Fourier Restriction Theory

Daniel Naylor

February 17, 2025

Contents

Lecture 1

1 What is Fourier Restriction Theory?

Main object: $f : \mathbb{R}^d \to \mathbb{C}$, $f(x) = \sum_{\xi \in \mathbb{R}} b_\xi e^{2\pi i x \cdot \xi}$, $b_\xi \in \mathbb{C}$.

Notation. We will write $e(x \cdot \xi) i e^{2\pi i x \cdot \xi}$.

$x \in \mathbb{R}^d$ is a spatial variable, and $\xi \in \mathbb{R}^d$ is the frequency variable.

The frequencies (or Fourier transform) of f is *restricted* to a set $\mathcal{R}$ (where $\mathcal{R}$ we will always be finite – so no need to worry about convergence issues).

Goal: Understand the behaviour of f in terms of properties of $\mathcal{R}$.

Example.

(i) Schrödinger equation:

$$u(x,t) = \sum_{n=1}^{N} b_n e(nx + n^2 t).$$

Easy: $(2\pi i \partial_+ - \Delta)u = 0$, with initial data $u(x,0) = \sum_{n-1}^{N} b_n e(nx)$.
$(x,t) = (x_1, x_2)$. Then since $e(nx+n^2t) i e((n,n^2) \cdot (x,t))$, we might consider $\mathcal{R} = \{(n,n^2) : n = 1, \ldots, N\}$.

(ii) Dirichlet polynomials:

$$D(t) = \sum_{n=N}^{2N} b_n e^{i(\log n)t}.$$

with $b_n \equiv 1$, partial sums of Riemann Zeta function. We might consider $\mathcal{R} = \{\frac{1}{2\pi} \log n\}_{n=N}^{2N}$.

Both avoid linear structure.
$\{\log n\}$ is a concave set (getting closer and closer together).
$\{(n,n^2)\}$ lie on a parabola.

Guiding principle: if properties of an object avoid (linear) structure, then we expect some random or average behaviour.

The above examples avoid linear structure using some notion fo curvature. See Bourgain $\Lambda(p)$ paper: $\to$ extra behaviour.

Square root cancellation: If we add ± 1 randomly N times, then we expect a quantity with size $N^{\frac{1}{2}}$.

Theorem 1.1 (Khinchin's inequality)**.** Assuming that:

- $\{\varepsilon_n\}_{n=1}^{N}$ be IID random variables with $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = \frac{1}{2}$

- $1 < p < \infty$
- $x_1, \dots, x_N \in \mathbb{C}$

Then

$$\left(\mathbb{E}\left|\sum_{n=1}^{N} \varepsilon_n x_n\right|^p\right)^{\frac{1}{p}} \sim_p \left(\sum_{n=1}^{N} |x_n|^2\right)^{\frac{1}{2}} = \|x_n\|_2.$$

Notation. $\sim_p$ means $\sim$ but the constant may depend on p.

Proof. Without loss of generality, $x_1, \dots, x_n \in \mathbb{R}$. Without loss of generality, $\|x\|_2 = 1$.

$p = 2$: want to show $\mathbb{E}\left(|\sum_n \varepsilon_n x_n|^2\right) \sim 1$.

$$\mathbb{E}\left(\sum_n \varepsilon_n x_n \overline{\sum_m \varepsilon_m x_m}\right) = \sum_{n,m} \mathbb{E}(\varepsilon_n \varepsilon_m x_n \overline{x_m}) = \sum_n |x_n|^2 + \sum_n \sum_{m \neq n} x_n \overline{x_m} \underbrace{\mathbb{E}\varepsilon_n}_{=0} \mathbb{E}\varepsilon_m.$$

What about general exponents p?

$$\mathbb{E}\left(\left|\sum_n \varepsilon_n x_n\right|^p\right) = \int_0^\infty \mathbb{P}\left(\left|\sum_n \varepsilon_n x_n\right|^p > \alpha\right) \mathrm{d}\alpha.$$

The equality here is the Layer cake formula, which is true for any $p \in (0, \infty)$.

Let $\lambda > 0$. Study the random variable $e^{\lambda \sum_n \varepsilon_n x_n} \in (0, \infty)$.

$$\mathbb{E}\left(e^{\lambda \sum_n \varepsilon_n x_n}\right) = \mathbb{E}\left(\prod_n e^{\lambda \varepsilon_n x_n}\right) = \prod_n \mathbb{E}e^{\lambda \varepsilon_n x_n} = \prod_n \left(\frac{1}{2}e^{\lambda x_n} + \frac{1}{2}e^{-\lambda x_n}\right).$$

Fact: $\frac{1}{2}e^z + \frac{1}{2}e^{-z} \leq e^{\frac{z^2}{2}}$ (to check, use the Taylor series). So we can get

$$\begin{aligned}
\alpha \mathbb{P}(e^{\lambda \sum_n \varepsilon_n x_n} > \alpha) &\leq \mathbb{E}(e^{\lambda \sum_n \varepsilon_n x_n}) && \text{(Chebyshev's inequality)} \\
&\leq \prod_n e^{\lambda^2 |x_n|^2/2} \\
&= e^{\lambda^2/2}
\end{aligned}$$

By symmetry,

$$\alpha \mathbb{P}(e^{|\lambda \sum_n \varepsilon_n x_n|} > \alpha) \lesssim e^{\lambda^2/2}.$$

Choose $\alpha = e^{\lambda^2}$:

$$\mathbb{P}\left(\left|\sum_n \varepsilon_n x_n\right| > \lambda\right) = \mathbb{P}\left(\left|\sum_n \varepsilon_n x_n\right|^p > \lambda^p\right) \lesssim e^{-\lambda^2/2}.$$

Use in Layer cake:

$$\mathbb{E}\left(\left|\sum_n \varepsilon_n x_n\right|^p\right) \lesssim \int_0^\infty e^{-\alpha^{2/p}/2}\mathrm{d}\alpha \sim_p 1.$$

Lower bound: use Hölder's inequality. $X = \sum_n \varepsilon_n x_n$.

$$\underbrace{\mathbb{E}(X\overline{X})}_{=1} \le \underbrace{(\mathbb{E}(|X|^p))^{1/p}}_{\lesssim_p 1}(\mathbb{E}|X|^q)^{1/q}.$$

$\frac{1}{p} + \frac{1}{q} = 1.$ □

Can you find a more intuitive proof? E-mail Dominique Maldague.

Corollary. $\mathbb{E}\left(\int \left|\sum_{n=1}^N \varepsilon_n f_n(x)\right|^p \mathrm{d}x\right) \sim_p \int \left|\sum_{n=1}^N |f_n(x)|^2\right|^{p/2} \mathrm{d}x.$

Useful for exercises!

Return to Fourier restriction context.

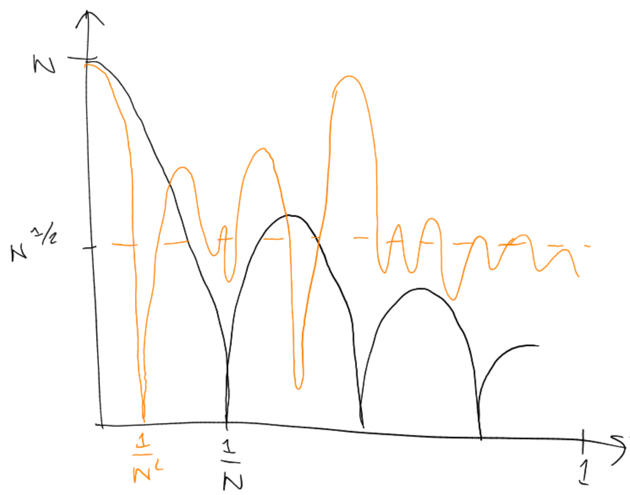
$$f(x) = \sum_{n=1}^N e(nx) \qquad \mathcal{R} \qquad = \{1, \dots, N\}$$

$$g(x) = \sum_{n=1}^N e(n^2x) \qquad \mathcal{R} \qquad = \{1^2, 2^2, \dots, N^2\}$$

Both f, g are 1-periodic. So study them on $\mathbb{T} = [0,1]$. $f(0) = N$, $|f(x)| \sim N$ for $\in \left[0, c\frac{1}{N}\right]$. $g(0)iN$, $|g(x)| \sim N$ for $x \in \left[0, c\frac{1}{N^2}\right]$.

$$\int_{[0,1]} |f(x)|^2\mathrm{d}x = \sum_{n,m}\int_{[0,1]} e((n - mx)\mathrm{d}x = N).$$

$$\int_{[0,1]} |g(x)|^2 = \sum_{n,m}\int_{[0,1]} e((n^2 - m^2)x)\mathrm{d}x = N.$$



$$\int_{[0,1]} |f(x)|^p \mathrm{d}x \geq N^{p/2} + N^{p-1}$$

$$\int_{[0,1]} |f(x)|^p \mathrm{d}x \geq Np/2 + N^{p-2}.$$

For the first one, N^{p-1} (organised behaviour) dominates as soon as $p > 2$, and for the second one, $N^{p/2}$ dominates for $2 \leq p \leq 4$ (“square root cancellation behaviour lasts for longer”).

Lecture 2

2 Exponential sums in L^p

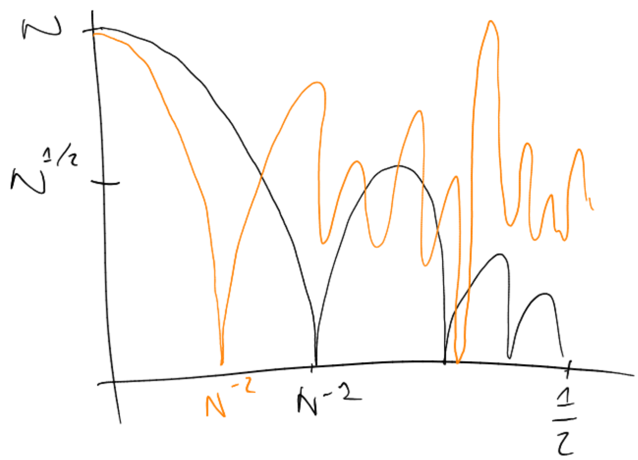
Recall: studying $f(x) = \sum_{\xi \in \mathcal{R}} e(\xi \cdot x)$. When $\mathcal{R}$ does not have (linear) structure, expect $|f(x)| \sim |\mathcal{R}|^{\frac{1}{2}}$ ("sqrt cancellation") in an appropriate sense (in L^p), *more* than when $\mathcal{R}$ is structured.

