Forcing and the Continuum Hypothesis

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Contents

Lecture 1

1 The Continuum Hypothesis

1398: Gödel showed $\text{Con}(\mathsf{ZFC}) \implies \text{Con}(\mathsf{ZFC} + \mathsf{CH})$.

1962: Cohen showed $\text{Con}(\mathsf{ZFC}) \implies \text{Con}(\mathsf{ZFC} + \neg\mathsf{CH})$.

Theorem (Hartogs's Theorem)**.** For every set X, there is a (least) cardinal α such that there is no injection from α to X.

We denote this by Hartogs's aleph of X, $\aleph(X)$.

Theorem. For every set X, there is no injection from $\mathcal{P}(X)$ into X. We denote this by $2^{|X|}$.

Using Axiom of Choice, well-order $\mathcal{P}(X)$ and get ordinal $2^{|X|}$.

We have

$$\aleph(X) \le 2^{|X|}.$$

Notation. Define:

$$\begin{aligned} \aleph_0 &:= \mathbb{N} \\ \aleph_{\alpha+1} &:= \aleph(\aleph_\alpha) \\ \aleph_\lambda &:= \bigcup_{\alpha<\lambda} \aleph_\alpha \\ \beth_0 &= \mathbb{N} \\ \beth_{\alpha+1} &= 2^{\beth_\alpha} \\ \beth_\lambda &= \bigcup_{\alpha<\lambda} \beth_\alpha \end{aligned}$$

Clearly, $\aleph_\alpha \le \beth_\alpha$.

Continuum Hypothesis (CH): $\aleph_1 = \beth_1$.

Generalised Continuum Hypothesis (GCH): $\forall\alpha \aleph_\alpha = \beth_\alpha$.

Why "continuum"?

Lemma. CH if and only if $\forall X \subseteq \mathbb{R}$, X is uncountable $\implies X \sim \mathbb{R}$ ($\sim$ means "there is a bijection").

Proof.

$\Rightarrow$ Obvious since $\beth_1 = |\mathbb{R}|$.

$\Leftarrow$ Suppose $|\mathbb{R}| > \aleph_1$. Well-order the reals in order type $\kappa > \aleph_1$:

$$\{r_\alpha : \alpha < \kappa\}.$$

Consider $X := \{r_\alpha : \alpha < w_1\} \subseteq \mathbb{R}$. This is a subset of the reals of cardinality $\aleph_1$, hence is an uncountable subset which is not in bijection with it. $\square$

Reminder:

Gödel showed $\mathrm{Con}(\mathsf{ZFC}) \implies \mathrm{Con}(\mathsf{ZFC} + \mathsf{CH})$.

Cohen showed $\mathrm{Con}(\mathsf{ZFC}) \implies \mathrm{Con}(\mathsf{ZFC} + \neg\mathsf{CH})$.

Relative consistency proofs.

By Completeness Theorem, this means:

If there is $M \models \mathsf{ZFC}$, then I can construct $N \models \mathsf{ZFC} + (\neg)\mathsf{CH}$.

Analogy from algebra:

$$\mathcal{L}_{\text{fields}} = \{0, 1, +, \cdot, -, {}^{-1}\}.$$

Axioms of fields: Fields.

Let $\varphi_{\sqrt{2}} := \exists x(x \cdot x = 1 + 1)$. Note $\mathbb{Q} \models \neg\varphi_{\sqrt{2}}$.

Idea: Start with $\mathbb{Q}$ and extend $\mathbb{Q}$ to get $F \models \mathrm{Fields} + \varphi_{\sqrt{2}}$.

Construct by *algebraic closure* (not in the usual sense – here we just mean adding in $\sqrt{2}$ and then adding everything else that this requires us to add).

Obtain $\mathbb{Q}(\sqrt{2}) \models \mathrm{Fields} + \varphi_{\sqrt{2}}$.

This is easy because everything that matters (Fields and $\varphi_{\sqrt{2}}$) is determined by equations; all formulas we need to check are atomic.

Definition 1.1 (absolute)**.** If $M \subseteq N$ and M, N are $\mathcal{L}$-structures and φ an $\mathcal{L}$-formula, then we say *φ is absolute between M and N* if for all $x_1, \ldots, x_m \in M$,

$$M \models \varphi(x_1, \ldots, x_n) \iff N \models \varphi(x_1, \ldots, x_n).$$

If the $\Rightarrow$ direction holds, then we say "upwards absolute", and if the $\Leftarrow$ direction holds, then we say "downwards absolute".

Theorem (Substructure Lemma)**.** All atomic formulas are absolute between substructures.

WHat if we have models of ZFC? Have

$$\mathcal{L}_\in = \{\in\}.$$

No function symbols nor constant symbols. So: almost nothing is atomic.

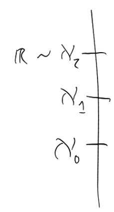
$M \subseteq N$ if and only if M is an $\mathcal{L}_\in$-substructure of N.

And: the formulas that we care about are definitely not atomic, but instead very complex.

Try to imagine a proof of:

If $M \models \mathsf{ZFC}$ then there is $N \supseteq M$ such that $N \models \mathsf{ZFC} + \mathsf{CH}$.

Without loss of generality $M \models \neg\mathsf{CH}$ (else we are done). For simplicity, let's consider the case $\beth_1 = \aleph_2$.



What can we do to "get rid of $\aleph_1$"?

Maybe a surjection $f : \mathbb{N} \to \aleph_1$. Maybe we can form $M[f] \supseteq M$ to get a smallest model $M[f] \models \mathsf{ZFC}$.

Clearly, in $M[f]$, $\aleph_1^M$ is not a cardinal anymore.

Does that show CH?

All sorts of things can happen

Assuming it is actually possible to form this smallest model $M[f]$, there are lots of ways that this might not end up being enough to deduce CH. For example:

- Maybe $\mathbb{R}^{M[f]} \neq \mathbb{R}^M$
- Maybe $\aleph_2^M$ is not a cardinal either

A fundamental problem of non-absoluteness

Consider $\varphi_\emptyset(x) := \forall z(z \notin x)$, which means "$x$ is empty".

Consider $M \models \mathsf{ZFC}$. Therefore there are e, e' such that $M \models \varphi_\emptyset(e)$ and $M \models \forall z(z \in e' \iff z = e)$.

Consider $N := M \setminus \{e\}$.

N is an $\mathcal{L}_\in$-substructure of M. But $N \models \varphi_\emptyset(e')$, even though $M \models \neg\varphi_\emptyset(e')$.

So $\varphi_\emptyset$ is not absolute between substructures.

Instead of substructures, we will restrict out attention to *transitive substructures*: in particular, to M transitive ($\forall x, x \in M \implies x \subseteq M$ or equivalently $x \in M \wedge y \in x \implies y \in M$) such that $M \models \mathsf{ZFC}$.

Lecture 2 Next time: theorems about absoluteness for transitive substructures.

Definition (Absolute formula)**.** We say φ *is absolute for* M if for all $x_1, \dots, x_n \in M$, we have

$$M \models \varphi(x_1, \dots, x_n) \iff \varphi(x_1, \dots, x_n) \text{ is true.}$$

Clearly, if φ is absolute for M and absolute for N, then it's absolute between M and N.

Cohen's proof becomes:

If M is a countable transitive set such that $M \models \mathsf{ZFC}$, then there is a a countable transitive set $N \supseteq M$ such that $N \models \mathsf{ZFC} + \neg\mathsf{CH}$.

2 Transitive Models

Observation: If M is transitive and $M \models$ "e is empty", then $e = \emptyset$. This is because if $w \in e$, then $w \in e$ and $e \in M$ gives us that $w \in M$ by transitivity, so $M \models w \in e$, so $M \models$ "e is not empty".

Lemma. Assuming that:

- M is transitive

Then

$$M \models \text{Extensionality} + \text{Foundation}.$$

Proof. Extensionality: $\forall x \forall y (\forall w (w \in x \leftrightarrow w \in y) \to x = y)$. Take $x \neq y$, $x, y \in M$. Without loss of generality take $z \in x \setminus y$. Then $z \in x$ and $x \in M$ so $z \in M$. So $M \models z \in x \wedge z \notin y$. So $M \models \neg \forall w (w \in x \leftrightarrow w \in y)$.

Foundation: $\forall x (x \neq \emptyset \to \exists m (m \in x \wedge \forall w \neg (w \in m \wedge w \in x)))$. Take $x \in M$. $M \models x \neq \emptyset$ so x is not empty. So find $m \in x$ which is $\in$-minimal. Then since $x \in M$ as well, we have $m \in M$. Therefore x has an $\in$-minimal element in M. □

2.1 Absoluteness for transitive models

Definition (Bounded quantifier)**.** We call a quantifier of the form $\exists x \in y, \varphi$ or $\forall x \in y, \varphi$ a *bounded quantifier*.
(Defined by $\exists x \in y, \varphi := \exists x (x \in y \wedge \varphi)$ and $\forall x \in y \varphi := \forall x (x \in y \to \varphi)$).

Definition (Closed under bounded quantification)**.** A class of formulas Γ is *closed under bounded quantification* if whenever φ is in Γ, then so are $\exists x \in y, \varphi$ and $\forall x \in y, \varphi$.

Definition (Delta0)**.** Δ_0 is the smallest class of formulas containing the atomic formulas that is closed under propositional connectives and bounded quantifiers.
Let T be any theory. Then Δ_0^T is the class of formulas equivalent to a Δ_0 formula in T.

Theorem. Δ_0 formulas are absolute for transitive models.

Proof. By induction:

(1) All atomic formulas are absolute by the substructure lemma.

(2) Propositional connectives: exactly the same proof as in the substructure lemma.

(3) Assume that φ is absolute and show that $\exists x \in y, \varphi$ and $\forall x \in y, \varphi$ are absolute.

- $\exists x \in y, \varphi$: If $\exists x \in y, \varphi$ is true for some $y \in M$, then pick a witness $x \in y$. Since $y \in M$, we have $x \in M$. By the induction hypothesis, we have that $M \models \varphi(x, y)$. Thus $M \models \exists x(x \in y \wedge \varphi(x, y))$.
  If $M \models \exists x(x \in y \wedge \varphi(x, y))$, then $x \in y \wedge \varphi(x, y)$ is true.
- $\forall x \in y, \varphi$: Similar. □

Corollary. Assuming that:

- T is any theory
- $M \models T$ is transitive

Then Δ_0^T-formulas are absolute for M.

Definition (*Sigma*1, *Pi*1)**.** A formula is called Σ_1 if it is of the form $\exists x_1, \dots \exists x_n, \varphi$ where φ is Δ_0.
It is called Π_1 if it is of the form $\forall x_1, \dots, \forall x_n, \varphi$ where φ is Δ_0.

(same for Σ_1^T, Π_1^T).

Proof. Just definition of the semantics of $\exists, \forall$. □

Example. What is Δ_0?

1. $x \in y$
2. $x = y$
3. $x \subseteq y$: $\iff \forall w \in x(w \in y)$
4. $z = \{x\}$: $\iff x \in z \wedge \forall w \in z(w = x)$
5. $z = \{x, y\}$
6. $z = (x, y) = \{\{x\}, \{x, y\}\}$
7. $z = \emptyset$: $\iff \forall w \in z(w \neq w)$
8. $z = x \cup y$: $\iff x \subseteq z \wedge y \subseteq z \wedge \forall w \in z(w \in x \vee w \in y)$
9. $z = x \cap y$
10. $z = x \setminus y$
11. $z = x \cup \{x\}$

12. z is transitive

13. $z = \bigcup x$

Definition (Absolute function)**.** Let M a transitive set, and $F : M^n \to M$ (note that this means that M is closed under F). We say F is *absolute for* M if there is a formula Φ which is absolute for M such that

$$F(x_1, \ldots, x_n) = z \iff \Phi(x_1, \ldots, x_n, z).$$

Observation: So, if M is closed under pairing ($\forall x, y \in M, \{x, y\} \in M$), then the pairing operation $x, y \mapsto \{x, y\}$ is absolute, and therefore $M \models$ Pairing.

Similarly for union.

Lemma. Assuming that:

- φ is absolute for M
- $F, G_1, \ldots, G_n$ are absolute operations on M

Then

$$\begin{aligned}\psi(x_1, \ldots, x_m) &:= \varphi(G_1(x_1, \ldots, x_m), \ldots, G_n(x_1, \ldots, x_m)) \\ H(x_1, \ldots, x_m) &:= F(G_1(x_1, \ldots, x_m), \ldots, G_n(x_1, \ldots, x_m))\end{aligned}$$

are absolute for M.

Proof. Check the definitions! □

Example. More examples:

(14) z is an ordered pair:

$$\exists s \in z, \exists d \in z, \exists x \in s, \exists y \in d, (\forall w \in s, (w = x) \wedge \forall v \in d, (v = x \wedge v = y) \wedge \forall w \in z, (w = s \vee w = d)).$$

(15) $z = a \times b$

(16) z is a relation

(17) $z = \operatorname{dom} x$

(18) $z = \operatorname{range} x$

(19) z is a function

(20) z is injective

(21) z is surjective

(22) z is bijective

Lecture 3

Ordinals

"x is an ordinal" means x is transitive and $(x, \in)$ is a well-order.

