Spectral Graph Theory

Lectured by Victor Souza

May 30, 2025

Contents

1 Setting **2**

2 Graphs and some of their matrices **7**

2.1 Irregular graphs 9

3 Expansion and Cheeger inequality **11**

4 Loewner order **17**

4.1 One-sided expanders 25

Index **43**

Lecture 1

"linear algebraic methods in combinatorics"

1 Setting

Ω a set (finite), $f : \Omega \to \mathbb{C}$, $l^2(\Omega) := \left\{ f : \Omega \to \mathbb{C} \mid \sum_x |f(x)|^2 < \infty \right\}$.

This generalises subsets: given $S \subseteq \Omega$, can consider $\mathbb{1}_S : \Omega \to \mathbb{C}$ (where $x \mapsto 1$ for $x \in S$, and $x \mapsto 0$ otherwise).

When S contains only a single element, we may use the shorthand $\mathbb{1}_x = \mathbb{1}_{\{x\}}$.

$l^2(\Omega)$ is a $\mathbb{C}$-vector space, equipped with the inner product $\langle \bullet, \bullet \rangle : l^2(\Omega) \times l^2(\Omega) \to \mathbb{C}$ defined by

$$\langle f, g \rangle := \sum_{x \in \Omega} \overline{f(x)} g(x),$$

and a norm $\|f\|_2^2 := \langle f, f \rangle = \sum_x |f(x)|^2$.

Matrix over Ω: $M : \Omega \times \Omega \to \mathbb{C}$. $M(x,y) = M_{x,y}$ is the x, y entry. M acts on $l^2(\Omega)$: for $f \in l^2(\Omega)$, $Mf \in l^2(\Omega)$ is given by $(Mf)(x) = \sum_y M(x,y) f(y)$. $M(\alpha f + \beta g) = \alpha M f + \beta M g$ (M is a linear map).

Given M, N, can calculate

$$(MNf)(X) = \sum_y \sum_z M(x,z) N(z,y) f(y),$$

so define $(MN)(x,y) = \sum_z M(x,z) N(z,y)$, so that the formula $(MNf)(x) = \sum_y (MN)(x,y) f(y)$ holds.

Eigenthings: for $M : \Omega \times \Omega \to \mathbb{C}$, we say that $\lambda \in \mathbb{C}$ is an eigenvalue with eigenfunction $\varphi \neq 0$ if $M\varphi = \lambda \varphi$.

Definition 1.1 (Hermitian)**.** M is *Hermitian* if $M = M^H$, where $M^H(x,y) = \overline{M(y,x)}$.
If M is Hermitian, then $\langle Mf, g \rangle = \langle f, Mg \rangle$.

Theorem 1.2 (Spectral theorem for Hermitian matrices)**.** Assuming that:

- Ω finite
- $M : \Omega \times \Omega \to \mathbb{C}$ Hermitian
- $|\Omega| = n$

Then there exist $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $\varphi_1, \dots, \varphi)n \in l^2(\Omega)$ non-zero such that

(1) $M\varphi_i = \lambda_i \varphi_i$

(2) $\langle \varphi_i, \varphi_j \rangle \mathbb{1}_{i=j}$

(3) $M = \sum_{i=1}^n \lambda \varphi_i \varphi_i^H$

(4) there exists U orthogonal such that $UMU^H = \operatorname{diag}(\lambda_i)$

(5) if M is real, then can take φ to be real ($\varphi : \Omega \to \mathbb{R}$)

Lemma 1.3. Any M has an eigenpair (λ, φ).

Proof. Want $Mf = zf$ for some $z \in \mathbb{C}$. So want $(zI - M)f = 0$ to have a non-trivial solution $f \neq 0$. This happens if and only if $zI - M$ is singular, which happens if and only if $\det(zI - M) = 0$.

$z \mapsto \det(zI - M)$ is a degree n polynomial in $\mathbb{C}$ (degree n since the leading term is z^n), so the fundamental theorem of algebra shows that there exists $\lambda \in \mathbb{C}$ such that $\det(\lambda I - M) = 0$. □

Lemma 1.4. Assuming that:

- M is Hermitian

Then all eigenvalues are real

Proof.

$$\begin{aligned}
\overline{\lambda}\langle \varphi, \varphi \rangle &= \langle \lambda\varphi, \varphi \rangle \\
&= \langle M\varphi, \varphi \rangle \\
&= \langle \varphi, M\varphi \rangle \\
&= \langle \varphi, \lambda\varphi \rangle \\
&= \lambda\langle \varphi, \varphi \rangle
\end{aligned}$$

Since $\varphi \neq 0$, $\langle \varphi, \varphi \rangle = \|\varphi\|^2 > 0$, hence $\lambda = \overline{\lambda}$, i.e. $\lambda \in \mathbb{R}$. □

Lemma 1.5. Assuming that:

- M Hermitian
- $\lambda_i \neq \lambda_j$ are eigenvalues of M with eigenvectors φ_i, φ_j

Then $\langle \varphi_i, \varphi_j \rangle = 0$.