Linear $\mathcal{R}$ $f(x) = \sum_{n=1}^{N} e(nx)$ vs convex $\mathcal{R}$ $g(x) = \sum_{n=1}^{N} e(n^2 x)$.

Have

$$\int_{[0,1]} |f(x)|^2 \mathrm{d}x = N = \int_{[0,1]} |g(x)|^2 \mathrm{d}x.$$

Have $|f(x)| \sim N$ on $[0, cN^{-1}]$, and $|g(x)| \sim N$ on $[0, cN^{-2}]$.



L^2 does not distinguish f, g. L^∞ does not distinguish. However, $2 < p < \infty$ does.

What about $1 \leq p \leq 2$? We don't usually study this range because the estimates tend to be trivial / not interesting.

Focus on $2 < p < \infty$. Preduct size of

$$\int_{[0,1]} |f|^p.$$

Square root cancellation lower bound:

$$N = \int_{[0,1]} |f|^2 \overset{\text{Hölder's}}{\leq} \left(\int_{[0,1]} |f|^{2 \cdot \frac{p}{2}} \right)^{\frac{2}{p}} (1)^{\square}$$

$\implies N^{p/2} \leq \int_{[0,1]} |f|^p$.

Constant integral lower bound:

$$\int_{[0,1]} |f|^p \geq \int_{[0,cN^{-1}]} |f|^p \gtrsim N^p N^{-1}$$
$$\int_{[0,1]} |g|^p \geq \int_{[0,cN^{-2}]} |g|^p \gtrsim N^p N^{-2}$$

$\int |f|^p \lesssim N^{p/2} + N^{p-1}$, $\int |g|^p \lesssim N^{p/2} + N^{p-2}$.

Note that for the f bound, N^{p-1} is bigger than N^{p-1} (as long as $p > 2$), so the constant integral dominates!

For g, if $2 \leq p \leq 4$, the square root cancellation dominates, but for $p > 4$ the constant integral takes over.

Assuming:

Theorem 2.1. - $p > 2$ - $b_n \in \mathbb{C}$ Then:

$$\int \left| \sum_n b_n e(nx) \right|^p \leq N^{\frac{p}{2}-1} \|b_n\|_2^p.$$

Proof. Consider: $\sum_{n=1}^N b_n e(nx)$, $b_n \in \mathbb{C}$.

$$\begin{aligned}
\int_{[0,1]} \left| \sum_n b_n e(nx) \right|^p \mathrm{d}x &\leq \left\| \sum_n b_n e(nx) \right\|_\infty^{p-2} \int_{[0,1]} \left| \sum_n b_n e(nx) \right|^2 \mathrm{d}x \\
&\leq (N^{\frac{1}{2}} \|b_n\|_2)^{p-2} \|b_n\|_2^2 && \text{CS} \\
&= N^{\frac{p}{2}-1} \|b_n\|_2^p && \square
\end{aligned}$$

Note that this is sharp when $b_n \equiv 1$.

$\sum_{n=1}^N b_n e(n^2 x)$. Focus on $p = 4$.

$$\int_{[0,1]} \left| \sum_n b_n e(n^2 x) \right|^4 \mathrm{d}x = \sum_{n_1} \sum_{n_2} \sum_{n_3} \sum_{n_4} b_{n_1} b_{n_2} \overline{b_{n_3} b_{n_4}} \int_{[0,1]} e((n_1^2 + n_2^2 - n_3^2 - n_4^2)x) \mathrm{d}x.$$

The integral vanishes unless $n_1^2 - n_3^2 = n_4^2 - n_2^2$.

Number Theory lemma: If $m \in \mathbb{Z}$, then

$$\# \text{ divisors of } m \lesssim \log m.$$

Follows from unique prime factorisation.

Warning. The above lemma is false.
See correction later.

For fixed n_1, n_3,

$$\# \underbrace{\{(n_2, n_4) : n_1^2 - n_3^2 = (n_4 + n_2)(n_4 - n_2)\}}_{=:\mathcal{S}_{n_1,n_3}} \lesssim \log N.$$

Hence

$$\begin{aligned}\int_{[0,1]} \left| \sum_n b_n e(n^2 x) \right|^4 \mathrm{d}x &\leq \sum_{n_1} \sum_{n_3} |b_{n_1} b_{n_3}| \left| \sum_{(n_2,n_4) \in \mathcal{S}_{n_1,n_3}} b_{n_2} \overline{b_{n_4}} \right| \\ &\lesssim (\log N) \|b_n\|_2^4\end{aligned}$$

We will now use $\lessapprox$ to mean $\lesssim$ up to powers of $\log N$.

$2 < p < 4$: $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{4}$, $\theta \in [0,1]$, $h = \sum_n b_n e(n^2 x)$, $\|h\|_p \leq \|h\|_2^\theta \|h\|_4^{1-\theta} \lessapprox \|b_n\|_2^\theta \|b_n\|_2^{1-\theta} \approx \|b_n\|_2$.

$p \geq 4$:

$$\int_{[0,1]} |h|^p \lessapprox \|h\|_\infty^{p-4} \|b_n\|_2^4 \overset{CS}{\leq} (N^{\frac{1}{2}} \|b_n\|_2)^{p-4} \|b_n\|_2^4 \approx N^{\frac{p}{2}-2} \|b_n\|_2^p.$$

Assuming:

Theorem 2.2. - $2 > p$ Then:

$$\int \left| \sum_n b_n e(n^2 x) \right|^p \mathrm{d}x \lessapprox (1 = N^{\frac{p}{2}-2}) \|b_n\|_2^p.$$

Sharp by $b_n \equiv 1$.

Positive take away: estimates are sharp, proofs are elementary. Easy to think of sharp examples.

Number Theory counting idea shows:

$$\int_{[0,1]^2} |u(x,t)|^6 \mathrm{d}x\mathrm{d}t \lessapprox \|b_n\|_2^6 \qquad u(x,t) = \sum_{\substack{|n| \sim N \\ n \in \mathbb{Z}}} b_n e(nx + n^2 t)$$

$$\int_{[0,1]^3} |u(x,t)|^4 \mathrm{d}x\mathrm{d}t \lessapprox \|b_n\|_2^4 u(x,t) \qquad = \sum_{\substack{|n| \sim N \\ n \in \mathbb{Z}^2}} b_n e(n \cdot x + |n|^2 t)$$

Sharp, $6, 4 = p_{\text{crit}}$.

Strichartz estimate for periodic Schrödinger equation, observed by Bourgain in 1990s.

Negatives: on $\mathbb{T}^3 \to p_{\text{crit}} = \frac{10}{3}$, but this technique can only work on even integer values of p.

$\mathbb{T}^1$, $\mathbb{T}^2$ only sharp Strichartz estimates per Schrödinger until 2015!

2015: Bourgain-Demeter proved (l^2, L^p) sharp decoupling estimate. Gives sharp Strichartz estimate for Schrödinger in $\mathbb{T}^d$ for all d.

Proved earlier:

$$N^{-2} \int_{[0,N^2]} \left| \sum_{n \sim N} e\left(\frac{n^2}{N^2} x \right) \right|^p \mathrm{d}x \leq (1 + N^{\frac{p}{2}-2}) \|b_n\|_2^p.$$

(where $n \sim N$ means $N \leq n \leq 2N$).

Conjecture:

$$⨍_{[0,N^2]} \left| \sum_{n \in N} b_n e(a_n x) \right|^p \mathrm{d}x \lesssim (1 + N^{\frac{p}{2}-2}) \|b_n\|_2^p$$

$a_n \in [0,1]$, $a_{n+1} - a_n \sim \frac{1}{N}$, $a_{n+2} - a_{n+1} - (a_{n+1} - a_n) \sim \frac{1}{N^2}$.

Lecture 3 Example: $\{a_n\} \sim \left\{ \frac{n^3}{N^3} \right\}_{n=N}^{2N}$.

3 Introduction to Fourier Transform

Correction for lecture 2: Number Theory Lemma.

True statements:

$$\frac{1}{N}\sum_{1\le n\le N} \# \div (n) \lesssim \log N,$$

and $\# \div (n) \lesssim_{\varepsilon} n^{\varepsilon}$.

See Terence Tao notes online.

Explaining $\# \div (n) \lesssim_{\varepsilon} n^{\varepsilon}$: $\forall \varepsilon > 0$, $\exists c_{\varepsilon} \in (0,\infty)$ such that $\# \div (n) \le C_{\varepsilon} n^{\varepsilon}$ for all $n \ge 1$.

We will be using $\lessapprox$ to mean "$\le$ but up to sub-polynomial in n".

Question:

$$\fint_{[0,N^2]} \left|\sum_{n\sim N} b_n e(a_n x)\right|^p \mathrm{d}x \lessapprox (1 + N^{\frac{p}{2}-2})\|b_n\|_2^2.$$

For example, $a_n = \frac{n^2}{N^2}$.

Recall: $n \sim N$ means $N \le n \le 2N$.

$\{a_n\} \subseteq [0,10]$, $a_{n+1} - a_n \sim \frac{1}{N}$, $(a_{n+2} - a_{n+1}) - (a_{n+1} - a_n) \sim \frac{1}{N^2}$.

Reasonable conjecture?

Yes, reasonable.