We know: being well-founded is not expressible in first-order logic (see Example Sheet 1).

Because all transitive models satisfy Foundation, we have that if M is transitive, then

$$M \models x \text{ is transitive} \wedge (x, \in) \text{ is linearly ordered.}$$

characterises ordinals. But this is clearly in Δ_0.

So: being an ordinal is absolute for transitive models.

Thus $M \cap \mathrm{Ord} = \{x \in M : M \models x \text{ is an ordinal}\}$. This is transitive, thus there is $\alpha \in \mathrm{Ord}$ such that $\alpha = M \cap \mathrm{Ord}$.

Also absolute:

- "x is a successor ordinal" $(\exists y \in TODO)$
- "x is a limit ordinal"
- "x is a non-zero limit ordinal"
- $x = \omega$ TODO

Cardinals

"x is a cardinal" if and only if

$$x \text{ is an ordinal} \wedge \forall f, \forall y \in x, f : y \to x \implies f \text{ is not a surjection}$$

Note that $\forall f$ is not bounded (while $\forall y \in x$ is bounded).

Observe: this is Π_1 and therefore downwards absolute.

Remark.

(1) We may not *want* this to be absolute. If it was, we couldn't change cardinal behaviour.

(2) We can't obviously bound $\forall f$, since the natural bound would be

$$\{h : h \to y \to x\}$$

or

$$\mathcal{P}(y \times x).$$

These, however, are not (yet??) on our list of absolute concepts.

(3) Not that neither (1) nor (2) is an argument, since there could be an equivalent formula that is Δ_0.

2.2 Non-absoluteness

Assume that $M \models \mathsf{ZFC}$ is transitive and countable. Then

$$M \cap \mathrm{Ord} = \alpha < \omega_1.$$

However, $M \models \mathsf{ZFC}$ implies $M \models$ there are uncountable cardinals.

Let $\beta < \alpha$ be such that $M \models \beta$ is the least uncountable cardinal.

But β is a countable ordinal, so not a cardinal.

Consequence: All cardinals in M except $\aleph_0$ are going to be fake.

So "x is a cardinal" can't be absolute.

Note. This also shows that "$x = \mathcal{P}(y)$" cannot be absolute:
Take y such that $M \models y = \mathcal{P}(\omega)$. Then $y \subseteq \mathcal{P}(\omega)$, but is countable since $y \subseteq M$.
Thus $y \neq \mathcal{P}(\omega)$. Therefore "$x = \mathcal{P}(y)$" is not absolute.

Recall the general proof strategy mentioned before:

> If M is a countable transitive set such that $M \models \mathsf{ZFC}$, then there is a a countable transitive set $N \supseteq M$ such that $N \models \mathsf{ZFC} + \neg\mathsf{CH}$.

Question: Is this really solving the original problem? i.e. $\mathrm{Con}(\mathsf{ZFC}) \implies \mathrm{Con}(\mathsf{ZFC} + \neg\mathsf{CH})$.

It's not obvious that $\mathrm{Con}(\mathsf{ZFC})$ implies that there is a countable transitive model (ctm) of ZFC.

Answer: That's not only not obvious, but fake.

Let's prove that Con(ZFC) $\not\Longrightarrow$ there is a countable transitive model of ZFC.

Why? Note that Con(ZFC), or Con(T) for any T is Δ_0. So, it's absolute for transitive models.

So if M is a countable transitive model of ZFC, then Con(ZFC) is true, so by absoluteness, $M \models$ Con(ZFC). So $M \models \mathsf{ZFC} + \text{Con}(\mathsf{ZFC})$. This contradicts Gödel's Incompleteness Theorem.

We can get a proof of

$$\text{Con}(\mathsf{ZFC}) \implies \text{Con}(\mathsf{ZFC} + \neg\mathsf{CH})$$

via a trick (Example Sheet 1).

Lemma (Cohen Lemma)**.** Assuming that:

- $T \subseteq \mathsf{ZFC}$

Then there is finite $T^* \subseteq \mathsf{ZFC}$ such that if M is a countable transitive model of T^*, then there is $N \supseteq M$ such that N is a countable transitive model of $T + \neg\mathsf{CH}$.

This reduces the problem to:

Find countable transitive models of T^* for sufficiently large finite $T^* \subseteq \mathsf{ZFC}$.

Definition (Hierarchy)**.** We call an assignment $\alpha \mapsto Z_\alpha$ a *hierarchy* if

(i) Z_α is a transitive set

(ii) $\text{Ord} \cap Z_\alpha = \alpha$

(iii) $\alpha < \beta \implies Z_\alpha \subseteq Z_\beta$

(iv) λ limit $\implies Z_\lambda = \bigcup_{\alpha<\lambda} Z_\alpha$

If $\{Z_\alpha : \alpha \in \text{Ord}\}$ is a hierarchy, we can define $Z := \bigcup_{\alpha\in\text{Ord}} Z_\alpha$. This is a proper class as $\text{Ord} \subseteq Z$. We also define $\rho_Z(x) := \min\{\alpha : x \in Z_\alpha\}$, a notion of *$Z$-rank*.

Paradigmatic example: von Neumann hierarchy V_α, and V is the entire universe.

Theorem 2.1 (Levy Reflection Theorem)**.** Assuming that:

- Z is a hierarchy
- φ is a formula

Then there are unboundedly many θ such that φ is absolute between Z_θ and Z.

Proposition 2.2 (Tarski-Vaught Test)**.** Assuming that:

- $\mathcal{M}$ is a substructure of $\mathcal{N}$

Then $\mathcal{M}$ is an elementary substructure if and only if for any formula $\phi(v, \overline{w})$ and $\overline{a} \in M$, if there is $b \in N$ such that $\mathcal{N} \models \phi(b, \overline{a})$, then there is $c \in M$ such that $\mathcal{N} \models \phi(c, \overline{a})$.

TVT$_\Phi$:

Let $M \subseteq N$ and Φ be a collection of formulas closed under subformulas. Then the following are equivalent:

(i) All formulas in Φ are absolute between M and N.

(ii) For all $\varphi \in \Phi$, the *TV-condition* holds: if $\varphi = \exists x \psi$, then for all $\overline{y} \in M$ if there is $a \in N$ sucht hat $N \models \psi(a, \overline{y})$, then there is $b \in M$ such that $N \models \psi(b, \overline{y})$.

Lecture 4

Warm-up: let $(M, \in) \models \mathsf{ZFC}$. Find countable $N \subseteq M$ such that $(N, \in) \prec (M, \in)$.

Suppose $\overline{p} = (p_0, \ldots, p_n) \in M$ and $M \models \exists y, \psi(y, \overline{p})$. Let $w(\psi, \overline{p})$ be a witness for this:

$$M \models \psi(w(\psi, \overline{p}), \overline{p})$$

(if necessary, use Axiom of Choice).

(if $M \models \neg \exists y, \psi(y, \overline{p})$, then let $w(\psi, \overline{p}) = \emptyset$).

Set:

$$\begin{aligned} N_0 &:= \emptyset \\ N_{i+1} &:= \{w(\psi, \overline{p}) : \psi \text{ formula and } \overline{p} \in N_1^{<\omega}\} \\ N &:= \bigcup_{i \in \omega} N_i \end{aligned}$$

Note:

(1) N is countable.

(2) $M \prec N$ by Tarski-Vaught Test.

Remark. In general, even if M is transitive, N is not.
For example, if $\omega_1 \in M$, then

$$\exists x, (x \text{ is the least countable ordinal})$$

is true in M.

$$w(\psi, \emptyset) = \omega_1.$$

So $\omega_1 \in N$. But $\omega_1 \subseteq N$, since N is countable.

Relevant later!

Also see Example Sheet 1.

Proof of Levy Reflection Theorem. Fix φ and let Φ be its collection of subformulas. This is a finite set!

Need to show: $\forall\alpha, \exists\theta > \alpha$ such that $Z_\theta \models \varphi \iff Z \models \varphi$.

For each $\psi \in \Phi$ and $\overline{p} = (p_0, \dots, p_n)$, write

$$\begin{aligned} o(\psi, \overline{p}) &:= \begin{cases} \text{least } \alpha \text{ such that } \exists y \in Z_\alpha \text{ with } Z \models \psi(y, \overline{p}) & \text{if it exists} \\ 0 & \text{otherwise} \end{cases} \\ o(\overline{p}) &:= \max_{\psi\in\Phi} o(\psi, \overline{p}) \\ \theta_0 &:= \alpha + 1 \\ \theta_{i+1} &:= \sup\{o(\overline{p}) : \overline{p} \in Z_{\theta_i}^{<\omega}\} \\ \theta &:= \sup_{i\in\omega} \theta_i \end{aligned}$$

Then Tarski-Vaught Test implies that Z_θ and Z agree on φ. □

Corollary. If $T \subseteq \mathsf{ZFC}$ is finite, then there is M transitive such that $M \models T$.

Proof. Let $\varphi := \bigwedge_{\psi\in T} \psi$. Since $\mathsf{ZFC} \vdash \varphi$, we have that φ is true. By Levy Reflection Theorem, we can find θ such that $V_\theta \models \varphi$. Note V_θ is transitive. □

Remark about the proof of Levy Reflection Theorem:

Can you do the same if Φ is infinite?

Of course not: otherwise we colud prove that there exists θ such that $V_\theta \models \mathsf{ZFC}$, and hence get Con($\mathsf{ZFC}$).

The problem is the case distinction in the definition of $o(\psi, \overline{p})$: it requires to check whether $\exists y, \psi$ is true.

Next goal: Obtain some $M \subseteq V_\theta$ countable such that $M \models \varphi$ and M is transitive.

TODO

Theorem (Mostowski's Collapsing Theorem)**.** Let r be a relation on a set a that is well-founded and extensional. Then there exists a transitive set b, adn a bijection $f : a \to b$ such that $(\forall x, y \in a)(x \; r \; y \iff f(x) \in f(y))$. Moreover, b and f are unique.

Proof. See Logic and Set Theory. $\square$

Corollary. For every $T \subseteq \mathsf{ZFC}$ finite, there is a countable transitive model of T.

Proof. Without loss of generality that T contains the axiom of extensionality. Form $M \models T$ transitive by Levy Reflection Theorem.

Use warm-up to obtain $N \prec M$ countable. This is extensional and well-founded, so by Mostowski find W transitive such that

$$(W, \in) \cong (N, \in).$$

Then $W \models T$ adn $|W| = ||$, so W is countable. $\square$

The next few lectures will be spent proving $\mathrm{Con}(\mathsf{ZFC} + \mathsf{CH})$ using Gödel's constructible universe.

Absoluteness is preserved under transfinite recursion.

Let F, G, H be three operations.

$$\begin{aligned} R(0, \overline{x}) &:= F(\overline{x}) \\ R(\alpha + 1, \overline{x}) &:= G(\alpha, R(\alpha, \overline{x}), \overline{x}) \\ R(\lambda, \overline{x}) &:= H(\lambda, \{R(\alpha, \overline{x}) : \alpha < \lambda\}, \overline{x}) \end{aligned} \tag{$*$}$$

Proof. Attempts: set functions satisfying the $(*)$.

L1 All attempts agree on their common domain.

L2 $\forall \alpha, \exists r$ attempt such that $(\alpha, \overline{x}) \in \mathrm{dom}(r)$.

$R(\alpha, \overline{x}) := y$ if and only if there exists attempt r with $(\alpha, \overline{x}) \in \mathrm{dom}(r)$ and $r(\alpha, \overline{x}) = y$. $\square$

Note that for F, G, H fixed, there is a finite fragment $T_{F,G,H} \subseteq \mathsf{ZFC}$ that proves the recursion theorem instance for F, G, H.

Theorem. If $T \supseteq T_{F,G,H}$ and F, G, H are absolute for transitive models of T, then so is R defined by $(*)$.

Lecture 5

Want $T_{F,G,H} \vdash L1(F, G, H)$, $T_{F,G,H} \vdash L2(F, G, H)$ and $T_{F,G,H}$ proves existence of R.

Convention: We say "T is sufficiently strong" if $T \subseteq \mathsf{ZFC}$ is finite and T proves hte existence of all relevant operations such that they are absolute for transitive models of T.

Proof. Observe that by assumption, being an "attempt" is absolute for transitive models of T.

Let $M \models T$ be transitive.

(1) To show: If $M \models R(\alpha, \overline{x}) = z$, then $R(\alpha, \overline{x}) = z$.

If $M \models R(\alpha, \overline{x}) = z$, then $M \models \exists r,\ r$ is an attempt and $r(\alpha, \overline{x}) = z$. Without the $\exists r$, this would be absolute. So when we include the existential quantifier, we get an upwards absolute sentence.

Thus: there is r such that r is an attempt and $r(\alpha, \overline{x}) = z$. So $R(\alpha, \overline{x}) = z$.

