $\mathbb{T}$ is replaced by $(\mathbb{C} \setminus \{0\}, \times)$ (because any finite subgroup of $(\mathbb{C} \setminus \{0\}, \times)$ must be a subgroup of $\mathbb{T}$).

Observe that if χ_1 and χ_2 are characters, then so is $\chi_1\chi_2$, and also that if χ is a character then so is $\overline{\chi} = \chi^{-1}$. Also, $\chi\overline{\chi} = \chi_0$, $\chi_0\chi = \chi$.

Thus, the characters on G form an Abelian group, called the *(Pontryagin) dual* $\hat{G}$ of G.

Notation 1.2. Let $f, g : G \to \mathbb{C}$. We write

$$\langle f, g\rangle = \mathbb{E}_{x\in G} f(x)\overline{g(x)},$$

where $\mathbb{E}_{x\in G}$ means $|G|^{-1}\sum_{x\in G}$. Then we also write $\|f\|_2 = \langle f, f\rangle^{\frac{1}{2}} = (\mathbb{E}_x|f(x)|^2)^{\frac{1}{2}}$. We also define $\|f\|_p = (\mathbb{E}_x|f(x)^p)^{\frac{1}{p}}$, $1 \le p < \infty$ and $\|f\|_\infty = \max_x |f(x)|$.

Lemma 1.3 (Orthogonality of characters)**.** The characters on G form an orthonormal set.

Proof. Let χ_1, χ_2 be characters and let $\chi = \chi_1\overline{\chi_2}$. We need to prove that $\mathbb{E}_x\chi(x)$ is 1 if $\chi = \chi_0$ and 0 otherwise. If $\chi = \chi_0$, then the result is clear. Otherwise, pick u such that $\chi(u) \ne 1$. Then

$$\mathbb{E}_x\chi(x) = \mathbb{E}_x\chi(ux) = \chi(u)\mathbb{E}_x\chi(x).$$

Since $\chi(u) \ne 1$, we get the result. □

This shows that $|\hat{G}| \le |G|$. To show the reverse inequality we appeal to the classification of finite Abelian groups.

Theorem. Every finite Abelian group is a product of cyclic groups.

Corollary 1.4. The characters on G form an orthonormal basis sof $\mathbb{C}^G$.

Proof. Since they form an orthonormal set, it remains to show that they span. Let

$$G = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_k\mathbb{Z}.$$

Given $r, x \in \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_k\mathbb{Z}$ ($r = (r_1, \ldots, r_k)$, $x = (x_1, \ldots, x_k)$). Let

$$\chi_r(x) = \prod_{j=1}^{k} e^{2\pi i \frac{r_j x_j}{m_j}}.$$

It is easy to check that χ_r is a character, and that if $r \neq s$ then $\chi_r \neq \chi_s$. □

Remark. This proof also demonstrates that $G \cong \hat{G}$. But the isomorphism is 'horrible': it doesn't only depend on G, but also on the choice of m_i and the homomorphism

$$G \cong \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_k\mathbb{Z}.$$

A very useful convention is to use the uniform probability measure on G and counting measure on $\hat{G}$. For example, if $\hat{f}, \hat{g} : \hat{G} \to \mathbb{C}$, we define

$$\langle \hat{f}, \hat{g} \rangle = \sum_{\chi} \hat{f}(\chi)\hat{g}(\chi)$$

and

$$\|\hat{f}\|_p = \left(\sum_{\chi} |\hat{f}(\chi)|^p \right)^{\frac{1}{p}}.$$

Definition 1.5 (Fourier transform)**.** Let $f : G \to \mathbb{C}$. The *Fourier transform* $\hat{f}$ of f is the function from $\hat{G}$ to $\mathbb{C}$ defined by

$$\hat{f}(\chi) = \mathbb{E}_{\chi} f(x)\overline{\chi(x)} = \langle f, \chi \rangle.$$

Lemma 1.6. The Fourier transofrm has the following properties:

(1) *Plancherel / Parseval identity:* $\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$.

(2) *Convolution identity:* Define $f * g$ by

$$f * g(x) = \mathbb{E}_{u+v=x} f(u)g(v).$$

Then

$$f \hat{*} g(\chi)\hat{f}(\chi)\hat{g}(\chi).$$

(3) *Inversion formula:*

$$f(x) = \sum_{\chi} \hat{f}(\chi)\chi(x).$$

Proof.

(1)

$$\begin{aligned}\langle \hat{f}, \hat{g}\rangle &= \sum_{\chi} \hat{f}(\chi)\overline{\hat{g}(\chi)} \\ &= \sum_{\chi} (\mathbb{E}_x f(x)\overline{\chi(x)})\overline{(\mathbb{E}_y g(y)\overline{\chi(y)})} \\ &= \mathbb{E}_x\mathbb{E}_y f(x)\overline{g(y)} \sum_{\chi} \chi(x^{-1}y)\end{aligned}$$

An examination of the proof of Corollary 1.4 shows straightforwardly that

$$\sum_{\chi} \chi(u) = \begin{cases} |G| & u = \text{identity} \\ 0 & \text{otherwise}\end{cases}$$

So $\sum_{\chi} \chi(x^{-1}y) = \Delta_{xy}$ where

$$\Delta_{xy} = \begin{cases} |G| & x = y \\ 0 & x \neq y\end{cases}$$

So we get

$$\mathbb{E}_x\mathbb{E}_y f(x)\overline{g(y)} \sum_{\chi} \chi(x^{-1}y) = \mathbb{E}_x\mathbb{E}_y f(x)\overline{g(y)}\Delta_{xy} = \mathbb{E}_x\mathbb{E}_y f(x)\overline{g(x)}.$$

(2), (3) Next time. □

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