Khinchin's inequality: May select $b_n \in \{\pm 1\}$ so that

$$\fint_{[0,N^2]} \left|\sum_{n} b_n e(a_n x)\right|^p \mathrm{d}x \sim \fint_{[0,N^2]} \left|\sum_{n} |b_n|^2\right|^{\frac{p}{2}} \mathrm{d}x.$$

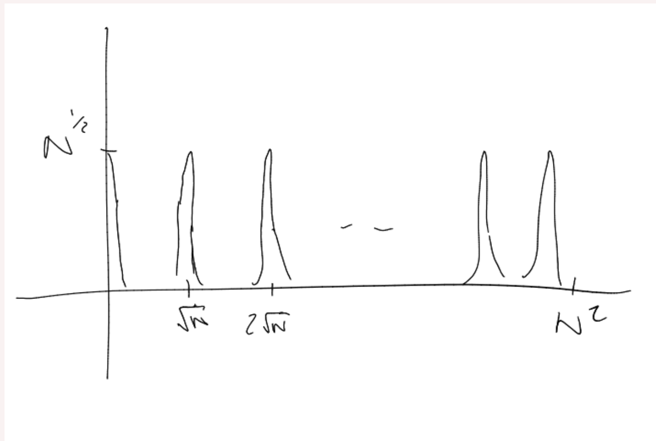
Constant integral: $b_n \equiv 1$. $(1,1,1,\dots,1)$.

$$\fint_{[0,N^2]} \left|\sum_{n} e(a_n x)\right|^p \mathrm{d}x \gtrsim \frac{1}{N^2}\int_{[0,c]} N^p \mathrm{d}x \sim N^{p-2} = N^{\frac{p}{2}-2}\|b_n\|_2^p.$$

Warning. Enemy scenario: $\{a_n\} = \left\{\sqrt{\frac{n}{N}}\right\}_{n=N}^{2N}$ (technically $\left\{-\sqrt{\frac{3N-n}{N}}\right\}$ if we want to satisfy the conditions mentioned above).

$(b_n) = (1, 0, \dots, 0, 1, 0, \dots, 0, 1, \dots, 0)$. Length N vector with $N^{\frac{1}{2}}$ many 1s. Have:

$$\fint_{[0,N^2]} \left| \sum_{N \le m^2 \le 2N} e\left(\sqrt{\frac{m^2}{N}} x\right) \right|^p \mathrm{d}x = \fint_{[0,N^2]} \left| \sum_{m^2 \sim N} e\left(\frac{m}{\sqrt{N}} x\right) \right|^p \mathrm{d}x$$



We can calculate that the above expression is in fact $\geq N^{\frac{p}{2} - \frac{1}{2}}$ (which breaks the conjecture until $p > 6$).

It turns out that this is (roughly speaking) the only problem.

Why do we care?

$b_n \equiv 1$, $p = 4$. Then

$$\fint_{[0,N^2]} \left| \sum_n e(a_n x) \right|^4 \mathrm{d}x \lessapprox N^2 = |\{a_n\}|^2.$$

$$\implies \#\{a_{n_1} + a_{n_2} = a_{n_3} + a_{n_4}\} \lessapprox |\{a_n\}|^2.$$

$\to$ Convex sequences have minimal additive energy.

Decoupling doesn't know how to take advantage of $b_n \equiv 1$.

3.1 Fourier Transform on $\mathbb{R}^n$

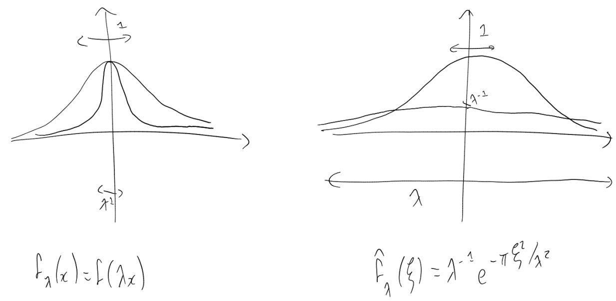
$f : \mathbb{R}^n \to \mathbb{C}$, $f \in \mathcal{S}(\mathbb{R}^n)$ Schwartz function: $\|x^\alpha \partial^\beta f\|_\infty < \infty$ for all α, β.

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) \mathrm{d}x.$$

x is the spatial variable, and ξ is the frequency variable.

Facts:

- If $f \in \mathcal{S}$, then $\hat{f} \in \mathcal{S}$.
- Plancherel's Theorem: $\int f(x)\overline{g(x)}\mathrm{d}x = \int \hat{f}(\xi)\overline{\hat{g}(\xi)}\mathrm{d}\xi$.
- $f(x) = e^{-\pi x^2}$, $\hat{f}(\xi) = e^{-\pi \xi^2}$
- $\lambda > 1$



On the left: mass of f is smashed by a factor of λ. On the right: the mass of $\hat{f}$ is stretched by a factor of λ, "L^1 normalized". $\|\hat{f}_\lambda\|_1$ is independent of λ.

There is a general formula for $\widehat{f \circ A}$ where A is an affine transformation.

- $\widehat{cf+g} = c\widehat{f} + \widehat{g}$.
- Translations are dual to modulations:

$$\begin{aligned} f_\tau(x) &= f(x-\tau) \\ \widehat{f_\tau}(\xi) &= e^{-2\pi i \xi\cdot\tau}\widehat{f}(\xi) \\ f^t au(x) &= e^{2\pi i \tau\cdot x} f(x) \\ \widehat{f^\tau}(\xi) &= \widehat{f}(\xi-\tau) \end{aligned}$$

Basic question about $\widehat{f}$: L^p to L^q boundedness?

Plancherel's:

$$\|\widehat{f}\|_2^2 = \int \widehat{f}\overline{\widehat{f}} = \int f\overline{f} = \|f\|_2^2$$

(isometry on L^2).

$p = 1$, $q = \infty$:

$$\left|\int e^{-2\pi i x\cdot\xi} f(x)\mathrm{d}x\right| \le \int |f(x)|\mathrm{d}x = \|f\|_1$$

(contraction from L^1 to L^∞).

By interpolation (Marcinkiewicz): $\|\widehat{f}\|_q \le \|f\|_p$ for $1 \le p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$ (Hausdorff-Young inequality).

Are there any other (p, q) for which
$$\|\widehat{f}\|_q \lesssim_{p,q} \|f\|_q?$$

Attempt 1: Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ (compactly supported smooth function on $\mathbb{R}^n$), with $\operatorname{supp}\varphi \subseteq B_1(0)$.

Consider $f(x) = \sum_{k=1}^N \varepsilon_k \varphi(x - v_k)$.

Choose v_k so that $\{B_1(v_k)\}$ are not overlapping. Then
$$\|f\|_p^p = \int \left| \sum_k \varepsilon_k \varphi(x - v_k) \right|^p = \sum_k \int |\varphi(x - v_k)|^p \mathrm{d}x = N \|\varphi\|_p^p.$$

Also
$$\widehat{f}(\xi) = \sum_{k=1}^N \varepsilon_k e^{-2\pi i \xi v_k} \widehat{\varphi}(\xi) = \left(\sum_{k=1}^N \varepsilon_k e^{-2\pi i \xi v_k} \right) \widehat{\varphi}(\xi).$$

We will use Khinchin's inequality.

Lecture 4 $\|\widehat{f}\|_q \sim N^{\frac{1}{2}} \|\widehat{\varphi}\|_q$.

4 Introduction to Fourier Restriction

Theorem (Hausdorff-Young inequality)**.** Assuming that:

- $1 \leq p \leq 2$
- $\frac{1}{p} + \frac{1}{q} = 1$

Then
$$\|\widehat{f}\|_q \leq \|f\|_p.$$

Proof. The inequality is true for $p = 1$, $q = \infty$ and for $p = 2$, $q = 2$ the inequality is true since we have equality (Plancherel).

For values in between we can interpolate. $\square$

Are there any other (p, q) for which
$$\|\widehat{f}\|_q \lesssim \|f\|_p? \tag{$*$}$$

We saw that $1 \leq p \leq 2$ was necessary (translations / modulations, Khinchin's inequality).

Scaling: Plug in $f_\lambda(x) = f(\lambda x)$ which is L^∞-normalised ($\|f_\lambda\|_\infty = \|f\|_\infty$). Then $\widehat{f_\lambda}(\xi) = \lambda^{-n}\widehat{f}(\lambda^{-1}\xi)$ which is L^1-normalised ($\|\widehat{f_\lambda}\|_1 = \|\widehat{f}\|_1$).

$$(\text{LHS of } (*))^q = \int_{\mathbb{R}^n} |\widehat{f_\lambda}(\xi)|^q \mathrm{d}\xi = \lambda^{-nq+n} \int |\widehat{f}(\xi)|^q \mathrm{d}\xi.$$

$$(\text{RHS of } (*))^p = \int_{\mathbb{R}^n} |f_\lambda(x)|^p \mathrm{d}x = \lambda^{-n} \int |f(x)|^p \mathrm{d}x.$$

So we need for all $\lambda > 0$:
$$\lambda^{-n+\frac{n}{q}} \|\widehat{f}\|_q \lesssim \lambda^{-\frac{n}{p}} \|f\|_p.$$

So we need $-n + \frac{n}{q} = -\frac{n}{p}$, i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

Classical questions

What is $C_{p,q}$ the smallest constant such that $\|\widehat{f}\|_q \leq C_{p,q}\|f\|_p$?

Which functions f satisfy $\frac{\|\widehat{f}\|_q}{\|f\|_p} = C_{p,q}$?

2014:
$$\|\widehat{f}\|_q \leq C_{p,q}\|f\|_p - \mathrm{dist}_p(f, \text{maximisers (Gaussian)}).$$

Fourier restriction asks which (p, q) permit estimates $\|\widehat{f}\|_{L^q(\mathcal{R})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$ ($\mathcal{R}$ is the restricted frequency set, $\mathcal{R} \subseteq \mathbb{R}^n$).

> **Example.** $\mathcal{R} = B_1(0) \subseteq \mathbb{R}^n$, unit ball $\to$: Basically, sitll governed by Hausdorff-Young inequality.

> **Example.** $\mathcal{R} = B_1(0) \cap \{x_n = 0\}$ (measure 0 subset of $\mathbb{R}^n$).

$R_{\mathcal{R}}(p \to q)$ means

$$\|\widehat{f}\|_{L^q(\mathcal{R})} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \qquad \forall f : \mathbb{R}^n \to \mathbb{C} \text{ Schwartz}$$

Always: $R_{\mathcal{R}}(1 \to \infty)$ true for all $\mathcal{R}$.

In the second example, only this trivial statement is true (i.e. $R_{\mathcal{R}}(p \to q)$ is false for all other values for p, q).