(2) Other direction. Assume r is an attempt with $r(\alpha, \overline{x}) = z$.

Since $T_{F,G,H} \vdash L2(F, G, H)$, we have

$$M \models \exists r',\ \underbrace{r' \text{ is an attempt and } (\alpha, \overline{x}) \in \operatorname{dom} r'}_{\text{absolute}}.$$

Since it is absolute, r' is a real attempt.

By ???, $r'(\alpha, \overline{x}) = r(\alpha, \overline{x})$. Hence $M \models R(\alpha, \overline{x}) = z$. □

Note. This uses the fact that "Δ_1" concepts are absolute.

Definition (Delta1T property)**.** A property is called Δ_1^T if it's both $\Sigma_1{}^T$ and $\Pi_1{}^T$.

Observe: Δ_0^T concepts are absolute (upwards from Σ_1 and downwards from Π_1).

Typical Applications

Bounding a quantifier by operation.

Let F be an operation and T strong enough to prove F is an operation and absolute.

$$T \vdash \forall x, \exists z, F(x) = z$$
$$T \vdash \forall x, \forall z, \forall z', F(x) = z \wedge F(x) = z' \to z = z'$$

Then the quantifiers $\exists y \in F(x)$ and $\forall y \in F(x)$ preserve absoluteness.

$$\begin{aligned} \exists y \in F(x)\psi &\iff \underbrace{\exists z\, (\underbrace{z = F(x) \wedge \exists y \in z\psi}_{\text{absolute}})}_{\text{upwards absolute}} \\ &\iff \underbrace{\forall z\, (\underbrace{z = F(x) \to \exists y \in z\psi}_{\text{absolute}})}_{\text{downwards absolute}} \end{aligned}$$

Applications

(1) Encode formulas as elements of $\omega^{<\omega}$.

$\in$	$=$	$($	$)$	$\wedge$	$\vee$	$\neg$	$\exists$	$\forall$	v_0	v_1	v_2	$\cdots$
0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$

$\forall v_0 \exists v_1 \neg v_0 \in v_1$ would be $(8, 9, 7, 10, 6, 8, 0, 10)$.

Fml $\subseteq \omega^{<\omega}$. So, Fml is absolute for some (sufficiently strong) finite fragment of ZFC (see Example Sheet 1).

(2) If X is any set then

$$\text{“}X \models \varphi\text{”}$$

(which means “$(X, \in) \models \varphi$”) is defined by the usual (Tarski) recursion and thus also absolute (Example Sheet 1).

2.3 The constructible hierarchy

Fix a set X. Define for each $\varphi \in \text{Fml}$ and each $p \in X^{<\omega}$ (parameter)

$$D(\varphi, p, X) := \{w \in X : X \models \varphi(p, w)\}$$

the subset of X defined by φ with parameter p.

For a sufficiently strong $T \subseteq \mathsf{ZFC}$ finite, we have that T proves that D is an absolute operation (see Example Sheet 1).

$$\mathcal{D}(X) := \{D(\varphi, p, X) : \varphi \in \text{Fml}, p \in X^{<\omega}\}.$$

This is absolute for a sufficiently strong theory (use Replacement to get $\mathcal{D}(X)$).

This $\mathcal{D}(X)$ is sometimes (misleadingly) called the “definable power set of X” (it is misleading because it is more like a “definable (by X) power set of X”).

$(\alpha \in \mathcal{D}(X) \iff \exists \varphi \in \text{Fml}, \exists p \in X^{<\omega}, a = D(\varphi, p, X))$

Obvious: $\mathcal{D}(X) \subseteq \mathcal{P}(X)$. Also: If X is transitive, then so is $\mathcal{D}(X)$.

$$\begin{aligned} L_0 &:= \emptyset \\ L_{\alpha+1} &:= \mathcal{D}(L_\alpha) \\ L_\lambda &= \bigcup_{\alpha<\lambda} L_\alpha \end{aligned}$$

The constructible hierarchy.

We usually write $L := \bigcap_{\alpha \in \text{Ord}} L_\alpha$.

Claim: L is a hierarchy (in the sense of Lecture). See Example Sheet 1.

By closure of absoluteness under transfinite recursion, the L-hierarchy is absolute for transitive models of $T \subseteq \mathsf{ZFC}$ where T is strong enough to prove that it exists.

i.e. if $M \models T$ transitive and $\alpha \in \mathrm{Ord} \cap M$ and $M \models X = L_\alpha$, then $X = L_\alpha$. So

$$\bigcup_{\alpha \in \mathrm{Ord} \cap M} L_\alpha \subseteq M.$$

The main theorem of next lecture will be:

If $L \models \mathsf{ZF}$ and $M \models \mathsf{ZF}$ transitive, then

$$\bigcup_{\alpha \in \mathrm{Ord} \cap M} L_\alpha \models \mathsf{ZF}.$$

(Minimal ZF-model).

Some first idea of what the L-hierarchy is like

Clearly, by induction, $L_\alpha \subseteq V_\alpha$, and clearly for $n \in \omega$, $L_n = V_n$. So $L_\omega = V_\omega$.

$$L_{\alpha+1} := \bigcup_{\varphi \in \mathrm{Fml}} \bigcup_{p \in L_\alpha^{<\omega}} \{D(\varphi, p, L_\alpha)\}.$$

If $\alpha \geq \omega$, then

$$|L_{\alpha+1}| \leq \aleph_0 \cdot |L_\alpha^{<\omega}| = \aleph_0 \cdot |L_\alpha|.$$

Thus $|L_\alpha| = |L_{\alpha+1}|$.

Therefore $\alpha < \omega_1$, $|L_\alpha| = \aleph_0$ and $|L_{\omega_1}| = \aleph_1$.

This means: $V_{\omega+1} \neq L_{\omega+1}$ (since the first has size $2^{\aleph_0}$, while the second has size $\aleph_0$).

Note: This does not mean $V \neq L$. ($V = L$ means $\forall x, \exists \alpha, x \in L_\alpha$).

Lecture 6 $V = L$ is called the "axiom of constructibility".

There is a finite fragment $T_{\mathbb{L}}$ of ZF[!] that proves that all of the operations occuring in the definition of $\mathbb{L}$, i.e. $\mathrm{Fml}, X^{<\omega}, \models, D, \mathcal{D}$ are well-defined and absolute.

Thus, if M is a transitive model of $T_{\mathbb{L}}$, then

$$\forall \alpha \in \mathrm{Ord} \cap M, \mathbb{L}_\alpha \in M$$

and thus $\mathbb{L}_\alpha \subseteq M$.

So $\bigcup_{\alpha \in \mathrm{Ord} \cap M} \mathbb{L}_\alpha \subseteq M$.

Axioms of ZF

Structural axioms:

- Extensionality:
$$\forall x, \forall y, (\forall w(w \in x \leftrightarrow w \in y) \to x = y).$$
- Foundation:
$$\forall x(\exists v, (v \in x) \to \exists m, (m \in x \wedge \forall w, \neg(w \in m \wedge w \in x))).$$
- Infinity:
$$\exists i, \exists e, (\forall v, \neg v \in e \wedge e \in i) \wedge \forall x, (x \in i \to \exists s, (s \in i \wedge \forall w, (w \in s \leftrightarrow w \in x \vee w = x))).$$

Functional axioms:

- Pairing:
$$\forall x, \forall y, \exists p, \forall w, (w \in p \leftrightarrow w = x \vee w = y).$$
- Union:
$$\forall x, \exists v, \forall w, (w \in v \leftrightarrow \exists z, (z \in x \wedge w \in z)).$$
- Powerset:
$$\forall x, \exists p, \forall w, (w \in p \leftrightarrow \forall v, (v \in w \to v \in x)).$$
- Separation φ:
$$\forall \overline{p}, \forall x, \exists s, \forall w, (w \in s \leftrightarrow w \in x \wedge \varphi(w, \overline{p})).$$
- Replacement φ:
$$\forall \overline{p}, [\forall x, \forall y, \forall z(\varphi(x, y, \overline{p}) \wedge \varphi(x, z, \overline{p}) \to y = z)] \to \forall x, \exists r, \forall w, (w \in r \leftrightarrow \exists y, (y \in x \wedge \varphi(y, w, \overline{p}))).$$

Now we check that these hold in $\mathbb{L}$.

In Lecture 2, we proved Extensionality and Foundation in all transitive structures, so also in $\mathbb{L}$.

Note that ω satisfies the condition of the axiom of infinity, so any M transitive with $\omega \in M$ will satisfy the axiom of infinity. TODO

Now do pairing and union.

Since the definitions of pairs and unions

$$z = \{x, y\}$$
$$z = \bigcup x$$

are absolute for transitive models, it's enough to show that

$$\forall x, y \in \mathbb{L}, \{x, y\} \in \mathbb{L}$$
$$\forall x \in \mathbb{L} \bigcup x \in \mathbb{L}$$

If $x, y \in \mathbb{L}_\alpha$, $\varphi(w, x, y) := w = x \vee w = y$,

$$\begin{aligned} D(\varphi, (x, y), L_\alpha) &= \{w \in L_\alpha : L_\alpha \models \varphi(w, x, y)\} \\ &= \{w \in L_\alpha : L_\alpha \models w = x \wedge w\} TODO \end{aligned}$$

Powerset axiom.

$$\forall x, \exists p, \underbrace{\forall w, (w \in p \leftrightarrow w \subseteq x)}_{*}$$

The problem here is that $*$ is not obviously absolute. In particular, $z = \mathcal{P}(x)$ is not absolute.

Consider $\mathbb{L}_{\omega+1}$: we have $\omega \in \mathbb{L}_{\omega+1}$ and $\mathcal{P} \cap \mathbb{L}_{\omega+1}$ is countable.

In $\mathbb{L}_{\omega+2}$, we find

$$\{a \in \mathbb{L}_{\omega+1} : a \subseteq \omega\}$$

which is the best possible answer to the question "what is the power set of ω?" that $\mathbb{L}_{\omega+1}$ can give, but unlikely to be the correct answer.

Consider instead $\mathcal{P}(\omega) \cap \mathbb{L} =: P$ and define

$$\Omega := \{\rho_{\mathbb{L}}(a) : a \in P\}.$$

(reminder: $\rho_{\mathbb{L}}(a)$ is the least α such that $a \in \mathbb{L}_{\alpha+1}$)

By Replacement, Ω is a set of ordinals, so find $\alpha > \Omega$. Then $P \subseteq \mathbb{L}_\alpha$.

Therefore $P = \{a \in \mathbb{L}_\alpha : a \subseteq \omega\} \in \mathbb{L}_{\alpha+1}$, so $P \in \mathbb{L}$.

Separation:

$$\forall \overline{p}, \forall x, \exists s, \forall w, (w \in s \leftrightarrow \underbrace{w \in x \wedge \varphi(w, \overline{p})}_{\varphi'(w, x, \overline{p})})$$

If $x \in \mathbb{L}_\alpha$, then

$$\begin{aligned} D(\varphi', x, L_\alpha) &:= \{w \in L_\alpha : L_\alpha \models \varphi'(w, x, \overline{p})\} \\ &= \{w \in L_\alpha : L_\alpha \models w \in x \wedge \varphi(w, \overline{p})\} \\ &\stackrel{?}{=} \{w \in \underbrace{\mathbb{L}}_{\text{not a problem}} : \underbrace{\mathbb{L}}_{\text{this is a problem}} \models w \in x \wedge \varphi(w, \overline{p})\} \end{aligned}$$

If φ is not absolute between L_α and L, this won't work.

Levy Reflection Theorem to the rescue: $\forall\varphi, \forall\alpha, \exists\theta > \alpha$ such that φ is absolute between L_θ and L.

Thus: form

$$\begin{aligned} D(\varphi', x, L_\theta) &= \{w \in L_\theta : L_\theta \models w \in x \wedge \varphi(w, \overline{p})\} \\ &\overset{\text{absolute}}{=} \{w \in L_\theta : L \models w \in x \wedge \varphi(w, \overline{p})\} \end{aligned}$$

Replacement:

This will be on Example Sheet 2. The proof is a combination of the ideas from power set and separation.

Corollary 2.3 (Minimality)**.** Assuming that:

- T is a transitive model of ZF

Then for all $\alpha \in T \cap \mathrm{Ord}$, $\mathbb{L}_\alpha \subseteq T$. Axiom of TODO.

