Algebraic Geometry

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0 Motivation

In Part II AG we defined an affine algebraic variety by:

Definition (Affine algebraic variety)**.** We fix an algebraically closed field k and defined affine n-space $\mathbb{A}^n_k := k^n$, and for an ideal $I \le k[x_1, \dots, x_n]$ we defined

$$Z(I) := \{(a_1, \dots, a_n) \in \mathbb{A}^n \mid f(a_1, \dots, a_n) = 0 \ \forall f \in I\} \subseteq \mathbb{A}^n_k.$$

We define a topology on $\mathbb{A}^n$ by taking the closed sets to be the sets of the form $Z(I) \subseteq \mathbb{A}^n$.

We will introduce *schemes*.

Why schemes?

Why not varieties?

(1) With varieties, we always work with algebraically closed fields. For example, take $k = \mathbb{R}$, $I = (x^2 + y^2 + 1) \subseteq \mathbb{R}[x, y]$. Then $Z(I) = \emptyset$.

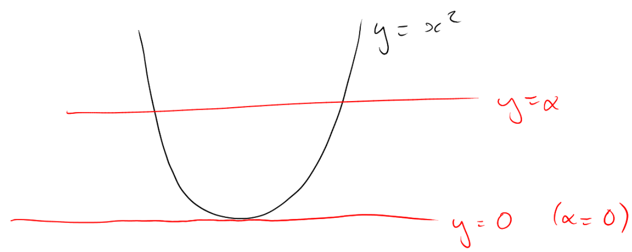
(2) Number theory? We study Diophantine equations, i.e. $I \subseteq \mathbb{Z}[x_1, \dots, x_n]$. So $Z(I) \subseteq \mathbb{Z}^n$.

(3) Even if k is algebraically closed, we lose information when we pass from I to $Z(I)$. For example, $I = (x^2) \subseteq \mathbb{C}[x]$. Then $Z(I) = \{0\} = Z(x)$.

Recall Hilbert's Nullstellensatz, which states that the set of polynomials in $k[x_1, \dots, x_n]$ vanishing on $Z(I)$ for $I \subseteq k[x_1, \dots, x_n]$ is the *radical* of I.

But it is natural ideals like (x^2).

$(y - x^2, y - \alpha) \subseteq \mathbb{C}[x, y]$ $(\alpha \in \mathbb{C})$.



If $x = 0$, $(x^2 - y, y - \alpha) = (x^2, y)$.

0.1 Categorical philosophy

(Read definition on Wikipedia or see CT).

Let **Sets** be the category of sets.

Sets is the category with objects being all sets, with morphisms between objects maps of sets. If X and Y are sets, we write $\mathrm{Hom}(X, Y)$ for the set of maps between X and Y. Note that there is a bijection

$$\mathrm{Hom}(\{*\}, X) \to X$$
$$(f : \{*\} \to X) \mapsto f(*)$$

Let's use this philosophy to understand points on affine algebraic varieties.

$\mathbb{A}^0_k$ is a point. If X is an affine variety, then the points of X should be in $1-1$ correspondence with $\mathrm{Hom}(\mathbb{A}^0_k, X)$. Giving a morphism between affine varieties is easy. Denote by $A(X)$ (sometimes $k[X]$) the coordinate ring ($A(X) = k[x_1, \ldots, x_n]/I(X)$, where $I(X)$ is the ideal of functions that vanish on X). This is a k-algebra. We showed that if X and Y are affine varieties, then $\mathrm{Hom}(X, Y) = \mathrm{Hom}_{k\text{-alg}}(A(Y), A(X))$ (see Part II AG or handout). So

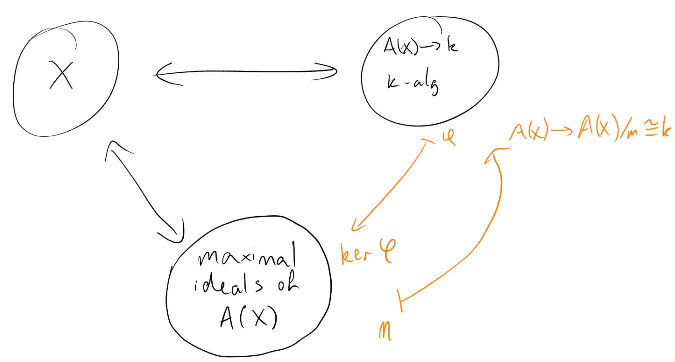
$$\mathrm{Hom}(\mathbb{A}^0_k, X) = \mathrm{Hom}_{k\text{-alg}}(k[x_1, \ldots, x_n]/I(X), k).$$

Note that giving a k-algebra homomorphism

$$k[x_1, \ldots, x_n] \to k$$

can be done by specifying the images of x_i can be done by specifying the images of x_i, say $x_i \mapsto a_i$ such that for any $f \in I(X)$, $f(a_1, \ldots, a_n) = 0$. So there is a $1-$ correspondence between such k-algebra homomorphisms and points of X.