Let $\mathcal{R} = S^{n-1}$ be the unit sphere in $\mathbb{R}^n$. Consider

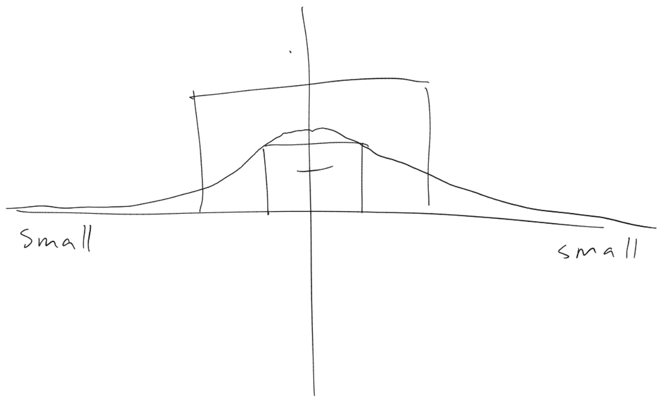
$$\|\widehat{f}\|_{L^q(S^{n-1})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

where $L^q(S^{n-1})$ uses the usual surface measure $\mathrm{d}\sigma$ on S^{n-1}.

> **Notation.**
>
> $$\begin{tikzcd} \underbrace{f(x)}_{\in\mathcal{S}(\mathbb{R}^n)} \arrow[r, "\widehat{\ }"] & \underbrace{\widehat{f}(x)}_{\in\mathcal{S}(\mathbb{R}^n)} \arrow[r, "\widehat{\ }"] \arrow[rr, bend right, "\check{\ }"'] & \underbrace{f(x)}_{\in\mathcal{S}(\mathbb{R}^n)} \arrow[r, "x\mapsto -x"] & \underbrace{f(x)}_{\in\mathcal{S}(\mathbb{R}^n)} \end{tikzcd}$$
>
> We may call $\check{\ }$ the "inverse Fourier transform".

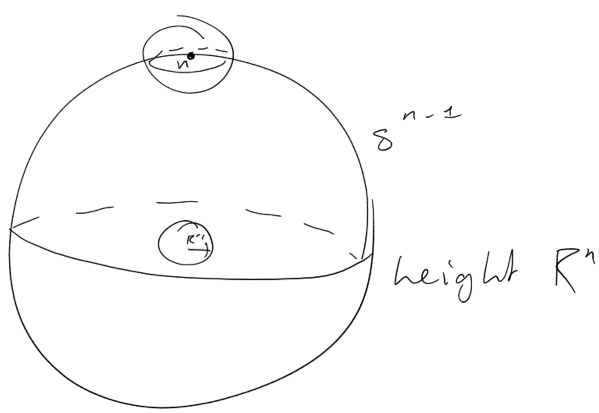
Let $\varphi \in C_c^\infty(\mathbb{R}^n)$, $\mathbb{R}$-valued, $\geq \frac{1}{2}$ on $B_1(0)$, with support in $B_2(0)$.

May also assume $\check{\varphi}$ is $\mathbb{R}$-valued, $\check{\varphi} \gtrsim 1$ on $B_c(0)$. $\check{\varphi}$ bounded, $|\check{\varphi}(x)| \lesssim_m (|x|^2+1)^{-m} \forall m$. $|\check{\varphi}(x)|$ behaves like $\chi_{B_1(0)}$ in L^p.



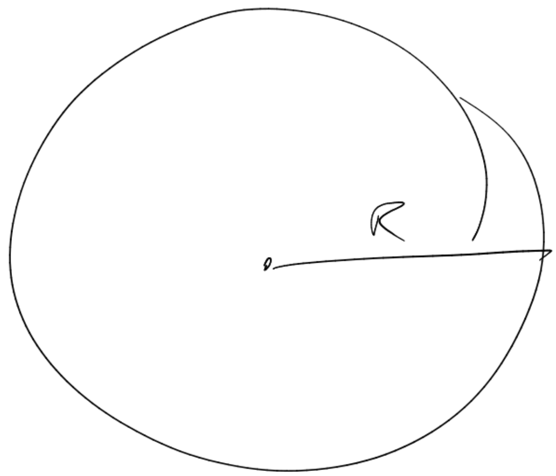
Consider dilates $f_R(x) = \check{\varphi}(R^{-1}x)e^{2\pi i x\cdot v_r}$, $R \gg 1$.

Frequency side $|\widehat{f_R}(\xi)| \sim R^n \chi_{B_{R^{-1}(0)}}(x)$ (L^1-norm).



$$\int_{S^{n-1}} |\widehat{f_R}(\xi)|^q \mathrm{d}\sigma(\xi) \sim \underbrace{R^{nq}R^{-(n-1)}}_{\to\sigma(B_{R^{-1}}(n)\cap S^{n-1})}.$$

$B_{R^{-1}}(n) \cap S^{n-1}$ cap of radius R^{-1}. Spacial side $|f_R(x)| \sim \chi_{B_R(0)}(x)$ (in L^∞-norm).

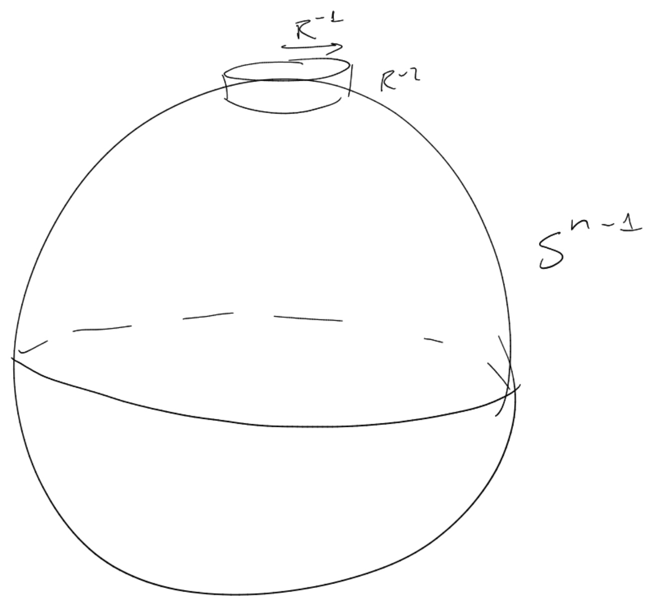


height 1, $\int_{\mathbb{R}^n} |f_R(x)|^p \mathrm{d}x \sim R^n$.

$R^{n-\frac{n-1}{q}} \lesssim R^{\frac{n}{p}}$, $n - \frac{n-1}{q} \leq \frac{n}{p}$.

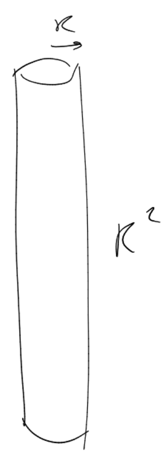
Consider: $g_{\mathcal{R}}(x) = e^{ix\cdot w_r}\check{\varphi}(R^{-1}x_1, \ldots, R^{-1}x_{n-1}, R^{-2}x_n)$.

Frequency: $|\widehat{g_r}(\xi)| \approx |\text{cylinder}|^{-1}\chi_{\text{cylinder}}(\xi)$. $|\text{cylinder}|^{-1}| \to (R{-}1 \cdot R^{-1}R^{-2})^{-1} = R^{n+1}$.



$$\int_{S^{n-1}} |\widehat{g_R}(\xi)|^q \mathrm{d}\sigma(\xi) \sim R^{(n+1)q} \sigma(\text{cap of radius } R^{-1}) = R^{(n+1)q-(n-1)}.$$

Spatial side: $|g_{R'}(x)| \approx \chi_{\text{cylinder}}(x)$ (L^∞-norm).



$\int |g_R(x)|^p = R^{n-1+2} = R^{n+1}$.

$\to R^{n+1-\frac{n-1}{q}} \lesssim R^{\frac{n+1}{q}}, \implies n+1-\frac{n-1}{q} \leq \frac{n+1}{p}$. Implies B.

Lecture 5 On Monday, we will build examples h such that $\widehat{h}$ sees *all* of S^{n-1}.

$\mathcal{R} = S^{n-1} \subseteq R^n$.

Consider the statement

$$\|\widehat{f}\|_{L^q(S^{n-1})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

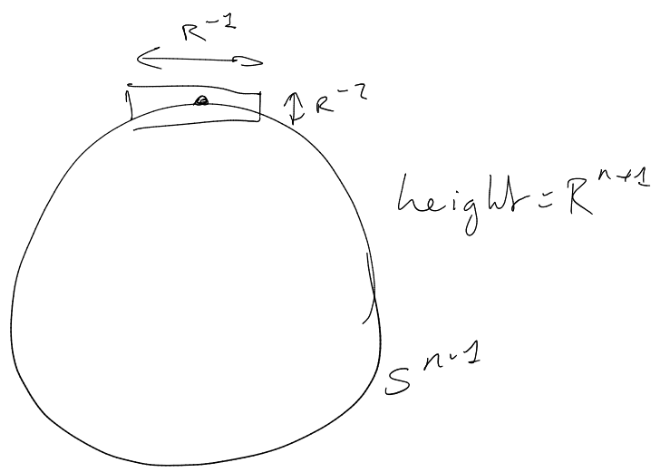
(recall that we called this $R_{S^{n-1}}(p \to q)$).

Fix $\varphi \in C_c^\infty(\mathbb{R}^n)$, $\varphi \sim \chi_{B_1(0)}$. For computing L^p norms, $|\widehat{\varphi}| \sim \chi_{B_c(0)}$.

Wave packet function with localised spatial and frequency behaviour.

Last time: $f_R(x) = e^{ix \cdot v_R} \check{\varphi}(R^{-1}x_1, \ldots, R^{-1}x_{n-1}, R^{-2}x_n)$.