Remark. Remark on the Axiom of Choice.
Gödel (1938): Con(ZF) $\to$ Con(ZFC).
Note first that everything we did so far only needed ZF in the universe. We will sketch that $\mathbb{L} \models$ AC. In fact a strong version of AC known as GLOBAL CHOICE: there is an absolutely definable bijective operation between $\mathbb{L}$ and Ord.
Sketch: Recursive construction of bijections $\pi_\alpha : \mathbb{L}_\alpha \to \eta_\alpha$ for some ordinal η_α, and such that for $\beta < \alpha$ we have $\pi_\alpha|_{\mathbb{L}_\beta} = \pi_\beta$.
If λ is a limit and π_α is defined for $\alpha < \lambda$, then let

$$\pi_\lambda(x) := \pi_\alpha(x)$$

if $x \in \mathbb{L}_\alpha$.
Suppose $\alpha = \beta + 1$ and π_β is given by $\pi_\beta : \mathbb{L}_\beta \to \eta_\beta$.
Consider $\mathrm{Fml} \times L_\beta^{<\omega}$. Well-order it in order type η'_β via the induced π_β well-order. Then if $x \in L_\alpha$, say

$$\pi_\alpha(x) := \begin{cases} \pi_\beta(x) & \text{if } x \in L_\beta \\ \eta_\beta \xi & \text{if } \xi \text{ is the ordinal corresponding to the least } (\varphi, \overline{p}) \text{ such that } x = D(\varphi, \overline{p}, TODO) \end{cases}$$

Lecture 7
Lecture 8
TODO

Cohen:

$\forall T \subseteq$ ZFC finite, $\exists T^* \subseteq$ ZFC finite such that if M is a countable transitive model of T^*, then there is $N \supseteq M$ countable transitive model of $T + \neg$CH. $(*)$

We have seen (Example Sheet 1) that $(*)$ implies Con(ZFC) $\implies$ Con(ZFC $+ \neg$CH).

Simplified: If M is a countable transitive model of ZFC, then there is $N \supseteq M$ countable transitive model of $\mathsf{ZFC}h = \neg\mathsf{CH}$.

Idea: If M is a countable transitive model of ZFC: $\alpha := \omega_1^M$; $\beta := \omega_2^M$; $f : \beta \to \mathcal{P}(\omega)$ injection. Force N such that $f \in N$ and $M \subseteq N$.

Observe that there is a countable transitive model N such that $f \in N$ and $M \subseteq N$. $M \cup \mathrm{tcl}(\{f\})$ is transitive and countable. Thus LSM gives N transitive countable with $M \cup \mathrm{tcl}(\{f\}) \subseteq N$. Know $N \models \exists g : \beta \to \mathcal{P}(\omega)$ is an injection, but no clue what $\aleph_1$ and $\aleph_2$ in N are.

How do we control what we add?

3 Forcing

Definition (Forcing). $(\mathbb{P}, \leq, \mathbb{1})$ is called a *forcing poset* or *forcing* if it is a partial order and $\mathbb{1} \in \mathbb{P}$ and $\mathbb{1}$ is the largest element. Elements of $\mathbb{P}$ are called *condition*.

$p \leq q$ is interpreted as

"p is stronger than q"
"q is weaker than p"

Note. Unconventional.

Alternative: "Jerusalem convention". Interpret $p \leq q$ as "q is stronger than p".

We are *not* following the Jerusalem convention.

Definition (Compatible). p and q are *compatible* if there is $r \leq p, q$. Otherwise, we say they are *incompatible* (which we write as $p \perp q$).

Definition (Antichain). $A \subseteq P$ is an *antichain* if any two distinct elements of A are incompatible.

Definition (Dense). $D \subseteq \mathbb{P}$ is *dense* if $\forall p \in TODO$.

Definition (Filter). $F \subseteq \mathbb{P}$ is called a *filter* if

(a) $\forall p, q \in F, \exists r \in F, r \leq p, q$

(b) $\forall p \in F, \forall q \in \mathbb{P}, q \geq p \implies q \in F$

If F only has property (a), we call it a *filter base*, and then

$$\{p \in \mathbb{P} : \exists q \in F, q \leq p\}$$

is the *filter generated from F*.

Note. Filters *cannot* contain incompatible elements.

Definition (D-generic). If $\mathcal{D}$ is a family of dense sets, then F is called $\mathcal{D}$*-generic* if F is a filter and $\forall D \in \mathcal{D}, D \cap F \neq \emptyset$.

Theorem 3.1. Assuming that:

- $\mathcal{D}$ is countable

Then there is a $\mathcal{D}$-generic filter.

Proof. Let $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$. Pick $p_0 \in D_0$ arbitrarily and define by recursion p_{i+1} by picking some $q \leq p_i$ with $q \in D_{i+1}$.

Then $\{p_i : i \in \mathbb{N}\}$ is a filter base, so let F be the filter generated by it. Then it is $\mathcal{D}$-generic by construction. □

Main example

Fix any sets X and Y

$$\mathrm{Fn}(X, Y) := \{p : p \text{ is a finite fucntion with } \mathrm{dom}(p) \subseteq X \text{ and } \mathrm{range}(p) \subseteq Y\}.$$

Define $p \leq q \iff p \supseteq q$ and $\mathbb{1} := \emptyset$.

What does $p \perp q$ mean?

$$p \perp q \iff \exists x \in X, x \in \mathrm{dom}(p) \cap \mathrm{dom}(q) \wedge p(x) \neq q(x).$$

Lemma 3.2. Assuming that:

- $F \subseteq \mathbb{P}$ is a filter

Then $\bigcup F$ is a function.

Lemma 3.3. $D_x := \{p \in \mathbb{P} : x \in \mathrm{dom}(p)\}$ is a dense set. If $\mathcal{D} := \{D_x : x \in X\}$, and F is $\mathcal{D}$-generic, then $\mathrm{dom}(\bigcup F) = X$.

Proof. If $x \in X$, find $p \in F \cap D_x$, then $x \in \mathrm{dom}(p)$, TODO □

3.1 Cohen Forcing

Example 1

$\mathbb{C} := \mathrm{Fn}(\omega, 2)$.

If F is $\mathcal{D}$-generic, then $\bigcup F : \omega \to 2$ (by above).

Lemma 3.4. Assuming that:

- Fix $f : \omega \to 2$ and define
$$N_f := \{p \in \mathbb{P} : \exists u, p(u) \neq f(u)\}.$$
- F is $\mathcal{D} \cup \{N_f\}$

Then $\bigcup F \neq f$.

Corollary 3.5. There is no F that is $\mathcal{D} \cup \{N_f \mid f : \omega \to 2\}$-generic.

However: if M is a countable transitive model and you consider

$$\mathcal{N}_f := \{N_f : f \in M\},$$

then by Theorem 3.1, there is a $\mathcal{D} \cup \mathcal{N}_M$-generic. And for this F, TODO

Example 2

$\text{Fn}(X, Y) =: \mathbb{P}$ as before.

$$R_y := \{p \in \mathbb{P} : y \in \text{range}(p)\}$$
$$\mathcal{R} := \{R_y : y \in Y\}$$

Lemma 3.6. Assuming that:

- F is $\mathcal{D} \cup \mathcal{R}$-generic

Then $\bigcup F : X \to Y$ is a surjection.

Corollary 3.7. Assuming that:

- $|Y| > |X|$

Then there is no $\mathcal{D} \cup \mathcal{R}$-generic.

However: if M is a countable transitive model and, e.g. TODO

Example 3

$\text{Fn}(X \times Y, 2)$. Assume X is infinite. Consider

$$E_{y,y'} := \{p \in \mathbb{P} : \exists x \in X, p(x, y) \neq p(x, y')\}.$$

This is dense for $y \neq y'$.

$$\mathcal{E} := \{E_{y,y'} : y \neq y' \in Y\}.$$

Lemma 3.8. Assuming that:

- F is $\mathcal{D} \cup \mathcal{E}$-generic

Then there is an injection from Y into $\mathcal{P}(X)$.

Proof. Fix y and define

$$A_y := \{x \in X : (\bigcup F)(x, y) = 1\}.$$

$E_{y,y'}$ guarantees that $y \mapsto Ay$ is an injection. □

Corollary 3.9. Assuming that:

- $|Y| > |\mathcal{P}(\omega)|$

Then there is no $\mathcal{D} \cup \mathcal{E}$-generic.

However: if M is a countable transitive model of $\mathsf{ZFC} + \mathsf{CH}$, α is countable and so a $\mathcal{D} \cup \mathcal{E}$-generic exists.

Lecture 9

Recap: M a countable transitive model of ZFC (or a sufficiently large finite fragment).

$\mathbb{P} \in M$: dense / filter / $\mathcal{D}$-generic.

Example. $\mathrm{Fn}(\omega, 2)$ produces a new function $f : \omega \to 2$. Cohen forcing.

Example. $\mathrm{Fn}(X, Y)$ produces a surjection $f : X \to Y$. $\mathrm{Fn}(\omega, Y)$ COLAPSE of Y.

Example. $\mathrm{Fn}(X \times Y, 2)$ produces an injection $f : Y \to \mathcal{P}(X)$.

If M is a countable transitive model of ZFC, then $\mathcal{D}_M := \{D \subseteq P \text{ dense} : D \in M\}$ is countable, so by Theorem, we have a $\mathcal{D}_M$-generic.

Definition 3.10 (P-generic over M)**.** We say F is $\mathbb{P}$*-generic over* M if it is $\mathcal{D}_M$-generic. These always exist if M is a countable transitive model.

GOAL: Build an extension $M[G]$ such that $M \subseteq M[G]$, $M[G]$ a countable transitive model of ZFC, $G \in M[G]$ and $M[G]$ is minimal.

Names

Idea: Think of elements of $\mathbb{P}$ as "truth values" for the von Neumann construction.

$$\begin{aligned}\text{Name}_0^{\mathbb{P}} &:= \emptyset \\ \text{Name}_{\alpha+1}^{\mathbb{P}} &:= \{\tau : \tau \subseteq \text{Name}_\alpha \times \mathbb{P}\} \\ \text{Name}_\lambda^{\mathbb{P}} &:= \bigcup_{\alpha<\lambda} \text{Name}_\alpha^{\mathbb{P}}\end{aligned}$$

Then

$$\text{Name}^{\mathbb{P}} := \bigcup_{\alpha\in\text{Ord}} \text{Name}_\alpha^{\mathbb{P}}$$

is the proper class of all names.

Consider: $\mathbb{P} = \{0,1\}$. Then $\text{Name}^{\mathbb{P}} \cong V$.

Since this is a recursive definition using absolute concepts, being a name is absolute for transitive models:

$$\{\tau : M \models \tau \text{ is a } \mathbb{P}\text{-name}\} = \text{Name}^{\mathbb{P}} \cap M.$$

TODO

Examples

$\emptyset \in \text{Name}_1^{\mathbb{P}}$,

$$\tau_p := \{(\emptyset, p)\} \in \text{Name}_2^{\mathbb{P}}$$

"The name for a set that contains $\emptyset$ with value p."

$$\tau_{pq} := \{(\tau_p, q)\}.$$

"The name for whatever τ_p describes with value q".

Interpretation

If $F \subseteq \mathbb{P}$, we interpret a $\mathbb{P}$-name as follows:

$$\text{val}(\tau, F) := \{\text{val}(\sigma, F) : \exists p \in F, (\sigma, p) \in \tau\}.$$

Important: This is a recursive definition.

Thus: the valuation is absolute for transitive models containing τ and F.

Example.

(1) $\emptyset$: clearly $\text{val}(\emptyset, F) = \emptyset$.

(2) τ_p:

$$\text{val}(\tau_p, F) := \begin{cases} \{\emptyset\} & p \in F \\ \emptyset & p \notin F \end{cases}$$

(3) τ_{pq}:

$$\text{val}(\tau_{pq}, F) := \begin{cases} \emptyset & q \notin F \\ \{\{\emptyset\}\} & q \in F \wedge p \in F \\ \{\emptyset\} & q \in F \wedge p \notin F \end{cases}$$

The relationship between p and q affects these possibilities.

Example: if $q \leq p$ and F is a filter, then $\{\emptyset\}$ is impossible.

Example: if $q = \mathbb{1}$ and F is a non-empty filter, then $\emptyset$ is impossible.

Example: if $p \perp q$ and F is a filter, then $\{\{\emptyset\}\}$ is impossible.

Definition 3.11 (Generic extension)**.** The *(generic) extension* for any countable transitive model M and any $F \subseteq \mathbb{P}$ where $\mathbb{P} \in M$ is

$$M[F] := \{\text{val}(\tau, F) : \tau \in \text{Name}^{\mathbb{P}} \cap M\}.$$

Obviously, $M[F]$ is a countable set with $\emptyset \in M[F]$ (Example (1)).

Also, by definition, $M[F]$ is transitive.

Note:

$$M[F] \models \text{Extensionality} + \text{Foundation}.$$

Need to show

(1) $M \subseteq M[F]$.

(2) $F \in M[F]$.

(3) $M[F] \models \mathsf{ZFC}$.

(4) $M[F]$ is minimal.

Canonical Names

Definition 3.12 (Canonical name)**.** Let $x \in M$. Define by recursion the *canonical name for* x by

$$\check{w} := \{(\check{y}, \mathbb{1}) : y \in x\}.$$

Lemma 3.13. $\mathrm{val}(\check{x}, F) = x$ if $\mathbb{1} \in F$.

Proof. Induction. □

Corollary 3.14. $M \subseteq M[F]$ if $\mathbb{1} \in F$.