If k is algebraically closed, the maximal ideals of $k[x_1, \ldots, x_n]$ are precisely the ideals of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for $(a_1, \ldots, a_n) \in \mathbb{A}^n$ (a form of Hilbert Nullstellensatz) and the maximal ideals of $A(X)$ are of the form $(x_1 - a_1, \ldots, x_n - a_n)$ mod $I(X)$ with $(a_1, \ldots, a_n) \in X$. Thus we have a $1-1$ correspondence between points of X and maximal ideals of $A(X)$.



Now suppose k is *not* algebraically closed. Let's consider k-algebra homomorphisms

$$k[x_1, \ldots, x_n]/I(X) = A(X) \to L$$

where L is an extension of k. These are given by $x_i \mapsto a_i$ with $f(a_1, \ldots, a_n) = 0$ for all $f \in I(X)$. Thus

$$\mathrm{Hom}_k(A(X), L) = \{(a_1, \ldots, a_n) \in \mathbb{A}^n_L \mid f(a_1, \ldots, a_n) = 0 \ \forall f \in I(X)\}.$$

In other words: the k-algebra homomorphisms $A(X) \to L$ correspond to L-valued points, i.e. points with coordinates in L.

Could work over $\mathbb{Z}$. Take an ideal $I \subseteq \mathbb{Z}[x_1, \ldots, x_n]$, and set

$$A = \mathbb{Z}[x_1, \ldots, x_n]/I.$$

Ring homomorphisms $A \to \mathbb{Z}$ are $1-1$ correspondence with $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ such that $f(a_1, \ldots, a_n) = 0 \ \forall f \in I$. Maps $A \to \mathbb{F}_p$ give "solutions" mod$p$, or maps $A \to \mathbb{Q}$ give rational solutions.

What we want: Given an extension of A,e we want to define a gadget $X = \mathrm{Spec}A$ (spectrum of A), and an R-valued point of X is a ring homomorphism $A \to R$. We write the set of R-valued points as

$$X(R) := \mathrm{Hom}_{\mathrm{Ring}}(A, R).$$

Morphisms $\mathrm{Spec}B \to \mathrm{Spec}A$ should be the same as ring homomorphisms $A \to B$.

Definition (Category of affine schemes)**.** The category of affine schemes is the opposite category of rings.

Reminder: All of our rings have 1 and are commutative, and ring homomorphisms $\varphi : A \to B$ satisfy $\varphi(1) = 1$.

Definition. A *scheme* is a geometric object which is locally an affine scheme.

Analogy: A manifold is something which locally looks like an open subset of $\mathbb{R}^n$.

Definition 0.1 (Spectrum)**.** Let A be a ring. Then

$$\mathrm{Spec} = \{p \subseteq A \mid P \text{ a prime ideal}\}.$$

Note: In general, if we have an L-valued point of $X = Z(I) \subseteq \mathbb{A}^n$, we get a ring homomorphism $\varphi : A(X) \to L$ has image an integral subdomain of L, so $\ker \varphi$ is prime.

Definition 0.2 ($V(I)$)**.** For $I \subseteq A$ an ideal, define

$$V(I) = \{P \in \mathrm{Spec}\, A \mid P \supseteq I\}.$$

Proposition. *The sets $V(I)$ form the closed sets of a topology on* Spec A*, called the Zariski topology.*

Proof.

(1) $V(A) = \emptyset$

(2) $V(0) = \operatorname{Spec} A$

(3) If $\{I_j\}_{j \in J}$ is a collection of ideals, then

$$\bigcap_{j \in J} V(I_j) = V\left(\sum_{r \in J} I_j\right).$$

(easy!)

(4) $V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$:

$\subseteq$ If $P \supseteq I_1$ or $P \supseteq I_2$, then $P \supseteq I_1 \cap I_2$.

$\supseteq$ If $P \supseteq I_1 \cap I_2$, then $P \supseteq I_1$ or $P \supseteq I_2$.