Proof.

$$\begin{aligned}
\lambda_i\langle \varphi_i, \varphi_j \rangle &= \langle M\varphi_i, \varphi_j \rangle \\
&= \langle \varphi_i, M\varphi_j \rangle \\
&= \lambda_j\langle \varphi_i, \varphi_j \rangle
\end{aligned}$$

Since we assumed $\lambda_i \neq \lambda_j$, this gives $\langle \varphi_i, \varphi_j \rangle = 0$. □

Lemma 1.6. Assuming: - M is real symmetric - λ is an eigenvalue Then: there exists $g : \Omega \to \mathbb{R}$ such that $Mg = \lambda g$.

Proof. Let $\varphi = f + ig$. Then $M\varphi = Mf + iMg = \lambda\varphi = \lambda f + i\lambda g$. Hence $Mf = M\lambda$ and $Mg = \lambda g$. So either f or g works. □

Notation. For $f, g \in l^2(\Omega)$, fg^H denotes the matrix $(fg^H)(x, y) = f(x)\overline{g}(y)$.
"$(fg^H)h = fg^H h = f\langle g, h\rangle$"

Proof of Spectral theorem for Hermitian matrices. Using the above lemmas, can find $\lambda_n \in \mathbb{R}$ and $\varphi_i : \Omega \to \mathbb{C}$ non-zero such that $M\varphi_n = \lambda_n\varphi_n$ and $\|\varphi_n\| = 1$.

Then let $M' = M - \lambda_n\varphi_n\varphi_n^H$. $l^2(\Omega) = \operatorname{span}(\varphi_n) \oplus \operatorname{span}(\varphi_n)^\top$. Then check that M' acts on $\operatorname{span}(\varphi_n)^\top$ and use induction. □

Theorem 1.7 (Courant-Fischer-Weyl Theorem)**.** Assuming: - $M : \Omega \times \Omega \to \mathbb{R}$ symmetric - eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$ Then:

$$\lambda_k = \min_{\substack{W \leq l^2(\Omega) \\ \dim W = k}} \max_{\substack{f \in W \\ f \neq 0}} \frac{\langle f, Mf\rangle}{\langle f, f\rangle} = F_k.$$

Definition 1.8 (Rayleigh quotient)**.** $\frac{\langle f, Mf\rangle}{\langle f, f\rangle}$ is called the *Rayleigh quotient*.

Proof. Let $W' = \operatorname{span}(\varphi_1, \ldots, \varphi_k)$. Note

$$F_k \leq \max_{\substack{f \in W \\ f \neq 0}} \frac{\langle f, Mf\rangle}{\langle f, f\rangle}.$$

For $f \in W$, $f = \sum_{i=1}^k \alpha_i\varphi_i$, so

$$\frac{\langle f, Mf\rangle}{\langle f, f\rangle} = \frac{\sum_{i=1}^k \alpha_i^2\lambda_i}{\sum_{i=1}^k \alpha_i^2} \leq \frac{\lambda_k\sum_{i=1}^k \alpha_i^2}{\sum_{i=1}^k \alpha_i^2} = \lambda_k.$$

So $F_k \leq \lambda_k$.

Now suppose W is a subspace with $\dim W = k$. Let $V = \mathrm{span}(\varphi_k, \dots, \varphi_n)$, and note $\div V = n - k + 1$. Note

$$\dim(V \cap W) = \dim V + \dim W - \dim(V + W) \geq k + (n - k + 1) - n = 1.$$

So for all such W, there exists $f \in V \cap W$, $f \neq 0$ such that $f = \sum_{i \geq k} \alpha_i \varphi_i$. Then

$$\langle f, Mf \rangle = \sum_{i \geq k} \alpha_i^2 \lambda_i \geq \lambda_k \sum_{i \geq k} \alpha_i^2 = \lambda_k \langle f, f \rangle,$$

so $F_k \geq \lambda_k$. □

Lecture 2

Notation 1.9. Define $Q_M : l^2(\Omega) \to \mathbb{C}$ by $Q_M(f) = \langle f, Mf \rangle = \sum_{x,y} f(x) M(x,y) f(y)$. Define $q_m(f) = \frac{Q_M(f)}{Q_I(f)}$.