Alternative construction of canonical names without $\mathbb{1}$ is on Example Sheet 2.

$$\Gamma := \{(\check{p}, p) : p \in \mathbb{P}\}.$$

Lemma 3.15. $\mathrm{val}(\Gamma, F) = F$.

Proof. Calculate:

$$\begin{aligned} \mathrm{val}(\Gamma, F) &= \{\mathrm{val}(\check{p}, F) : p \in F\} \\ &= \{p : p \in F\} && \text{(by previous lemma)} \\ &= F \end{aligned}$$

□

Corollary 3.16. $F \in M[F]$.

Remark. If N is a countable transitive model with $M \subseteq N$ and $F \in N$, then $M[F] \subseteq N$. (by absoluteness of $\mathrm{val}(\tau, F)$).

Warm-up: Suppose $\sigma, \tau \in \mathrm{Name}^{\mathbb{P}}$. Define

$$\mathrm{up}(\sigma, \tau) := \{(\sigma, \mathbb{1}), (\tau, \mathbb{1})\}.$$

Pairing: Then

$$\mathrm{val}(\mathrm{up}(\sigma, \tau), F)) = \{\mathrm{val}(\sigma, F), \mathrm{val}(\tau, F)\}.$$

("up" stands for unordered pair).

Corollary 3.17. $M[F] \models$ Pairing (if $\mathbb{1} \in F$).

Union: If τ is a name, define

$$u_\tau := \{(\sigma', r) : \exists \sigma, p, q, \text{s.t.} (\sigma, p) \in \tau, (\sigma', q) \in \sigma, r \le p, q\}.$$

Claim: $\mathrm{val}(u_\tau, F) = \bigcup \mathrm{val}(\tau, F)$ if F is a filter.

Proof. Suppose $z \in \mathrm{val}(u_\tau, F)$.

So $z = \mathrm{val}(\sigma', F)$ for some $(\sigma', r) \in u_\tau$ with $r \in F$.

So $\exists \sigma, p, q$ with $(\sigma, p) \in \tau$, $(\sigma', q) \in \sigma$, $r \le p, q$.

So $p, q \in F$. Hence $\mathrm{val}(\sigma, F) \in \mathrm{val}(\tau, F)$, $z = \mathrm{val}(\sigma', F) \in \mathrm{val}(\sigma, F)$.

So $z \in \bigcup \mathrm{val}(\tau, F)$.

Conversely, suppose $z \in \bigcup \mathrm{val}(\tau, F)$. Then $\exists y, z \in y \in \mathrm{val}(\tau, F)$ ($z \to (\sigma', q) \in \sigma$ with $q \in F$, $y \to (\sigma, \rho) \in \tau$ with $p \in F$).

Hence since F a filter, find $r \le p, q$ with $r \in F$.

Then $(\sigma', r) \in u_\tau$, so $z \in \mathrm{val}(u_\tau, F)$. □

Lecture 10

TODO

Further Recap

We proved

$$M[G] \models \text{Extensionality} + \text{Foundation} + \text{Pairing} + \text{Union}.$$

Homework was: Think about why power set is not easy.

“Union” proof was:

collect all natural candidates of names for elements and assign the natural values.

Problem: If you try to do this for power set, neither the “natural candidates for names” nor the “natural values” are obvious. It’ll turn out that they *are* obvious in the end, but that requires some assistance.

Note: Separation and Replacement are even worse.

One remaining easy axiom: AC. By well-ordering theorem, AC holds if and only if $\forall x, \exists \alpha, \exists i, i : x \to \alpha$ injection. If $x \in M[F]$, then there is $\sigma \in \mathrm{Name}$ such that $x = \mathrm{val}(\sigma, F)$.

Notation. I write $\mathrm{dom}(\sigma) := \{\tau : \exists p, (\tau, p) \in \sigma\}$.

Consider $\mathrm{dom}(\sigma)$. In M, I have $i : \mathrm{dom}(\sigma) \to \alpha$ for some ordinal α. In $M[F]$, define

$$y \mapsto \min\{i(\tau) : \mathrm{val}(\tau, F) = y, \tau \in \mathrm{dom}(\sigma)\}.$$

Call this i^*. Then $i^* : x \to \alpha$ is an injection. So, AC holds in $M[F]$.

Forcing

Definition (Forcing language)**.** Fix M a countable transitive model of ZFC, $\mathbb{P} \in M$ a forcing poset.
We call the language

$$\mathcal{L}_{\text{forcing}} := \mathcal{L}_{\in} \cup \{\tau : \mathbb{P}\textbf{-names}\}$$

the *forcing language (over M)*.

Definition (Interpretation of forcing language)**.** If G is a $\mathbb{P}$-generic over M and φ is in the forcing language, we say

$$M[G] \models \varphi$$

if and only if

$$M[G], v \models \varphi$$

where $v(\tau) := \mathrm{val}(\tau, G)$.

Definition (Semantic forcing predicate)**.** $p \Vdash_{M,\mathbb{P}} \varphi$: Forall G $\mathbb{P}$-generic over M with $p \in G$, $M[G] \models \varphi$.
"p forces φ"
We often omit $M, \mathbb{P}$.

Two theorems at the heart of forcing:

(1) **Forcing Theorem:** If G is a $\mathbb{P}$-generic over M, then

$$M[G] \models \varphi \iff \exists p \in G, p \Vdash \varphi.$$

(2) **Definability Theorem:** "$p \Vdash \varphi$" is absolutely definable for transitive models containing M.

We are going to do the following:

(a) Define the *syntactic forcing predicate* $\Vdash^*$ that is absolutely definable.

(b) Prove the Forcing Theorem for $\Vdash^*$.

(c) Derive that $\Vdash \iff \Vdash^*$.

Lectures 11 and 12.

Observations (under the assumption of the Forcing Theorem):

(1) If $q \le p$ and $p \Vdash \varphi$, then $q \Vdash \varphi$.

(2) If $p \Vdash \exists x, \varphi$, then there is a name τ and $q \le p$ such that $q \Vdash \varphi(\tau)$.

[if $p \Vdash \exists x, \varphi(x)$ and $p \in G$, then by definition $M[G] \models \exists x, \varphi(x)$, so there is a name τ such that $M[G] \models \varphi(\tau)$. By Forcing Theorem, find $r \in G$ such that $r \Vdash \varphi(\tau)$. Find $q \le p, r$, $q \in G$. By (1), $q \Vdash \varphi(\tau)$.]

Proof of Separation in $M[G]$. Let $\varphi(x, x_1, \dots, x_n)$ be an $\mathcal{L}_\in$-formula (not in the forcing language) and $x \in M[G]$. Need a name for

$$A := \{z \in x : M[G] \models \varphi(z, \overline{p})\}.$$

[For readability, we now drop parameters $\overline{p}$].

Fix σ such that $\mathrm{val}(\sigma, G) = x$. Define

$$\varrho := \{(\pi, p) : \pi \in \mathrm{dom}(\sigma) \wedge p \Vdash \pi \in \sigma \wedge \varphi(\pi)\}.$$

By the Definability Theorem, ϱ is a name in M.

Claim: $\mathrm{val}(\varrho, G) = A$.

"$\subseteq$" If $z \in \mathrm{val}(\varrho, G)$, then there is $(\pi, p) \in \varrho$ such that $z = \mathrm{val}(\pi, G)$, $p \in G$. Since $(\pi, p) \in \varrho$, we get $\pi \in \mathrm{dom}(\sigma)$, $p \Vdash \pi \in \sigma \wedge \varphi(\pi)$. Together with $p \in G$, we get

$$M[G] \models \underbrace{\pi \in \sigma \wedge \varphi(\pi)}_{z \in x \wedge \varphi(z)}.$$

Hence $z \in A$.

"$\supseteq$" If $z \in A$, then $z \in x$ and $M[G] \models \varphi(z)$. So there is $\pi \in \mathrm{dom}(\sigma)$, $z = \mathrm{val}(\pi, G)$.

By the Forcing THeorem, $\exists p \in G$, $p \Vdash \varphi(\pi)$. Also, there is $q \in G$ and $q \Vdash \pi \in \sigma$. Find $r \in G$, $r \le p, q$ such that $r \Vdash \pi \in r \wedge \varphi(\pi)$.

Hence $(\pi, r) \in \varrho$, so $z = \mathrm{val}(\pi, G) \in \mathrm{val}(\varrho, G)$. □

Lecture 11

Proof of power set. Example Sheet 3. □

Proof of Replacement. Since we already have Separation, it's enough to show the following:

if φ is a functional formula and $x \in M[G]$, then there is $R \in M[G]$ such that

$$M[G] \models \forall y \in x, \exists z \in R, \varphi(y, z, \not{p}) \qquad (*)$$

(as before, we suppress parameters for notational ease).

We work in M and identify a name ρ for R. Fix σ such that $x = \text{val}(\sigma, G)$. Find α large enough such that $\text{dom}(\sigma) \subseteq V_\alpha$. Consider $\psi(p, \pi) := \exists \mu, p \Vdash \varphi(\pi, \mu)$ ($p \in \mathbb{P}$, π a name). By Definability Theorem, this is an $\mathcal{L}_\in$ formula.

By Levy Reflection Theorem, find $\theta > \alpha$ such that ψ is absolute between V_θ and $V = M$. Define

$$\rho := \{(\mu, \mathbb{1}) : \mu \in V_\theta\}$$

and $R := \text{val}(\rho, G)$.

Now we verify $(*)$ holds: Fix $y \in x$, $y = \text{val}(\pi, G)$. Since φ is functional, we know that $\varphi(\pi, \mu)$ holds in $M[G]$ for some μ. By Forcing Theorem, there is $p \in G$ such that $p \Vdash \varphi(\pi, \mu)$. So $M \models \psi(p, \pi)$.

By absoluteness, $V_\theta \models \psi(p, \pi)$. This means $\exists \mu \in V_\theta$ such that $p \Vdash \varphi(\pi, \mu)$.

Thus: $\text{val}(\mu, G) \in \text{val}(\rho, G) = R$. □

Definition (Dense below p)**.** $D \subseteq \mathbb{P}$ is called *dense below* p if $\forall q \le p, \exists r \le q, r \in D$.

Lemma 3.18. Assuming that:

- G is $\mathbb{P}$-generic over M
- $E \subseteq \mathbb{P}$
- $E \in M$

Then

(i) If E is dense below p, $q \le p$, then E is dense below q.

(ii) If $\{r : E \text{ is dense below } r\}$ is dense below p, then E is dense below p

(iii) Either $G \cap E \neq \emptyset$ or $\exists q \in G, \forall r \in E, r \perp q$.

(iv) If $p \in G$, E is dense below p, then $G \cap E \neq \emptyset$.

Proof. Example Sheet 3 □

Definition of the syntactic forcing relation

$p \Vdash^* \varphi(\overline{\tau})$.

Two recursions:

First "$\Vdash^* =$" by recursion on Name_α.

Then "$\Vdash^* \in$" without recursion.

Then the rest by recursion on formula complexity.

$p \Vdash^* \tau_0 = \tau_1$: if and only if

$$\begin{aligned}&\forall(\pi_0, s_0) \in \tau_0\\ &\qquad \{q \le p : q \le s_0 \to \exists(\pi_1, s_1) \in \tau_1, (q \le s_1 \wedge q \Vdash^* \pi_0 = \pi_1)\}\\ &\qquad \text{is dense below } p\end{aligned}$$

and

$$\begin{aligned}&\forall(\pi_1, s_1) \in \tau_1\\ &\qquad \{q \le p : q \le s_1 \to \exists(\pi_0, s_0) \in \tau_0, (q \le s_0 \wedge q \Vdash^* \pi_0 = \pi_1)\}\\ &\qquad \text{is dense below } p\end{aligned}$$

Remark. This is a recursion on Name_α.

$p \Vdash^* \tau_0 \in \tau_1$: if and only if

$$\{q \le p : \exists(\pi, s) \in \tau_1, (q \le s \wedge q \Vdash^* \pi = \tau_0)\}$$

is dense below p.

Remark. No recursion involved, just "$\Vdash^* =$".

Recursion on complexity of formulas:

- $p \Vdash^* \varphi \wedge \psi$: if and only if $p \Vdash^* \varphi$ and $p \Vdash^* \psi$.
- $p \Vdash^* \neg\varphi$: if and only if $\forall q \le p$, $q \nVdash^* \varphi$.
- $p \Vdash^* \exists x, \varphi(x)$: if and only if $\{r : \exists \sigma, r \Vdash^* \varphi(\sigma)\}$.

Remark. These definitions remind us of Kripke semantics for intuitionistic logic.

Lemma 3.19. The following are equivalent:

(i) $p \Vdash {}^*\varphi$.

(ii) $\forall r \leq p, r \Vdash {}^*\varphi$.

(iii) $\{r : r \Vdash {}^*\varphi\}$ is dense below ρ.

Proof. If this is true for φ of the form $\tau_0 = \tau_1$ and of the form $\tau_0 \in \tau_1$, then it's true for all formulas.