See Atiyah + MacDonald, Prop 1.11 ii) [Try to prove it for yourself!] $\square$

Example. $A = k[x_1, \ldots, x_n]$ with k algebraically closed. For $I \subseteq A$, the maximal ideals of A corresponding to points of $Z(I)$ are precisely the maximal ideals containing I.

1 Sheaves

Fix a topological space X.

Definition 1.1 (Presheaf)**.** A *preasheaf* $\mathcal{F}$ on X consists of data:

(1) For every open set $U \subseteq X$, an abelian group $\mathcal{F}(U)$.

(2) Whenever $V \subseteq U \subseteq X$ open, there is a restriction homomorphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\rho_{UU} = \mathrm{id}_{F(U)}$ and if $W \subseteq V \subseteq U \subseteq X$, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

Remark. This is precisely a contravariant functor from the category of open sets [i.e. objects are open sets $U \subseteq X$, morphisms are inclusions $V \subseteq U$] to the category of abelian groups. Can replace the category of abelian groups with your favourite category.

Definition 1.2 (Morphism of presheaves)**.** If $\mathcal{F}, \mathcal{M}$ are presheaves on X, then a *morphism* of presheaves $f : \mathcal{F} \to \mathcal{M}$ is data of, for each $U \subseteq X$ open, a group homomorphism $f_U : \mathcal{F}(U) \to \mathcal{M}(U)$ such that whenever $V \subseteq U$, we have a commutative diagram

$$\begin{array}{ccccc} s & & \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{M}(U) \\ \downarrow & & \downarrow \rho^{\mathcal{F}}_{UV} & & \downarrow \rho^{\mathcal{M}}_{UV} \\ s|_V & & \mathcal{F}(V) & \xrightarrow{f_V} & \mathcal{M}(V) \end{array} .$$

Example. $\mathcal{F}(U) = \{f : U \to \mathbb{R} \text{ continuous}\}$. $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is restriction of functions.

Definition 1.3 (Sheaf)**.** A presheaf $\mathcal{F}$ on X is a sheaf if it satisfies:

(1) If $U \subseteq X$ is covered by $\{U_i\}$ ($U, U_i \subseteq X$ open) and $s \in \mathcal{F}(U)$ such that $s|_{U_i} = \rho_{UU_i}(s) = 0$, then $s = 0$.

(2) If $U, \{U_i\}$ are as is (1), and $s_i \in \mathcal{F}(U_i)$ for each i such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i (gluing axiom).

Remark.

(1) If $\mathcal{F}$ is a sheaf, then $\mathcal{F}(\emptyset) = 0$ since the empty cover is a cover of $\emptyset$.

(2) Properties (1) and (2) of the definition of a sheaf together can be stated by saying

$$0 \to F(U) \xrightarrow{\alpha} \bigoplus_{i \in I} \mathcal{F}(U_i) \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \bigoplus_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

is *exact*, for all $U \subseteq X$ open and open covers $\{U_i\}$ of U.

Here,

$$\alpha(S) = (S|_{U_i})_{i \in I}$$
$$\beta_1((S_i)_{i\in I}) = (S_i|_{U_i \cap U_r})_{i,j \in I}$$
$$\beta_2((S_i)_{i\in I}) = (S_j|_{U_i \cap U_r})_{i,j \in I}$$

Exactness means:w

(1) α is injective (this is property (1) of a sheaf)

(2) $\beta_1 \circ \alpha = \beta_2 \circ \alpha$ (obvious)

(3) For any $(S_i) \in \bigoplus_{i\in I} \mathcal{F}(U_i)$ with $\beta_1((S_i)) = \beta_2((S_i))$, there exists an $s \in \mathcal{F}(U)$ with $\alpha(S) = (S_i)$. (this is property (2) of a sheaf)

Remark. α is the *equalizer* of β_1, β_2 (from category theory).

This definition works, even if e.g. we a set rather than abelian group.

Example.

(1) X any topological space

$$\mathcal{F}(U) = \{\text{continuous functions } f : U \to \mathbb{R}\}$$

is a sheaf.

(2) $X = \mathbb{C}$ with the Euclidean topology. Then for

$$\mathcal{F}(U) = \{f : U \to \mathbb{C} \mid f \text{ bounded and holomorphic}\}$$

gluing fails because one may not preserve boundedness.