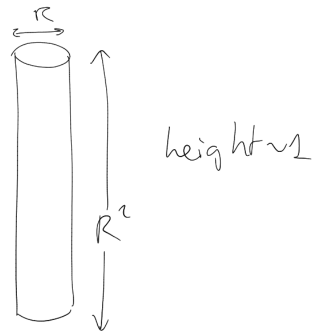
Frequency: $|\widehat{f_R}(\xi)|$



$\int_{S^{n-1}} |\widehat{f_R}|^q \mathrm{d}\sigma$

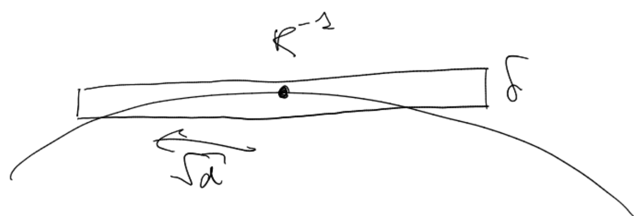
$$\boxed{n + 1 - \frac{n-1}{q} \leq \frac{n+1}{q}}$$

Spatial: $|f_R(x)|$



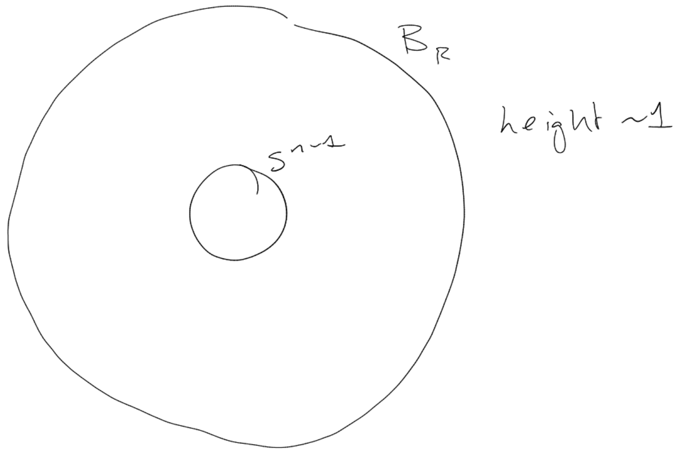
$\int_{\mathbb{R}^1} |f_R|^p$

Note: sphere near $\mathbf{n}$ looks like $(\xi^1, 1 - \frac{1}{2}|\xi'|^2)$, $S^{n-1} \cap \operatorname{supp} \widehat{f_R} \sim$ cap of radius R^{-1}.



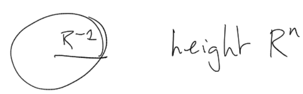
Naive attempt: $g_R(x) = R^n \check{\varphi}(Rx)$

Frequency: $|\widehat{g_R}(\xi)|$



$\int_{S^{n-1}} |\widehat{g_R}(\xi)|^q \mathrm{d}\xi \sim 1$

Spatial: $|g_R(x)|$



$\int |g_R|^p \sim R^{-n} \cdot R^{np}$.

Deduce: $1 \lesssim R^{-\frac{n}{p}+n}$, so

$$\boxed{p \geq 1}$$

(trivial).

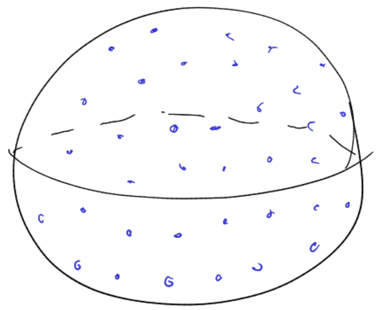
$|\widehat{g_R}| \sim 1$ on S^{n-1} made $\|\widehat{g_R}\|_{L^q(S^{n-1})}$ easy to compute.

Could we improve things?

Could think about $R \sim 1$: then $\|\widehat{g_R}\|_{L^q(S^{n-1})} \sim 1$, $\|g_R\|_p \sim 1$. This is more efficient, but we can't take a limit. So not so useful.

Build a function $H(x)$ which satisfies $|\widehat{H}(\xi)| \sim 1$ on S^{n-1}.

Let $\{\theta\}$ be a maximal collection of $\sim R^{-1}$-spaced points on S^{n-1} ($\#\{\theta\} \sim R^{n-1}$).



For each θ, let $A_\theta^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ be an affine map which sends

$$B_1(0) \to R^{-1} \times \cdots \times R^{-1} \times R^{-2} \text{ ellipsoid centered at } \theta, \text{ tangent to } S^{n-1} \text{ at } \theta.$$

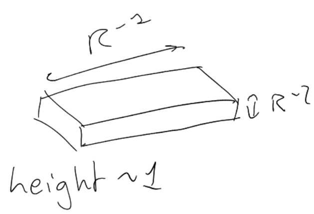
Define $\varphi_\theta = \varphi \circ A_\theta$, $H(x) = \sum_\theta \widecheck{\varphi_\theta}(x)$.

$\widehat{H}(\xi) = \sum_\theta \varphi_\theta(\xi) \sim 1$ on S^{n-1} (actually on R^{-2}-neighbourhood of S^{n-1}).

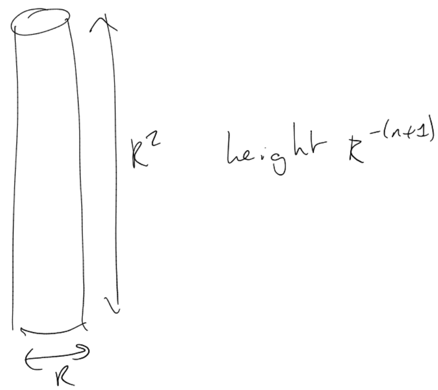
$\int_{S^{n-1}} |\widehat{H}(\xi)|^q \mathrm{d}\sigma \sim 1$.

$H(x) = \sum_\theta \widecheck{\varphi_\theta}(x)$.

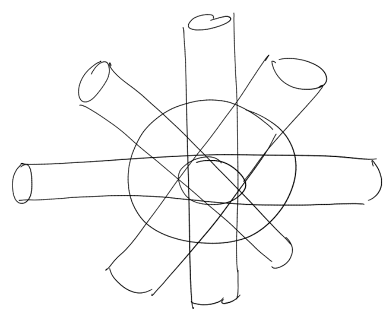
$|\varphi_\theta(x)|$ Frequency:



$|\widecheck{\varphi_\theta}(x)|$ Spatial:



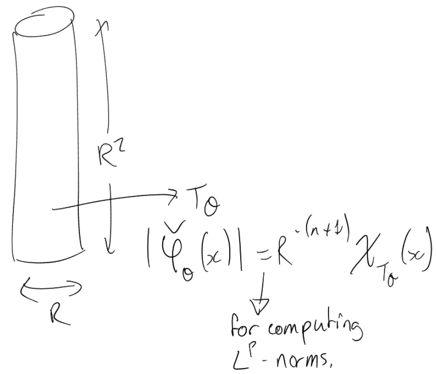
ess supp $H \supseteq \bigcup_\theta$ ess supp $\widecheck{\varphi}_\theta$.



“bush of tubes”

$\sim R^{n-1}$ many $R \times \cdots \times R \times R^2$ tubes in R^{-1}-separated directions

$$\int_{\mathbb{R}^n} |H(x)|^p \mathrm{d}x = \int_{\mathbb{R}^n} \left| \sum_\theta \underbrace{\widecheck{\varphi}_\theta(x)}_{\mathbb{C}\text{-valued function}} \right|^p \mathrm{d}x$$



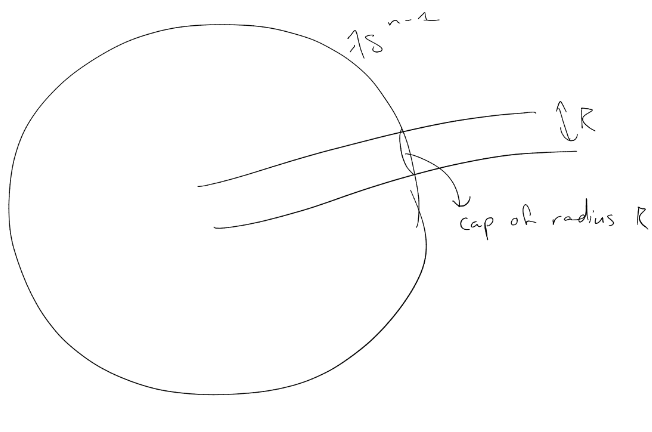
Compute

$$\int_{\mathbb{R}^n} \left| \sum_\theta \chi_{T_\theta}(x)^2 \right|^{\frac{p}{2}} = \int_{\mathbb{R}^n} \left| \sum_\theta \chi_{T_\theta}(x) \right|^{\frac{p}{2}}$$
$$= \underbrace{\int_{B_R} \left| \sum \chi_{T_\theta} \right|^{\frac{p}{2}}}_{(R^{n-1})^{\frac{p}{2}} \cdot R^n} + \underbrace{\int_{R_{R^2} \setminus B_R} \left| \sum \chi_{T_\theta} \right|^{\frac{p}{2}}}_{(*)}$$

Consider overlap of T_θ on λS^{n-1} ($\lambda \in (R, R^2)$).

Average overlap on λS^{n-1}:

$$\fint_{\lambda S^{n-1}} \sum_\theta \chi_{T_\theta} \sim \lambda^{-(n-1)} \sum_\theta \int_{\lambda S^{n-1}} \chi_{T_\theta} \sim \lambda^{-(n-1)} \sum_\theta R^{n-1} = \lambda^{-(n-1)} R^{2(n-1)}.$$



Not too hard to check that the number of active T_θ on λS^{n-1} is $\sim \lambda^{-(n-1)} R^{2(n-1)}$.

Now calculate:

$$(*) \sim \sum_{R<\lambda<R^2} \int_{\lambda<|x|<2\lambda} \underbrace{\left|\sum_\theta \chi_{T_\theta}\right|^{\frac{p}{2}}}_{[\lambda^{-(n-1)}R^{2(n-1)}]^{\frac{p}{2}}\lambda^n} \, \mathrm{d}x.$$

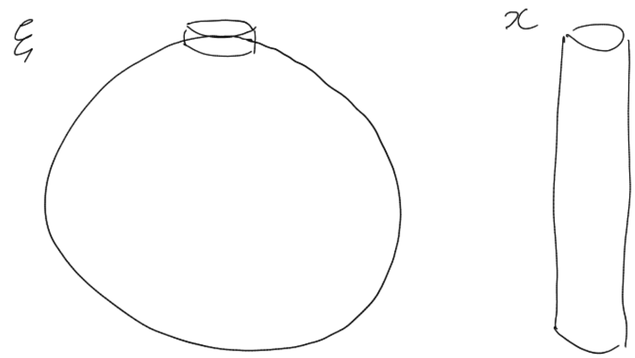
$$1 \lesssim R^{-(n+1)p}[R^{2n} + R^{(n-1)\frac{p}{2}}R^n].$$

Lecture 6 Two cases: either R^{2n} dominates or the other term dominates. So $\boxed{p \leq \dfrac{2n}{n+1}}$.