For φ atomic, we get that (ii) $\implies$ (i) and (ii) $\implies$ (iii) are trivial.

(i) $\implies$ (ii) follows from Lemma 3.18(i).

(iii) $\implies$ (i) follows from Lemma 3.18(ii). □

Strategy:

Theorem 3.20 (Syntactic Forcing Theorem)**.** $M[G] \models \varphi$ if and only if $\exists p \in G, M \models r \Vdash {}^*\varphi$.

Corollary 3.21 (Definability Theorem)**.** $p \Vdash \varphi$ if and only if $p \Vdash {}^*\varphi$.

($p \Vdash_M \varphi \iff M \models p \Vdash {}^*\varphi$)

Corollary 3.22 (Forcing Theorem)**.** $M[G] \models \varphi$ if and only if $\exists p \in G, p \Vdash \varphi$.

This corollary is immediate from combining the two previously stated results.

Proof of Definability Theorem from Syntactic Forcing Theorem.

$\Leftarrow$ This is just Syntactic Forcing Theorem.

$\Rightarrow$ Suppose $p \Vdash \varphi$. By Lemma 3.18, we need to show

$$\{r \leq p : r \Vdash {}^*\varphi\}$$

is dense below p.
Suppose not. Then I find $q \leq p$ such that $\forall r \leq q$, $r \nVdash {}^*\varphi$. So $q \Vdash {}^*\neg\varphi$.
If $q \in G$, then $p \in G$. Since $p \Vdash \varphi$, $M[G] \models \varphi$. Since $q \Vdash {}^*\neg\varphi$, $M[G] \models \neg\varphi$. Contradiction! □

Proof of the Syntactic Forcing Theorem. The Theorem for $=$ can be proved by induction on Name_α.

The Theorem for $\in$ can be proved once the Theorem has been established for $=$.

Rest is induction on formula complexity.

Let's do $\neg$ and leave the rest to Example Sheet 3. Assume we have proved it for φ. We will prove it for $\neg\varphi$.

- $\Rightarrow$ $M[G] \Vdash \neg\varphi$. Consider
$$D := \{p : p \Vdash^* \varphi \text{ or } p \Vdash^* \neg\varphi\}.$$
By definition of $\Vdash^* \neg$, this is dense. Find $p \in G \cap D$.
By assumption, $p \nVdash^* \varphi$, so $p \Vdash^* \neg\varphi$.
- $\Leftarrow$ Let $p \in G$ such that $p \Vdash^* \neg\varphi$. By definition, $\forall q \le p$, $q \nVdash^* \varphi$. If $M[G] \not\models \neg\varphi$, thus $M[G] \models \varphi$. By induction hypothesis, find $r \in G$, $r \Vdash^* \varphi$. So $q \le r, p, q \in G$.
By Lemma, $q \Vdash^* \varphi$, contradiction to $\forall q \le p, q \nVdash^* \varphi$.

□

Lecture 12

> **Note.** The Forcing Theorem is very useful! The inner details of the proof are not very important though. We won't really discuss these details once we have proved the theorem.

Continuing the proof of Syntactic Forcing Theorem. Assume Syntactic Forcing Theorem for "=" and prove if for "∈".

Want to show:

$p \Vdash^* \tau_0 \in \tau_1$ if and only if
$$\{q : \exists (\pi, s) \in \tau_1, (q \le s \wedge q \Vdash^* \pi = \tau_0)\}$$
is dense below p.

Proof.

- $\Rightarrow$ Assume that $M[G] \models \tau_0 \in \tau_1$, i.e. $\mathrm{val}(\tau_0, G) \in \mathrm{val}(\tau_1, G)$. Thus, there is $(\pi, s) \in \tau_1$ such that $\mathrm{val}(\pi, G) = \mathrm{val}(\tau_0, G)$ ($\iff M[G] \models \pi = \tau_0$) and $s \in G$. So by Syntactic Forcing Theorem for =, find $r \in G$ such that $r \Vdash^* \pi = \tau_0$.
Find $p \le r, s$ such that $p \in G$.
Claim that $\mathcal{D}$ is dense below p. Let $q \in D$ and pick the above (π, s). Then we have $q \le p \le s$ and $q \Vdash^* \pi = \tau_0$ (since $q \le p \le r$).
- $\Leftarrow$ Assume $p \in G$, $p \Vdash^* \tau_0 \in \tau_1$, i.e. D is dense below p. By Lemma 3.18(iv), find $q \in G \cap D$, thus there is $(\pi, s) \in \tau_1$ such that $q \le s$, $q \Vdash^* \pi = \tau_0$. Since $q \in G$, $s \in G$, we have $\mathrm{val}(\pi, G) \in \mathrm{val}(\tau_1, G)$. By Syntactic Forcing Theorem for "=", we have $\mathrm{val}(\pi, G) = \mathrm{val}(\tau_0, G)$. So $\mathrm{val}(\tau_0, G) \in \mathrm{val}(\tau_1, G)$.

□

Now we prove it for "=".

We prove this by induction on name rank, so assume that Syntactic Forcing Theorem for "=" is true for names π_0, π_1 with smaller name rank.

Reminder: we need to show $M[G] \models \tau_0 = \tau_1$ if and only if $\exists p \in G$, $p \Vdash {}^*\tau_0 = \tau_1$.

TODO: double check the below

Proof.

- $\Leftarrow$ Assume $p \in G$ such that $p \Vdash {}^*\tau_0 = \tau_1$. Need to prove $\text{val}(\tau_0, G) = \text{val}(\tau_1, G)$.

  We'll show that the fact that D_0 is dense below p implies $\text{val}(\tau_0, G) \subseteq \text{val}(\tau_1, G)$. The other direction is the same proof but with 0 and 1 flipped.

  Proof of this: Let $x \in \text{val}(\tau_0, G)$, so find $(\pi_0, s_0) \in \tau_0$ such that $s_0 \in G$ and $x = \text{val}(\pi_0, G)$. So, the corresponding D_0 is dense below p. Find $q \leq s_0, p$ such that $q \in G$. Then D_0 is dense below q. So find $r \leq q$ such that $r \in G \cap D_0$. (Note that $r \leq q \leq s_0$, so $r \leq s_0$).

  Since $r \in D_0$ and $r \leq s_0$, find $(\pi_1, s_1) \in \tau_1$ such that $r \leq s_1 \wedge r \Vdash {}^*\pi_0 = \pi_1$. By induction hypothesis, $r \in G$ and $r \Vdash {}^*\pi_0 = \pi_1$, which implies $x = \text{val}(\pi_, v) = \text{val}(\pi_1, G)$ (since $(\pi_1, s_1) \in \tau_1$ and $s_1 \in G$).

- $\Rightarrow$ Assume $\text{val}(\tau_0, G) = \text{val}(\tau_1, G)$. Consider the set

  $$\begin{aligned} D := \{ & r : r \Vdash {}^*\tau_0 = \tau_1 \text{ or} \\ \Phi_r^0 \quad & \exists(\pi_0, s_0) \in \tau_0 (r \leq s_0 \wedge \forall(\pi_1, s_1) \in \tau_1, \forall q((q \leq s_1 \wedge q \Vdash {}^*\pi_0 = \pi_1) \to q \perp r) \\ \Phi_r^1 \quad & \exists(\pi_1, s_1) \in \tau_1 (r \leq s_1 \wedge \forall(\pi_0, s_0) \in \tau_0, \forall q((q \leq s_0 \wedge q \Vdash {}^*\pi_0 = \pi_1) \to q \perp r) \\ \} & \end{aligned}$$

  Claim: D is dense. If $p \Vdash {}^*$, then $p \in D$, so nothing to show. If $p \nVdash {}^*\tau_0 = \tau_1$, then there is some D_0/D_1 that fails to be dense below p.

  We'll show: if D_0 is not dense below p, then we can find $r \leq p$ such that Φ_r^0 holds.

  (Other proof: D_1 not dense below $p \to \Phi_r^1$ is flipping 0 and 1).

  Suppose $p \nVdash {}^*\tau_0 = \tau_1$ and there is $(\pi_0, s_0) \in \tau_0$ such that D_0 is not dense below p. This means: there is $r \leq p$ such that

  $$\forall q \leq r(q \leq s_0 \wedge \forall(\pi_1, s_1) \in \tau_1 \neg(q \leq s_1 \wedge q \Vdash {}^*\pi_0 = \pi_1).$$

  So, we get

  $$r \leq s_0 \wedge \forall(\pi_1, s_1) \in \tau_1$$

  for any q that satisfies $q \leq s_1 \wedge q \Vdash {}^*\pi_0 = \pi_1$. It can't be compatible with r (finishes the proof of the claim that D is dense).

  Summary: We now have that D is dense.

  $$D = \{r : r \Vdash {}^*\tau_0 = \tau_1 \text{ or } \Phi_r^0 \text{ or } \Phi_1^r\}.$$

Claim 2: If $r \in G$, then neither Φ_r^0 nor Φ_r^1 holds (again, just do Φ_r^0 and then flip 0 and 1 for Φ_r^1).

If Φ_r^0 holds, then find $(\pi_0, s_0) \in \tau_0$ such that $r \leq s_0$ and the incompatibility statement holds. $r \in G \to s_0 \in G$, so $\mathrm{val}(\pi_0, G) \in \mathrm{val}(\tau_0, G)$. If $(\pi_1, s_1) \in \tau_1$ such that TODO

Put everything together: since D is dense by Claim 1, find $r \in D \cap G$. Therefore, by Claim 2, Φ_r^0, Φ_r^1 do not hold.

Thus $r \Vdash {}^*\tau_0 = \tau_1$. □

Lecture 13

Summary: If G is $\mathrm{Fn}(\omega \times \aleph_2^M, 2)$-generic, then $M[G] \models \mathsf{ZFC}$+there is an injection from $\aleph_2^M$ into $\mathcal{P}(\omega)$.

This implies $M[G] \models \mathsf{ZFC} + 2^{\aleph_0} \geq \aleph_2$ if we have $\aleph_1^M = \aleph^{M[G]}$, $\aleph_2^M = \aleph_2^{M[G]}$. This is the goal for today's lecture.

Note that our proof above is *not good enough* for the statement $(*)$ from Lecture 8:

> For all $T \subseteq \mathsf{ZFC}$ finite, there exists $T^* \subseteq \mathsf{ZFC}$ finite such that if M is a countable transitive model of T^*, then there is $N \supseteq M$ countable transitive model of $T^* + \neg\mathsf{CH}$.

For this, we need to look more carefully at the proof of the GMT:

$$M \models \mathsf{ZFC} \implies M[G] \models \mathsf{ZFC}.$$

The proof proceeds AXIOM BY AXIOM and thus for each $\varphi \in \mathsf{ZFC}$, we find finite S_φ such that $M \models S_\varphi$ implies $M[G] \models \varphi$.

Let $S \subseteq \mathsf{ZFC}$ finite such that S proves that all relevant notions (name, value, ...) are well-defined and absolute. Then for $T \subseteq \mathsf{ZFC}$ finite, define

$$T^* := S \cup \bigcup_{\varphi \in T} S_\varphi.$$

Then, the proof shows $(*)$.

Let's prove $\neg\mathsf{CH}$ in $M[G]$.

Definition (Preserves cardinals)**.** We say $\mathbb{P}$ *preserves cardinals* if $\forall G$ $\mathbb{P}$-generic over M, "κ is a cardinal" is absolute between M and $M[G]$.

Definition 3.23 (Countable chain condition)**.** We say $\mathbb{P}$ has the *countable chain condition* (c.c.c.) if every antichain (note the anti!) in $\mathbb{P}$ is countable.

Theorem 3.24. Assuming that:

- $\mathbb{P}$ has countable chain condition

Then $\mathbb{P}$ preserves cardinals.

Theorem 3.25. $\mathrm{Fn}(\omega \times \aleph_2^M, 2)$ has the countable chain condition.

Corollary 3.26. Assuming: - G is $\mathrm{Fn}(\omega \times \aleph_2^M, 2)$-generic over M, then

$$M[G] \models \mathsf{ZFC} + 2^{\aleph_0} \geq \aleph_2.$$

Lemma 3.27. Assuming that:

- $M \models \mathbb{P}$ has countable chain condition
- $X, Y \in M$
- G is $\mathbb{P}$-generic over M
- $f : X \to Y$, $f \in M[G]$

Then there is $F \in M$ such that $\forall x \in X, F(x) \subseteq Y$, $\forall x \in X, f(x) \in F(x)$ and $M \models \forall x \in X, F(x)$ is countable.

Proof of Theorem 3.24. Suppose $M \models \kappa$is a cardinal, $M[G] \models \kappa$ is not a cardinal, so there is $\lambda < \kappa$ and $f \in M[G]$, $f : \lambda \to \kappa$, f is a surjection. Apply the lemma to get F.

Define $R := \bigcup_{\alpha<\lambda} F(\alpha)$.

Since $\forall x, f(x) \in F(x)$, $R = \kappa$.