(3) Let G be a group, and set $\mathcal{F}(U) = G$ for all $U \subseteq X$. $\rho_{UV} = \mathrm{id}$. This is a presheaf, known as the *constant presheaf.* If we give G the discrete topology, set

$$\mathcal{F}'(U) = \{f : U \to G \text{continuous}\}.$$

These are locally constant functions. Obviously a sheaf, called the *constant sheaf.*

(4) If X is a variety, denote by $\mathcal{O}_X(U)$ the set of regular functions $f : U \to k$. $\mathcal{O}_X$ is a sheaf, called the *structure sheaf* of X. See Part II AG for definitions of these.

Definition 1.4 (Stalk / germ)**.** Let $\mathcal{F}$ be a presheaf on X, $p \in X$. Then the *stalk* of f at p is

$$\mathcal{F}_p := \{(U, s) \mid U \text{ an open neighbourhood of } p,\ s \in \mathcal{F}(U)\}/\equiv$$

where $(U,s) \equiv (V,t)$ if there exists $W \subseteq U \cap V$ a neighbourhood of p such that $s|_W = t|_W$. The equivalence class of $(U,s) \in \mathcal{F}_p$ is written as s_p, and is the *germ* of s at p.

Note that given a morphism $f : \mathcal{F} \to \mathcal{G}$, we obtain $f_p : \mathcal{F}_p \to \mathcal{G}_p$ via $f_p(U,s) = (U, f_U(s))$.

Note that a morphism of sheaves is just a morphism of presheaves.

Proposition. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then f is an isomorphism if and only if f_p is an isomorphism for all $p \in X$.

Proof.

$\Rightarrow$ Obvious.

$\Leftarrow$ Assume f_p is an isomorphism for all p. We will show $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for all U, and can then define the inverse to f by $(f^{-1})_U = (f_U)^{-1}$.

f_U is injective: Suppose $s \in \mathcal{F}(U)$, $f_U(s) = 0$. Then for all $p \in U$, $f_p((U,s)) = (U, f_u(s)) = (U,0) = 0 \in \mathcal{G}_p$. Thus $s_p = 0$ since f_p is injective. Thus there exists an open neighbourhood $V_p \subseteq U$ of p such that $s|_{V_p} = 0$. But $\{V_p\}_{p \in U}$ covers U, so by (1) in the definition of a sheaf, we get $s = 0$.

f_U is surjective: Let $p \in \mathcal{G}(U)$. Then for all $p \in U$, there exists $s_p \in \mathcal{F}_p$ such that $f_p(s_p) = t_p$, i.e. there exists an open neighbourhood of $p \in U$ and a germ $(V_p, \tilde{s}_p)$ representating s_p such that $(V_p, f_{V_p}(\tilde{s}_p)) = (U,t) = t_p$. Shrinking V_p if necessary, we can assume $f_{V_p}(\tilde{s}_p) = t|_{V_p}$ on $V_p \cap V_q$. Then

$$f_{V_p \cap V_q}(\tilde{s}_p|_{V_p \cap V_q} - \tilde{s}_q|_{V_p \cap V_q}) = t|_{V_p \cap V_q} - t|_{V_p \cap V_q} = 0.$$

Since we have shown $f_{V_p \cap V_q}$ is injective, we get

$$\tilde{s}_p|_{V_p \cap V_q} = \tilde{s}_q|_{V_p \cap V_q},$$

and by (2) in the definition of a sheaf, there exists $s \in \mathcal{F}(U)$ such that $s|_{V_p} = \tilde{s}_p$ for all p. Now

$$f_U(s)|_{V_p} = f_{V_p}(s|_{V_p}) = f_{V_p}(\tilde{s}_p) = t|_{V_p}.$$

Thus $f_U(s) - t = 0$ by (1) in the definition of a sheaf, i.e. $f_U(s) = t$. Thus f_U is surjective. □

Remark. If instead $f_p : \mathcal{F}_p \to \mathcal{G}_p$ is injective for all p, then $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is still injective. If instead $f_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective for all p, we need not have $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ surjective (we will see examples later).

Sheafification: Given a presheaf $\mathcal{F}$, there exists a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ satisfying the following universal property:

For any sheaf $\mathcal{G}$ and a morphism $\varphi : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}$ such that

$$\begin{array}{ccc} F & \xrightarrow{\theta} & F^+ \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \varphi^+} \\ & & \mathcal{G} \end{array}$$

commutes. The pair $(\mathcal{F}^+, \theta)$ is unique up to unique isomorphism.

Also, $\mathcal{F}_p \cong \mathcal{F}_p^+$ via θ_p for all $p \in X$.

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