5 Equivalent Versions of Fourier Restriction

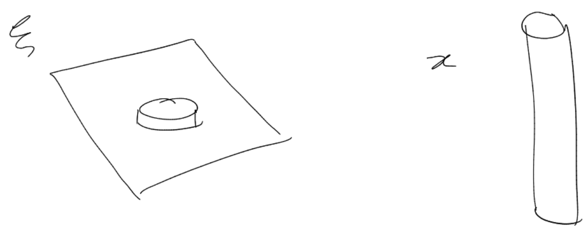
Searching for (p, q) for which

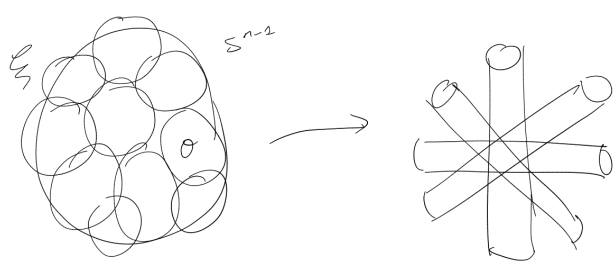
$$\|\widehat{f}\|_{L^q(S^{n-1})} \leq C_{p,q,n} \|f\|_{L^p(\mathbb{R}^n)}.$$



(1)

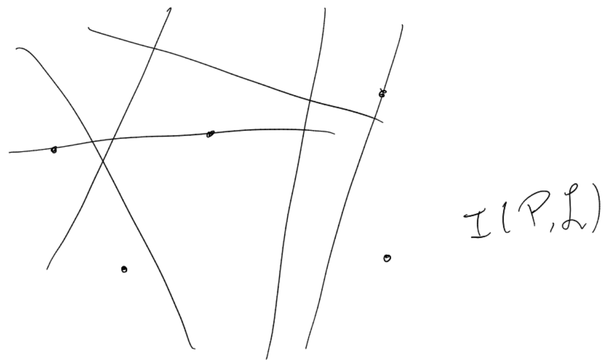
$\mathcal{R} = [0,1]^{n-1} \times \{0\}$



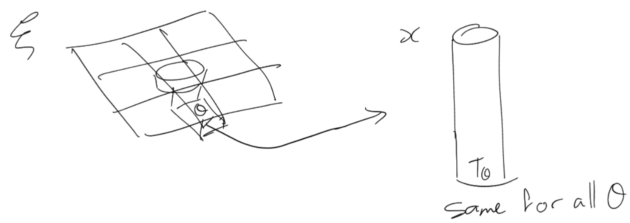


(2)

$\rightarrow \int_{B_{R^2}} |\sum_\theta \chi_{T_\theta}|^{\frac{p}{2}}$. "continuum incidence geometry problem".



integral equals $= \sum_{B_R \subseteq B_{R^2}} \int_{B_R} |\sum_\theta \chi_{T_\theta}|^{\frac{p}{2}}$.
"fractal geometry".



Reminder: $R_{S^{n-1}}(p \to q)$ means

$$\|\widehat{f}\|_{L^q(S^{n-1})} \le C_{p,q,n} \|f\|_{L^q(\mathbb{R}^n)} \qquad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

Conjecture (Restriction Conjecture)**.** $R_{S^{n-1}}(p \to q)$ if and only if $n + 1 - \frac{n-1}{q} \le \frac{n+1}{p}$ and $p < \frac{2n}{n+1}$.

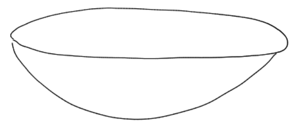
First proved for $S^1 \subseteq \mathbb{R}^2$ by Fefferman (1970) and Zygmund (1974).

Special things happen in $\mathbb{R}^2$, classical harmonic analysis techniques apply.

Same conjecture for $R_{P^{n-1}}(p \to q)$, where

$$P^{n-1} = \{(\xi, |\xi|^2) \in \mathbb{R}^n : |\xi| < 1\}.$$

$\|\widehat{f}\|^q_{L^q(P^{n-1})} = \int_{|\xi|<1} |\widehat{f}(\xi, |\xi|^2)|^q \mathrm{d}\xi$.



For $n \geq 3$, open and active!

Restriction theory can be used to deduce continuum incidence geometry estimates.

Surprisingly, we can go the other way too (very recent progress, whereas the above direction has been well-known since at least the 90s).

Equivalent formulations of $R_{P^{n-1}}(p \to q)$

Dual version is called "Fourier extension":

$$\|\widehat{f}\|_{L^q(P^{n-1})} = \sup_{\substack{g \in L^{q'}(P^{n-1}) \\ \|g\|_{L^{q'}(P^{n-1})}=1}} \left| \int_{\substack{|\xi|<1 \\ \xi \in \mathbb{R}^{n-1}}} \widehat{f}(\xi, |\xi|^2) g(\xi) \mathrm{d}\xi \right|.$$

$(\frac{1}{q} + \frac{1}{q'} = 1)$. The integral equals:

$$\int_{|\xi|<1} \int_{\mathbb{R}^n} e^{-2\pi i x \cdot (\xi, |\xi|^2)} f(x) \mathrm{d}x g(\xi) \mathrm{d}\xi = \int_{\mathbb{R}^n} f(x) \underbrace{\int_{|\xi|<1} e^{-2\pi i x \cdot (\xi, |\xi|^2)} g(\xi) \mathrm{d}\xi}_{Eg(x)} \mathrm{d}x$$

$$\|Eg\|_{L^{p'}(\mathbb{R}^n)} \leq C_{p,q,n} \|g\|_{L^{q'}(P^{n-1})} \qquad \forall \mathcal{S}(P^{n-1})$$

Call the last inequality $R^*_{P^{n-1}}(q' \to p')$.

Local, dual version: allows us to work with functions, F.T.

For any $R \geq 1$, any $B_R \subseteq \mathbb{R}^n$, we have

$$\left(\int_{\mathbb{R}^n} |f(x)|^{p'} \right)^{\frac{1}{p'}} \lesssim R^{\frac{1}{q}} \left(\int |\widehat{f}(\xi)|^{q'} \mathrm{d}\xi \right)^{\frac{1}{q'}}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp} \widehat{f} \subseteq N_{R^{-1}}(P^{n-1})$.

Call this $R^{*,\mathrm{loc}}_{P^{n-1}}(q' \to p')$.

$R^*_{P^{n-1}}(q' \to p') \implies R^{*,\mathrm{loc}}_{p^{n-1}}(q' \to p')$. First thing bounds Eg, while secound thing bounds f when we have $\operatorname{supp} \widehat{f} \subseteq N_{R^{-1}}(P^{n-1})$.

Let $f \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp} \widehat{f} \subseteq N_{R^{-1}}(P^{n-1})$. Try to express f in trems of ext. op.

$$
\begin{aligned}
f(x) &\overset{\text{Fourier inversion}}{=} \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} \widehat{f}(\xi)\mathrm{d}\xi \\
&= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} e^{2\pi i x\cdot(\xi',\xi_n)} \widehat{f}(\xi',\xi_n)\mathrm{d}\xi_n\mathrm{d}\xi' \\
&= \int_{\mathbb{R}} e^{2\pi x_n\xi_n} \int_{\mathbb{R}^{n-1}} e^{2\pi i x\cdot(\xi',|\xi'|^2)} \underbrace{\widehat{f}(\xi',|\xi'|^2+\xi_n)}_{=:g_{\xi_n}(\xi')} \mathrm{d}\xi'\mathrm{d}\xi_n \qquad \xi_n \mapsto \xi_n + |\xi'|^2 \\
&= \int_{[-R^{-1},R^{-1}]} e^{2\pi x_n\xi_n} \underbrace{\int_{|\xi'|<1} e^{2\pi i x\cdot(\xi',|\xi'|^2)} \underbrace{\widehat{f}(\xi',|\xi'|^2+\xi_n)}_{=:g_{\xi_n}(\xi')} \mathrm{d}\xi'}_{Eg_{\xi_n}(x)} \mathrm{d}\xi_n
\end{aligned}
$$

$$
\begin{aligned}
\int_{\mathbb{R}^n} |f(x)|^{p'}\mathrm{d}x &= \int_{\mathbb{R}^n} \left| \int_{I_{R^{-1}}} e^{2\pi i x_n\xi_n} Eg_{\xi_n}(x)\mathrm{d}\xi_n \right|^{p'} \mathrm{d}x \qquad R^* \text{ bounds } \int |Eg|^{p'} \\
&\overset{\text{Hölders's}}{\le} \int_{\mathbb{R}^n} |I_{R^{-1}}|^{p'-1} \int_{I_{R^{-1}}} |Eg_{\xi_n(x)}|^{p'}\mathrm{d}\xi_n\mathrm{d}x \\
&\lesssim \underbrace{(R^{-1})^{p'-1} \int_{I_{R^{-1}}} \left(\int_{|\xi'|<1} |g_{\xi_n}(\xi')|^{q'}\mathrm{d}\xi' \right)^{\frac{p'}{q'}} \mathrm{d}\xi_n}_{(*)} \qquad \text{using } R^*_{P^{n-1}}(q' \to p')
\end{aligned}
$$

Goal is to bound this last expression by

$$
R^{-\frac{p'}{q}} \underbrace{\int_{I_{R^{-1}}} \int_{|\xi'|<1} |g_{\xi_n}(\xi')|^{q'}\mathrm{d}\xi'\mathrm{d}\xi_n}_{\|\widehat{f}\|_{L^{q'}(N_{R^{-1}}(P^{n-1}))}}
$$

Lucy case: $\frac{p'}{q'} < 1$, i.e. $1 < \frac{q'}{p'}$. Then

$$
(*) \le R^{-p'+1} \int_{I_{R^{-1}}}
$$

$$
\int_A h \overset{\text{Hölder's}}{\le} |A|^{1-\frac{1}{s}} \left(\int_A h^s \right)^{\frac{1}{s}},
$$

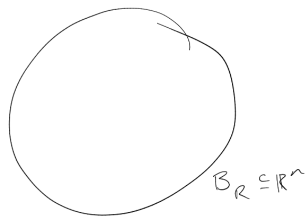
Lecture 7 $\quad s \ge 1$ (actually $\approx$ since h approximately constant on A)

$R^*_{\mathcal{P}^{n-1}}(q' \to p') \implies R^{*,\mathrm{loc}}_{\mathcal{P}^{n-1}}(q' \to p')$ continued.