But $M \models |R| = \aleph_0 \cdot \lambda = \lambda < \kappa$. But then M thinks that κ is not a cardinal, contradiction. □

Now we prove the lemma:

Proof. Let $F(x) := \{y \in Y : \exists p \in \mathbb{P}, p \Vdash \tau(\check{x}) = \check{y}\}$.

Fix τ a name for f.

By Definability Theorem , $F \in M$.

Then $\forall x \in X, F(x) \subseteq Y$ follows from definition.

$\forall x \in X, f(x) \in F(x)$ follows from Forcing Theorem.

Let's look at $M \models F(x)$ is countable: If $y \in F(x)$, let p_y be such that $p_y \Vdash \tau(\check{x}) = \check{y}$.

If $y \neq y'$, then $p_y \perp p_{y'}$. Thus

$$\{p_y : y \in F(x)\}$$

is an antichain.

By countable chain condition, it's countalbe. So $F(x)$ was countable. □

TODO

Proof of Theorem 3.25. Actually, we prove $\mathrm{Fn}(X, Y)$ has countable chain condition whenever Y is countable.

Δ-systems form Example Sheet 3 Q33 are also called *quasi-disjoint families*.

(A family of finite sets $\mathcal{D}$ is called a Δ-system if there is a finite set R (called the root of the Δ-system) such that for all $D, D' \in \mathcal{D}$, if $D \neq D'$ then $D \cap D' = R$).

Δ-system lemma: Any countable family of finite sets contains an uncountable Δ-system.

Take any $A \subseteq \mathbb{P}$ uncountable and prove that it's not an antichain.

If $p \in A$, then $\mathrm{dom}(p) \subseteq X$ finite. Consider

$$S := \{\mathrm{dom}(p) : p \in A\}.$$

That's an uncountable family of finite sets, so by Δ-system lemma, find Δ-system $\mathcal{D} \subseteq S$ uncountable.

Let $r \subseteq X$ finite be the root of $\mathcal{D}$.

Since Y is countable, there are only countably many functions $q : r \to Y$. Since D is uncountable, by pigeonhole principle, there are p, q such that $\mathrm{dom}(p), \mathrm{dom}(q) \in D$ and $p|_r = q|_r$. But since $\mathrm{dom}(p) \cap \mathrm{dom}(q) = r$, p and q are compatible.

So A is not an antichain. □

We got $M[G] \models 2^{\aleph_0} \geq \aleph_2$.

Next time: What is the size of $2^{\aleph_0}$ in $M[G]$?

Remember: LST Example Sheet 4: $\mathsf{ZFC} \vdash 2^{\aleph_0} \neq \aleph_\omega$.

What if $\aleph_2$ was one of the forbidden values?

Remark. Obtaining $M[G] \models 2^{\aleph_0} = \aleph_2$ cannot be quite as general as this proof: if $M \models 2^{\aleph_0} > \aleph_2$, then this will remain true in $M[G]$.

Remark. If G is $\mathrm{Fn}(\omega \times \aleph_\alpha^M, 2)$-generic over M, then

$$M[G] \models 2^{\aleph_0} \geq \aleph_\alpha.$$

Lecture 14

Previous lecture: Suppose M a countable transitive model of ZFC.

TODO

Question: $(*)$ What are the possible values for $2^{\aleph_0}$?

Mentioned last lecture: not all values are possible. In particular, $2^{\aleph_0} \neq \aleph_\omega$.

Definition (Cofinal)**.** $C \subseteq \kappa$ is *cofinal* (= unbounded) if $\forall \lambda < \kappa, \exists \gamma \in C, \gamma \geq \lambda$. We can then define:

$$\mathrm{cf}\,\kappa := \{|C| : C \text{ is cofinal}\}.$$

Example 3.28. $\mathrm{cf}\,\aleph_1 = \aleph_1$, $\mathrm{cf}\,\aleph_\omega = \aleph_0$.

Lemma 3.29 (Kőnig's Lemma)**.** $\kappa^{\mathrm{cf}\,\kappa} > \kappa$.

Then LST Example Sheet 4 Q10 is the special case $\kappa = \aleph_\omega$, $\mathrm{cf}\,\kappa = \aleph_0$ of Kőnig's Lemma.

Consequence for $2^{\mathrm{cf}\,\kappa}$: $(2^{\mathrm{cf}\,\kappa})^{\mathrm{cf}\,\kappa} = 2^{\mathrm{cf}\,\kappa}$, thus $2^{\mathrm{cf}\,\kappa} \neq \kappa$.

Preview of answer to $(*)$: Every value not prohibited by Kőnig's Lemma is possible.

Now: $\aleph_2$!

Definition. If $\mathbb{P}$ is any forcing, let

$$\mathrm{OL} := \{A_n : n \in \omega\}$$

be any ω-sequence of antichains in $\mathbb{P}$.
Let

$$\tau_{\mathrm{OL}} := \{(\check{n}, p) : p \in A_n\}.$$

We call these *nice names*.

If $|\mathbb{P}| = \kappa$ and $\mathbb{P}$ has countable chain condition, then there are at most $\kappa^{\aleph_0}$ many antichains and thus at most $(\kappa^{\aleph_0})^{\aleph_0} = \kappa^{\aleph_0 \cdot \aleph_0} = \kappa^{\aleph_0}$ many ω-sequences of antichains and thus nice names.

Theorem 3.30. Assuming that:

- $M[G] \models x \subseteq \omega$

Then there is a nice name τ such that $\mathrm{val}(\tau, G) = x$.

Note. The Theorem does not need any assumptions about $\mathbb{P}$.

Proof. Start with M such that $\mathrm{val}(\mu, G) = x$ (possibly not nice). Fix $n \in \omega$. Fix either a well-ordering of $\mathbb{P}$ (possibly using AC to get one) and build a maximal antichain A_n such that $\forall p \in A_n$, $p \Vdash \check{n} \in \mu$.

TODO.

Claim: $\mathrm{val}(\mu, G) = \mathrm{val}(\tau_{\mathrm{OL}}, G)$.

$\supseteq$: If $n \in \mathrm{val}(\tau_{OL}, G)$, then there is $p \in G$ such that $(\check{n}, p) \in \tau_{\mathrm{OL}}$, and then $p \in A_n$, so $p \Vdash \check{n} \in \mu$. So $n \in \mathrm{val}(\mu, G)$.

$\subseteq$: If $n \in \mathrm{val}(\mu, G)$, then by Forcing Theorem, get $q \in G$ such that $q \Vdash \check{n} \in \mu$.

Subclaim: $A_n \cap G \neq \emptyset$. Indeed, by our lemma on compatibility (Example Sheet 3), get $q' \in G$ such that $q' \perp p$ for all $p \in A_n$. Find $r \le q, q'$. Then $r \Vdash \check{n} \in \mu$. But that is in contradiction to A_n being maximal.

So find $p \in G \cap A_n$. By definition, $(\check{n}, p) \in \tau_{\mathrm{OL}}$ and $p \in G$. So $n \in \mathrm{val}(\tau_{\mathrm{OL}}, G)$. □

Corollary 3.31. Assuming that:

- $\mathbb{P}$ has countable chain condition
- $M \models |\mathbb{P}| = \kappa \wedge \lambda = \kappa^{\aleph_0}$

Then $M[G] \models 2^{\aleph_0} \le \lambda$.

Proof. Follows directly from:

(a) Theorem.

(b) Calculation of the number of nice names.

□

Main Application

If $\mathbb{P} = \mathrm{Fn}(\omega \times \aleph_2^M, 2)$, then $|\mathbb{P}| = \aleph_2^M$.

Calculate in M, $\aleph_2^{\aleph_0}$.

Hausdorff's Formula:

$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_{\alpha+1} \cdot \aleph_\alpha^{\aleph_\beta}.$$

So in particular:

$$\begin{aligned} \aleph_1^{\aleph_0} &= \aleph_1 \cdot \aleph_0^{\aleph_0} \\ &= 2^{\aleph_0} \\ \aleph_2^{\aleph_0} &= \aleph_2 \cdot \aleph_1^{\aleph_0} \\ &= \aleph_2 \cdot 2 \end{aligned}$$

So $\aleph_2^{\aleph_0} = \max(\aleph_2, 2^{\aleph_0})$.

By this calculation, if $M \models 2^{\aleph_0} \le \aleph_2$, then $M[G] \models 2^{\aleph_0} \le \aleph_2$.

Corollary 3.32. If $M \models \mathsf{CH}$, then $M[G] \models 2^{\aleph_0} = \aleph_2$.

Remark. This proof also shows that if

$$M \models 2^{\aleph_0} \le \aleph_n$$

and G is $\mathbb{P}$-generic over M where $\mathbb{P} = \mathrm{Fn}(\omega \times \aleph_2^M, 2)$, then

$$M[G] \models 2^{\aleph_0} \le \aleph_n.$$

Corollary: If $M \models \mathsf{CH}$, then $M[G] \models 2^{\aleph_0} = \aleph_n$.

Remark. What happens at $\aleph_\omega$?

$$\mathbb{P} := \mathrm{Fn}(\omega \times \aleph_\omega^M, 2).$$

By general theory, $M[G] \models 2^{\aleph_0} \ge \aleph_\omega$, but Kőnig's Lemma gives $2^{\aleph_0} \ge \aleph_{\omega+1}$.
What about the lower bound?
Our theorem and counting of nice names yields

$$M[G] \models 2^{\aleph_0} \le \underbrace{\aleph_\omega^{\aleph_0}}_{>\aleph_\omega}.$$

If $M \models \mathsf{GCH}$, then

$$\aleph_\omega^{\aleph_0} \le \aleph_\omega^{\aleph_\omega} = \aleph_{\omega+1}.$$

So by Kőnig's Lemma, $\aleph_\omega^{\aleph_0} = \aleph_{\omega+1}$.
Therefore, if $M \models \mathsf{GCH}$, then $M[G] \models 2^{\aleph_0} = \aleph_{\omega+1}$.

TODO

First limit cardinal that is a possible value of $2^{\aleph_0}$ is $\aleph_{\omega_1}$.

Clearly, $\mathbb{P} = \mathrm{Fn}(\omega \times \aleph_{\omega_1}^M, 2)$ adds injection from $\aleph_{\omega_1}^M$ into $\mathcal{P}(\omega)$. So $M[G] \models 2^{\aleph_0} \geq \aleph_{\omega_1}$.

Count nice names: $|\mathbb{P}|^{\aleph_0} = \aleph_{\omega_1}^{\aleph_0}$.

If for all $\alpha < \omega_1$, $\aleph_\alpha^{\aleph_0} \leq \aleph_{\omega_1}$ $(*)$, then $\aleph_{\omega_1}^{\aleph_0} = \aleph_{\omega_1}$.

$$\begin{aligned} \aleph_{\omega_1}^{\aleph_0} &= \aleph_{\omega_1} \\ &= \underbrace{\{f \mid f : \omega \to \aleph_{\omega_1}\}}_{=:X} \\ &= \bigcup_{\alpha < \omega_1} \{f \mid f : \omega \to \aleph_\alpha\} \end{aligned}$$

(since ω_1 has no cofinal ω-sequence).

Thus $|X| \leq \aleph_1 \cdot \aleph_{\omega_1} = \aleph_{\omega_1}$ (by assumption $(*)$).

Lecture 15 Thus $M[G] \models 2^{\aleph_0} = \aleph_{\omega_1}$.

TODO

What about $2^{\aleph_1}$?

More on nice names:

Definition (*lambda*-nice names)**.** Generalise "nice names" to λ-*nice names*:
Let

$$\mathrm{OL} = \{A_\alpha : \alpha < \lambda\}$$

be a family of λ many maximal $\mathbb{P}$-antichains:

$$\tau_{\mathrm{OL}} := \{(\check{\alpha}, p) : p \in A_\alpha\}$$

for $\alpha \in \lambda$.
These are names for subsets of λ.

Observe that our theorem "every $A \subseteq \lambda$ in $M[G]$ has a λ-nice name" still goes through.

If κ is an M-cardinal such that every antichain of $\mathbb{P}$ has size $\leq \kappa$. [On Example Sheet 3, this is called the κ^+-chain condition.]

Let $\mu := |\mathbb{P}|$ (in M). Then

$$(\mu^\kappa)^\lambda = \mu^{\kappa \cdot \lambda}$$

is an upper bound on the number of λ-nice names.

Thus

$$M[G] \models 2^\lambda \leq (\mu^{\kappa\cdot\lambda})^M.$$

Question: Forcing with $\mathbb{P} := \mathrm{Fn}(\omega \times \aleph_2, 2)$ and calculate $2^{\aleph_1}$. Assume $M \models \mathsf{GCH}$. Then:

$$\begin{aligned} \mu = |\mathbb{P}| &= \aleph_2 \\ \lambda &= \aleph_1 \\ \lambda &= \aleph_0 \end{aligned}$$

(since $\mathbb{P}$ has countable chain condition). So

$$M[G] \models 2^\lambda \leq \aleph_2^{\aleph_1\cdot\aleph_0} = (\aleph_2^{\aleph_1})^M.$$

Calculate $(\aleph_2^{\aleph_1})^M$:

$$\aleph_2^{\aleph_1} \overset{\text{Hausdorff's formula}}{=} \aleph_2 \cdot 2^{\aleph_1} \overset{\mathsf{GCH}}{=} \aleph_2 \cdot \aleph_2 = \aleph_2.$$

Together:

$$M[G] \models 2^{\aleph_1} = \aleph_2 = 2^{\aleph_0}.$$

Question: Is it possible to get PSA, i.e.

$$\forall \kappa, \lambda, \quad \kappa < \lambda \to 2^\kappa < 2^\lambda$$

without CH.

In particular, can we get

$$\begin{aligned} 2^{\aleph_0} &= \aleph_2 \\ 2^{\aleph_1} &= \aleph_3 \end{aligned}$$

First idea: Force with $\mathrm{Fn}(\aleph_1 \times \aleph_3, 2)$.

(1) Yields $\aleph_3$ many subsets of $\aleph_1$.

(2) Still has countable chain condition, so all cardinals are preserved.

(3) How many $\aleph_1$-nice names are there:

$$\aleph_3^{\aleph_1\cdot\aleph_0} = \aleph_3^{\aleph_1} \overset{\text{Hausdorff}}{=} \aleph_3 \cdot 2^{\aleph_1} \overset{\mathsf{GCH}}{=} \aleph_3.$$

Get: $2^{\aleph_1} = \aleph_3$ in $M[G]$.

Unfortunately,

$$M[G] \models 2^{\aleph_0} = \aleph_3.$$

Interpret the generic object G as $f_\alpha \to \aleph_1 \to 2$ for $\alpha < \aleph_3$.

Define $g_\alpha := f_\alpha|_\omega$.

Claim: For $\alpha \neq \alpha'$, $g_\alpha \neq g'_\alpha$. This is since

$$\mathcal{D}_{\alpha,\alpha'} := \{p : \exists u \in \omega, g_\alpha(u) \neq g_{\alpha'}(u)\}$$

is still dense.

So, forcing with $\mathrm{Fn}(\aleph_1 \times \aleph_3, 2)$ gives the same situation as forcing with $\mathrm{Fn}(\omega \times \aleph_3, 2)$ for $2^{\aleph_0}$, $2^{\aleph_1}$.

Idea:

$$\mathrm{Fn}(X, Y, \kappa) := \{p : \mathrm{dom}(p) \subseteq X, \mathrm{range}(p) \subseteq Y, |p| \leq \kappa\}.$$

Thus $\mathrm{Fn}(X, Y) = \mathrm{Fn}(X, Y, \aleph_0)$.

Consider

$$\mathbb{P} := \mathrm{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1).$$

Properties:

(1) We still have $M[G] \models 2^{\aleph_1} \geq \aleph_3^M$.

(2) Not clear that this forcing is preserving cardinals!

First goal: *What about preserving cardinals?*

Clearly, $\mathrm{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1)$ does not have the countable chain condition anymore.

With example (38) (on Example Sheet 3), we need to figure out the chain condition of $\mathrm{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1)$.

We need a Δ-system lemma for this: If λ is regular ($\mathrm{cf}\,\lambda = \lambda$) and OL is a family of sets of size $< \lambda$ of size λ^+. Then there is a Δ-system $\mathcal{D} \subseteq$ OL of size λ^+.

This Δ-system lemma gives with the same proof as before:

$\mathrm{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1)$ has the $\aleph_2^M$-chain condition.

So: $\mathrm{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1)$ preserves cardinals $\geq \aleph_2^M$.

Closure

Definition (*lambda*-closed)**.** A forcing $\mathbb{P}$ is called λ-*closed* if any family $\{p_\alpha : \alpha < \gamma\}$ for $\gamma < \lambda$ that is a descending chain:

$$\alpha < \beta \implies p_\beta < p_\alpha$$

there is q such that $q \leq p_\alpha$ for all $\alpha < \gamma$.

Example. $\text{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1)$ is $\aleph_1$-cloesd.
[If $\{p_\alpha\}$ is a descending chain, then $\bigcup_{\alpha<\gamma} p_\alpha$ is a condition in $\text{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1)$.]

Theorem 3.33. Assuming that:

- $\mathbb{P}$ is λ-closed and $\kappa < \lambda$

Then

$$\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap M[G].$$

Corollary 3.34. Forcing with $\text{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1)$

(a) Does not change $\mathcal{P}(\omega)$.

(b) Therefore preserves $\aleph_1$ (see Example Sheet 1 and the relation between codes for countable well-orders and preserving $\aleph_1$).

Summary: Forcing with $\text{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1)$ over a model of GCH gives $M[G]$ with:

(1) The same cardinals (cardinals $\geq \aleph_2$ preserved by $\aleph_2$-chain condition; $\aleph_1$ preserved by $\aleph_1$-closure).

(2) $2^{\aleph_1} \geq \aleph_3$ (standard).

(3) $2^{\aleph_0} = \aleph_1$ (by corollary 1 to the closure theorem).

(4) Calculate number of nice names:

$$\aleph_3^{\aleph_1 \cdot \aleph_1} = \aleph_3^{\aleph_1} = \aleph_3 \cdot 2^{\aleph_1} = \aleph_3 \cdot \aleph_2 = \aleph_3.$$

Lecture 16 Hence $2^{\aleph_1} = \aleph_3$.

Forcing	Property	Preservation	Arithmetic
$\text{Fn}(\omega \times \aleph_2, 2)$	countable chain condition	all cardinals	$2^{\aleph_0} = \aleph_2$, $2^{\aleph_1} = \aleph_2$
$\text{Fn}(\omega \times \aleph_3, 2)$	countable chain condition	all cardinals	$2^{\aleph_0} = \aleph_3$, $2^{\aleph_1} = \aleph_3$, $2^{\aleph_2} = \aleph_3$
$\text{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1)$	$\aleph_1$-closed. If $M \models$ CH, then $\aleph_2$-chain condition	$\aleph_1$ preserved. (Closure lemma [not yet proved]) $\kappa \geq \aleph_2$ preserved	If $M \models$ CH, then $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_3$.

Note GCH $\to$ PSA, but our model of $\neg$CH fails PSA.

Question: Can we have $\aleph_1 < 2^{\aleph_0} < 2^{\aleph_1}$?

Closure lemma:

Theorem. If $\mathbb{P}$ is λ-closed and $\kappa < \lambda$, then $\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap M[G]$.

λ-closed: every descending sequence of length $< \lambda$ has a lower bound.

Proof. Let $f \in M[G]$, $f : \kappa \to 2$ and assume towards contradiction that $f \notin M$. $\to f \notin B$.

$$B := \{f \in M \mid f : M \to 2\}.$$

Let τ be a name for f.

By Forcing Theorem, there is $p \in G$ such that $p \Vdash \tau : \check{\kappa} \to \check{2} \wedge \tau \notin \check{B}$.

Construct a κ-sequence of conditions p_α TODO

TODO

The sequence $\{p_\alpha : \alpha \leq \kappa\}$ is defined in M (by Definability Theorem), so we can define

$$g(\alpha) = 1 :\iff p_{\alpha+1} \Vdash \tau(\check{\alpha}) = \check{1}.$$

Then $g \in M$.

But now $p_\kappa \Vdash \tau(\check{\alpha}) = \check{1}$ or $p_\kappa \Vdash \tau(\check{\alpha}) = \check{0}$ for all α.

uso $p_k \Vdash \tau = \check{g}$. Hence $p_\kappa \Vdash \tau \in \check{B}$.

But $p_\kappa \leq p \Vdash \tau \notin \check{B}$. Contradiction. ☐

Note that while $\mathrm{Fn}(X, Y, \aleph_1)$ is *always* $\aleph_1$-closed, the chain condition depended on the value of $\aleph_1^{\aleph_0}$.

The partial order $\mathrm{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1)$ has in general the $(2^{\aleph_0})^+$-chain condition.

CH implies $(2^{\aleph_0})^+ = \aleph_2$, so *all* cardinals are preserved.

However, if $2^{\aleph_0} > \aleph_1$, then there is a gap and we do not know whether TODO.

If $\mathcal{M} \models 2^{\aleph_0} = \aleph_2$, does

$$\mathrm{Fn}(\aleph_1^M \times \aleph_3^M, 2, \aleph_1^M)$$

preserve $\aleph_2^M$?

Answer: $\mathrm{Fn}(\lambda^+ \times \kappa, 2, \lambda^+)$ *always adds* a surjection from λ^+ to $2^\lambda \cap M$. $(*)$

Application: If $\lambda = \aleph_2$ and $M = 2^{\aleph_0} = \aleph_2$, then $\mathrm{Fn}(\aleph_1 \times \kappa, 2, \aleph_1^M$ adds a surjection from $\aleph_1^M$ onto $\mathcal{P}(\aleph_0) \cap M$, i.e. $\aleph_2^M$. So $|\aleph_2^M| = \aleph_1^{M[G]}$.

[Proof of (∗).] A generic for $\mathbb{P}$ "is" a map $f : \lambda^+ \times \kappa \to 2$.

Define $h : \lambda^+ \to 2^\lambda$ by
$$h(\alpha)(\beta) = 1 :\iff f(\alpha, \beta) = 1.$$
Claim: h is a surjection onto $2^\lambda \cap M$.

If $g \in M$, $g : \lambda \to 2$, consider
$$\mathcal{D}_g := \{p \mid \exists \alpha < \lambda^+, \forall \beta < \lambda, p(\alpha, \beta) = 1 \iff g(\beta) = 1\}.$$
This is dense, and thus $g \in \text{range}(h)$. □

Back to our question: Can we get $\aleph_1 < 2^{\aleph_0} < 2^{\aleph_1}$?

Start with $M \models \mathsf{GCH}$. Consider
$$\mathbb{P} := \text{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1) \cap M$$
and let G be $\mathbb{P}$-generic over M. Consider
$$\mathbb{Q} := \text{Fn}(\omega \times \aleph_2, 2) \cap M[G],$$
and let H be $\mathbb{Q}$-generic over $M[G]$.

Claim: $M[G][H] \models \aleph_1 < 2^{\aleph_0} < 2^{\aleph_1}$.

($M[G][H]$ is a *forcing iteration*).

(1) $\mathbb{P}$ is cardinal preserving over M since $M \models \mathsf{GCH}$. So $\aleph_n^M = \aleph_n^{M[G]}$.

(2) $\mathbb{Q}$ has countable chain condition, so is cardinal preserving: $\implies$
$$\aleph_n^{M[G][H]} = \aleph_n^{M[G]} = \aleph_n^M.$$

(3) $M[G] \models 2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_3$ (lecture 15; since $M \models \mathsf{CH}$).

(4) Then $M[G][H] \models 2^{\aleph_0} = \aleph_2 \wedge 2^{\aleph_1} \geq \aleph_3$ (using a nice name analysis, we could calculate $2^{\aleph_1} = \aleph_3$).

This proves the claim.

Important: The order of forcings matters!

Suppose $M \models \mathsf{GCH}$.
$$\mathbb{Q}' := \text{Fn}(\omega \times \aleph_2, 2) \cap M.$$
H $\mathbb{Q}'$-generic over M.
$$\mathbb{P}' := \text{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1) \cap M[H].$$
G $\mathbb{P}'$-generic over $M[H]$.

Consider $M[H][G]$. Then
$$M[H] \models 2^{\aleph_0} = \aleph_2 = 2^{\aleph_1}.$$
But that means that forcing with $\mathbb{P}'$ *will* collapse $\aleph_2^M = \aleph_2^{M[H]}$.

Since $\mathbb{P}'$ is $\aleph_1$-closed,
$$\mathcal{P}(\omega) \cap M[H] = \mathcal{P}(\omega) \cap M[H][G].$$
In particular, $M[H][G] \models \mathsf{CH}$. So, this order does not achieve what we want.

Final Remark on Forcing CH

Assume $M \models 2^{\aleph_0} = \aleph_2$.

Question: Can you obtain $M[G] \models \mathsf{CH}$?

The natural forcing would be
$$\mathbb{P} := \mathrm{Fn}(\omega, \aleph_1^M).$$
This collapses $\aleph_1^M$; it does not have the countable chain condition, but since it has size $\aleph_1^M$, it has the $\aleph_2^M$-chain condition, so all cardinals $\geq \aleph_2^M$ are preserved.

Clearly therefore:
$$M[G] \models |\mathcal{P}(\omega) \cap M| = \aleph_2^M = \aleph_1^{M[G]}.$$
But: is $|\mathcal{P}(\omega) \cap M| = |\mathcal{P}(\omega) \cap M[G]|$?

Nice names: gives upper bound of
$$(\aleph_1^M)^{\aleph_1^M \cdot \aleph_0} = (2^{\aleph_1^M})^M.$$
That's not surprising, since any $A \subseteq \aleph_1$ in M becomes a new subset of ω in $M[G]$ via the new bijection between ω and $\aleph_1^M$.

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