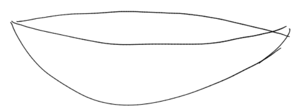
Spatial: $x, f(x), Eg(x)$

$$
\boxed{\int_{\mathbb{R}^n} |f(x)|^{p'}\mathrm{d}x}
$$

$$\int_{\mathbb{R}^n} |Eg(x)|^{p'} \mathrm{d}x$$



Frequency ξ:



$g(\xi', |\xi'|^2) = g(\xi')$, $\operatorname{supp} \widehat{f} \subseteq \mathcal{N}_{R^{-1}}(\mathcal{P}^{n-1})$,

$$\int_{\mathcal{N}_{R^{-1}}(\mathcal{P}^{n-1})} |\widehat{f}(\xi)|^{q'} \mathrm{d}\xi$$

$$\int_{|\xi'|<1} |g(\xi')|^{q'} \mathrm{d}\xi'.$$

$$\int_{B_R} |f(x)|^{p'} \mathrm{d}x \leq \cdots \overset{(*)}{\leq} R^{-(p'-1)} \int_{I_{R^{-1}}} \left(\int_{|\xi'|<1} |g_{\xi_n}(\xi')|^{q'} \mathrm{d}\xi' \right)^{\frac{p'}{q'}} \mathrm{d}\xi_n.$$

Aiming for

$$(*) \leq R^{-\frac{p'}{q}} \left(\int_{I_{R^{-1}}} \int_{|\xi'|<1} |g_{\xi_n}(\xi')|^{q'} \mathrm{d}\xi' \right)^{\frac{p'}{q'}}.$$

Lucky case: $\frac{p'}{q'} \leq 1$.

$$(*) \overset{\text{Hölder's inequality in } \xi_n}{\leq} R^{-(p'-1)} |I_{R^{-1}}|^{1-\frac{p'}{q'}} \left(\int_{I_{R^{-1}}} \int_{|\xi'|<1} |g_{\xi_n}(\xi')|^{q'} \mathrm{d}\xi' \mathrm{d}\xi_n \right)^{\frac{p'}{q'}}$$

$R^{-(p'-1)} (R^{-1})^{1-\frac{p'}{q'}} = R^{-\frac{p'}{q}}$ $(1 = \frac{1}{q} + \frac{1}{q'})$

Unlucky case: $\frac{p'}{q'} > 1$. Hölder's inequality goes in the wrong direction:

$$\int_A h \leq |A|^{1-\frac{1}{s}} \left(\int_A h^s \right)^{\frac{1}{s}}$$

for all $s \geq 1$, $h \geq 0$.

This is equality if h is constant on A.

PAUSE THIS.

Useful Harmonic Analysis tool

The locally constant property.

Convolution: Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Define $f * g \in \mathcal{S}(\mathbb{R}^n)$ by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}y = \int_{\mathbb{R}^n} f(Y)g(x-y)\mathrm{d}y.$$

See Young's convolution:

$$\|f * g\|_{L^r} \leq \|f\|_p \|g\|_q$$

when $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Example. $g = \chi_B$ (B is the unit ball in $\mathbb{R}^n$),

$$f * \chi_B(x) = \int_{y \in B} f(x-y)\mathrm{d}y$$

RHS is "average value of f on $B(x)$".
"$f * \chi_B$ is approximately constant on balls of radius ~ 1".

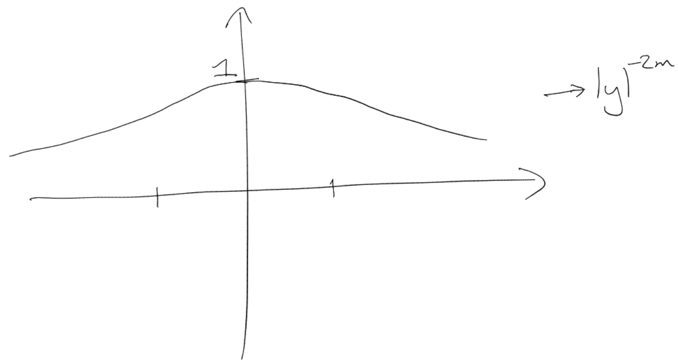
Support property: $\operatorname{supp} f = A$, $\operatorname{supp} g = B$, $\operatorname{supp} f * g \subseteq A + B = \{a + b : a \in A, b \in B\}$.

Convolution and Fourier Transform: $\widehat{fg} = \widehat{f} * \widehat{g}$.

Locally constant property: Let $f \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp} \widehat{f} \subseteq B_1(0)$. Then for any unit ball $B' \subseteq \mathbb{R}^n$ and any $x \in B'$,

$$\|f\|_{L^\infty(B')} \lesssim_m \int_{\mathbb{R}^n} |f|(x-y) \frac{1}{(1+|y|^2)^m} \mathrm{d}y.$$

$\frac{1}{(1+|y|^2)^m}$ is ~ 1 on $|y| \lesssim 1$.

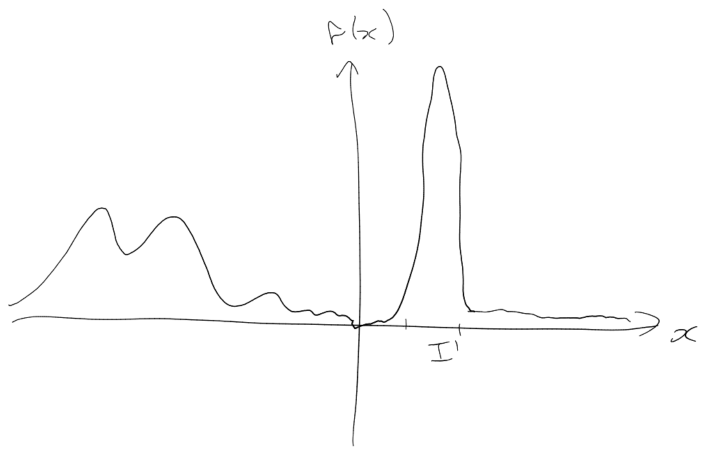


Digesting $f \geq 0$.

For any unit interval $I' \subseteq \mathbb{R}^n$, $x \in I'$,

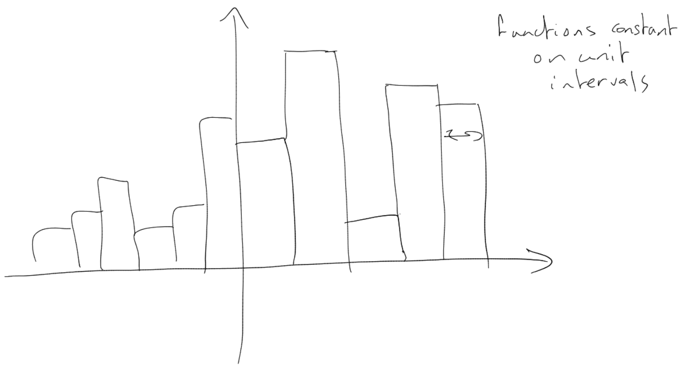
$$\|f\|_{L^\infty(I')} \leq \int_{(-\frac{1}{2},\frac{1}{2})} f(x-y)\mathrm{d}y.$$

Suppose this:



LHS has to be constant.

$$\|f\|_{L^\infty(I')} \lesssim \int_{(-\frac{1}{2},\frac{1}{2})} f(x-y)\mathrm{d}y.$$



$$\|f\|_{L^\infty(I')} \overset{(*)}{\lesssim} \int_{(-\frac{1}{2},\frac{1}{2})} f(x-y)\mathrm{d}y$$

Lemma (Locally constant property)**.** Assuming that:

- $f \in \mathcal{S}(\mathbb{R}^n)$
- $\operatorname{supp} \widehat{f} \subseteq B_1(0)$

Then for any unit ball $B^1 \subseteq \mathbb{R}^n$ and any $x \in B'$,

$$\|f\|_{L^\infty(B')} \lesssim_m \int_{\mathbb{R}^n} |f|(x-y)\frac{1}{(1+|y|^2)^m}\mathrm{d}y.$$

The proof of this fact is more important than the statement – we will be using the strategy in future.

Proof of locally constant property. Let $f \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp} \widehat{f} \subseteq B_1(0)$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi \equiv 1$ on $B_1(0)$, $\operatorname{supp} \varphi \subseteq B_2(0)$.

By Fourier inversion:

$$f = (\widehat{f})^\vee = (\widehat{f}\varphi)^\vee = (\widehat{f})^\vee * \check{\varphi} = f * \check{\varphi}.$$

Let $x_0, x \in B^1$ (unit ball in $\mathbb{R}^n$).

$$\begin{aligned}
|f(x_0)| &= \left|\int f(x_0 - y)\check{\varphi}(y)\mathrm{d}y\right| \\
&\leq \int |f|(x_0 - y)|\check{\varphi}(y)|\mathrm{d}y \\
&= \int |f|(x - y)|\check{\varphi}|(y - (y - x_0))\mathrm{d}y \\
&\lesssim_m \int |f|(x-y)\frac{1}{(1 + |y - \underbrace{(x - x_0)}_{|x-x_0|<1}|^2)^m}\mathrm{d}y \\
&\lesssim_m \int |f|(x-y)\frac{1}{(1+|y|^2)^m}
\end{aligned}$$

□

Returning to $R^*_{\mathcal{P}^{n-1}}(q' \to p') \implies R^{*,\mathbf{loc}}_{\mathcal{P}^{n-1}}(q' \to p')$

$$\int_{B_R} |f(x)|^{p'}\mathrm{d}x \lesssim \int_{B_R} |f\varphi_{B_R}|^{p'}\mathrm{d}x$$

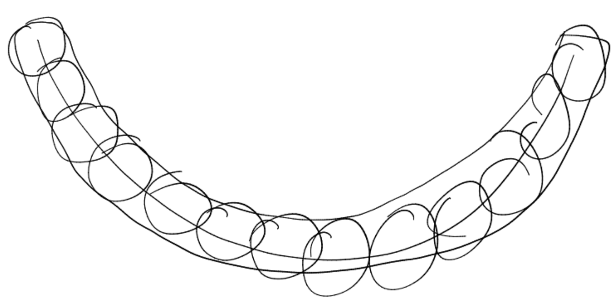
where $\varphi_{B_R} \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\varphi_{B_R} \sim 1$ on B_R, $\operatorname{supp}\widehat{\varphi_{B_R}} \subseteq B_{R^{-1}}(0)$.

$$\widehat{f\varphi_{B_R}} = \underbrace{\widehat{f}}_{R^{-1} \text{ neighbourhood of } \mathcal{P}^{n-1}} * \underbrace{\widehat{\varphi_{B_R}}}_{R^{-1} \text{ ball}} (\xi)$$

supported in $2R^{-1}$-neighbourhood of $\mathcal{P}^{n-1}$.

Repeat steps of proof:

$$\int_{B_R} |f(x)|^{p'}\mathrm{d}x \lesssim R^{-(p'-1)} \int_{I_{R^{-1}}} \left(\int_{|\xi'|<1} |h_{\xi_n}(\xi')|^{q'}\mathrm{d}\xi'\right)^{\frac{p'}{q'}} \mathrm{d}\xi_n$$



$$h(\xi_n) = \boxed{\widehat{f} * \widehat{\varphi_{B_R}}(\xi', |\xi'|^2 + \xi_n)}$$

Lecture 8

6 Tube incidence implications of Fourier restriction

Last time: LOCALLY CONSTANT PROPERTY (think of it more as "heuristic").

If $\operatorname{supp} \widehat{f} \subseteq B_1$ then

(1) Imagine $|f| \sim$ const on unit balls.

(2) $f = (\widecheck{\widehat{\varphi_{B_1}}}) = f * \widecheck{\varphi_{B_1}}$.

What if $\operatorname{supp} \widehat{f} \subseteq B_\lambda(0)$ (so $|f| \sim$ const on λ^{-1}-balls)?

$$f = (\widecheck{\widehat{f}\varphi_{B_\lambda}}) = f * \widecheck{\varphi_{B_\lambda}},$$

where $\varphi_{B_\lambda}(\xi) = \varphi_{B_1}(\lambda^{-1}\xi)$.

$$\widecheck{\varphi_{B_\lambda}}(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\varphi_{B_1}(\lambda^{-1}\xi)\mathrm{d}\xi = \lambda^n \int_{\mathbb{R}^n} e^{2\pi i(\lambda x)\cdot\xi}\varphi_{B_1}(\xi)\mathrm{d}\xi = \lambda^n \widecheck{\varphi_{B_1}}(\lambda x).$$

$*|\widecheck{\varphi_{B_1}}|$ is approximately averaging over a 1-ball. $*|\widecheck{\varphi_{B_\lambda}}|$ is approximately averaging over λ^{-1}-balls.

What about $\operatorname{supp} \widehat{f} \subseteq B_\lambda(\overline{v})$?

Same thing happens, because $e^{ix\cdot\text{something}}f$ will have Fourier support in $B_\lambda(0)$, and taking absolute values means we don't notice the $e^{ix\cdot\text{something}}$ (modulation).

Returning to $R^*_{\mathcal{P}^{n-1}}(q' \to p') \implies R^{*,\mathrm{loc}}_{\mathcal{P}^{n-1}}(q' \to p')$.

$$\int_{B_R} |f(x)|^{p'}\mathrm{d}x \lesssim \int_{B_R} |f(x)\varphi_{B_R}(x)|^{p'}\mathrm{d}x$$

$\varphi_{B_R}(x) \in \mathcal{S}(\mathbb{R}^n)$, $|\varphi_{B_R}| \sim 1$ on B_R, $\operatorname{supp}\widehat{\varphi_{B_R}} \subseteq B_{R^{-1}}(0)$.

Last lecture $\to$

$$\int_{I_{R^{-1}}} \left(\int_{\substack{|\xi'|<1 \\ \xi'\in\mathbb{R}^{n-1}}} |\widehat{f} * \widehat{\varphi_{B_R}}(\xi', |\xi'|^2 + \xi_n)|^{q'}\mathrm{d}\xi' \right) \overset{(*)}{\lesssim} (R^{-1})^{1-\frac{p'}{q'}} \left(\int_{\xi\in\mathbb{R}^n} |\widehat{f}(\xi)|^{q'}\mathrm{d}\xi \right)^{\frac{p'}{q'}}.$$

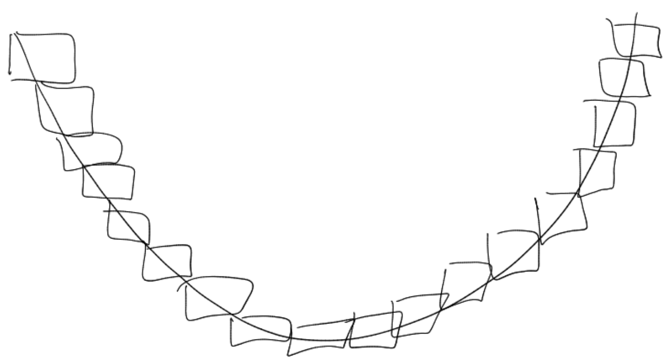
Can choose $\widehat{\varphi_{B_R}}$ such that

- $|\widehat{\varphi_{B_R}}| \sim R^n \chi_{B_{R^{-1}}(0)}$.
- $\|\widehat{\varphi_{B_R}}\|_1 \sim 1$.

Case 1: $\frac{p'}{q'} \leq 1$. LHS of $(*)$ (using Hölder) is

$$
\begin{aligned}
&\leq |I_{R^{-1}}|^{1-\frac{p'}{q'}} \left(\int_{I_{R^{-1}}} \int_{|\xi'|<1} (|\widehat{f}| * |\widehat{\varphi_{B_R}}|^{\frac{1}{q'}+\frac{1}{q}}(\xi', |\xi'|^2+\xi_n))^{q'} \mathrm{d}\xi' \mathrm{d}\xi_n \right)^{\frac{p'}{q'}} \\
&\sim |I_{R^{-1}}|^{1-\frac{p'}{q'}} \left(\int_{\mathbb{R}^n} (|\widehat{f}| * |\widehat{\varphi_{B_R}}|(\xi))^{q'} \mathrm{d}\mathrm{d}\xi \right)^{\frac{p'}{q'}} \\
&\leq \left(\int |\widehat{f}|^{q'}(\xi-\eta)|\widehat{\varphi_{B_R}}|(\eta)\mathrm{d}\eta \right)^{\frac{1}{q'}} \underbrace{\left(\int |\widehat{\varphi_{B_R}}|^{\frac{q}{q}} \right)^{\frac{1}{q}}}_{\sim 1} && \text{(Holder)} \\
&\lesssim |I_{R^{-1}}|^{1-\frac{p'}{q'}} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\widehat{f}|^{q'}(\xi-\eta)|\widehat{\varphi_{B_R}}|(\eta)\mathrm{d}\eta\mathrm{d}\xi \right)^{\frac{p'}{q'}} \\
&\sim |I_{R^{-1}}|^{1-\frac{p'}{q'}} \left(\int_{\mathbb{R}^n |\widehat{f}|^{q'}(\xi)\mathrm{d}\xi} \right)^{\frac{p'}{q'}}
\end{aligned}
$$

Case 2: $\frac{p'}{q'} > 1$. Use $\mathcal{P}^1 \subseteq \mathbb{R}^2$ for intuition.



Imagine a function g which is approximately constant on each R^{-1} cube Q. Think of g as $g = \sum_Q g_Q$.

$$\int_{I_{R^{-1}}} \left(\int_{|t|<1} g(t, t^2+\xi_2)\mathrm{d}t \right)^{\frac{p'}{q'}} \mathrm{d}\xi_2.$$

Note

$$\int_{|t|<1} g(t, t^2+\xi_2)\mathrm{d}t \sim g_Q.$$

Therefore,

$$\int_{|t|<1} g(t, t^2+\xi_2)\mathrm{d}t \sim C$$

if $|\xi_2| \le cR^{-1}$. ($\le C$ for all $\xi_2 \in I_{R^{-1}}$).

$|(P' + (0,\xi_2)) \cap Q| \sim R^{-1}$ for $|\xi_2| \le cR^{-1}$.

$\sim |I_{R^{-1}}|^{1-\frac{p'}{q}+\frac{p}{q'}} C^{\frac{p'}{q'}} = |I_{R^{-1}}|^{1-\frac{p'}{q'}} (|I_{R^{-1}}|C)^{\frac{p'}{q'}} \left(\int_{R^{-1}} \int_{|t|<1} g(t, t^2+\xi_2) \mathrm{d}t \mathrm{d}\xi_2\right)^{\frac{p'}{q'}}$

Important: locally constant property means we didn't need $\frac{p'}{q'} < 1$, like before.

Make the intuition rigorous.

$$\text{LHS of } (*) \lesssim |I_{R^{-1}}| \max_{\xi_2 \in I_{R^{-1}}} \left(\int_{|\xi'|<1} (|\widehat{f}| * |\widehat{\varphi_{B_R}}|(\xi', |\xi'|^2 + \xi_n))^{q'} \mathrm{d}\xi' \right)^{\frac{p'}{q'}}$$

Consider the integral:

$$\begin{aligned}
\int_{|\xi'|<1} (|\widehat{f}| * |\widehat{\varphi_{B_R}}|(\xi', |\xi'|^2 + \xi_n))^{q'} \mathrm{d}\xi' &= \int_{|\xi'|<1} \left(\int_{\mathbb{R}^n} |\widehat{f}|(\eta) |\widehat{\varphi_{B_R}}|((\xi', |\xi'|^2 + \xi_n) - \eta) \mathrm{d}\eta \right)^{q'} \mathrm{d}\xi' \\
&\le \int_{|\xi'|<1} \left(\int_{\mathbb{R}^n} |\widehat{f}|^{q'}(\eta) |\widehat{\varphi_{B_R}}|((\xi', |\xi'|^2 + \xi_n) - \eta) \mathrm{d}\eta \right) \mathrm{d}\xi' \quad \text{Same pointwise Holder} \\
&\sim R^n (R^{-1})^{n-1} \int_{\mathbb{R}^n} |\widehat{f}|^{q'}(\eta) \mathrm{d}\eta \\
&\sim R^1 \int_{\mathbb{R}^n} |\widehat{f}|^{q'} \\
&\lesssim (R^{-1})^1 \cdot R^{\frac{p'}{q'}} \left(\int |\widehat{f}|^{q'} \right)^{\frac{p'}{q'}}
\end{aligned}$$

Lecture 9

Index