$\lambda_1(M) = \min_{f \neq 0} q_M(f)$ and it is attained only on

Lemma 1.10. eigenfunctions of λ_1.

Proof. Let $f = \sum_i \alpha_i \varphi_i$ and $Mf = \sum_i \alpha_i \lambda_i \varphi_i$. Then

$$Q_M(f) = \langle f, Mf \rangle$$
$$\sum_{i,j} \alpha_i \alpha_j \lambda_i \langle \varphi_i, \varphi_j \rangle$$
$$= \sum_i \alpha_i^2 \lambda_i$$

and

$$q_M(f) = \frac{\sum_i \alpha_i^2 \lambda_i}{\sum_i \alpha_i^2} \geq \lambda_i.$$

Equality occurs here if and only if $\sum_i (\lambda_1 - \lambda_i) \alpha_i^2 = 0$. So $\alpha_i = 0$ whenever $\lambda_i > \lambda_1$. □

Assuming:

Lemma 1.11. - φ_1 is an eigenfunction of λ_1. Then: $\lambda_2(M) = \min_{\substack{f \perp \varphi_1 \\ f \neq 0}} q_M(f)$, and it is attained only on eigenfunctions of $\lambda_2(M)$.

Proof.

$$q_M(f) = \frac{\sum_i \alpha_i^2 \lambda_i}{\sum_i \alpha_i^2} \geq \frac{\alpha_1^2 \lambda_1 + \left(\sum_{i \geq 2} \alpha_i^2 \right) \lambda_2}{\sum_i \alpha_i^2}.$$

So

$$\min_{\substack{f \perp \varphi_i \\ f \neq 0}} q_M(f) \geq \lambda_2.$$

Deal with the equality case similarly to before. □

In general:

$$\lambda_k(M) = \min_{\substack{f \perp \varphi_1, \dots, \varphi_{k-1} \\ f \neq 0}} q_M(f).$$

(and equality case is similar to before). Also,

$$\lambda_n(M) = \max_{f \neq 0} q_M(f)$$
$$\lambda_{n-k}(M) = \max_{\substack{f \perp \varphi_n, \dots, \varphi_{n-k+1} \\ f \neq 0}} q_M(f)$$

(and equality case is similar to before).

2 Graphs and some of their matrices

Graph $G = (V, E)$: set of vertices V, $|V| = n$, E is a set of (unordered) pairs of vertices.

Definition 2.1 (Adjacency matrix)**.** The *adjacency matrix* of a graph G is the matrix $A_G : V \times V \to \mathbb{R}$ defined by

$$A_G(x, y) = \begin{cases} 1 & \{x, y\} \in E(G) \\ 0 & \{x, y\} \notin E(G) \end{cases}$$

Definition 2.2 (Degree matrix)**.** The *degree matrix* of a graph G is the matrix $D_G : V \times V \to \mathbb{R}$ defined by

$$D_G(x, y) = \begin{cases} \deg(x) & x = y \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.3 (Laplacian matrix)**.** The *Laplacian matrix* of a graph G is defined by $L_G = D - A$.

We can now calculate:

$$
\begin{aligned}
Q_A(f) &= \sum_{x,y} f(x)A(x,y)f(y) \\
&= 2\sum_{x\sim y} f(x)f(y) \\
Q_D(f) &= \sum_{x,y} f(x)D(x,y)f(y) \\
&= \sum_{x} f(x)^2 \deg(x) \\
&= \sum_{x} f(x)^2 \sum_{y} A(x,y) \\
&= \sum_{x,y} f(x)^2 A(x,y) \\
&= \frac{1}{2}\sum_{x,y} (f(x)^2 + f(y)^2)A(x,y) \\
&= \sum_{x\sim y} (f(x)^2 + f(y)^2) \\
Q_L(f) &= Q_{D-A}(f) \\
&= Q_D(f) - Q_A(f) \\
&= \sum_{x\sim y} (f(x)^2 + f(y)^2 - 2f(x)f(y)) \\
&= \sum_{x\sim y} (f(x) - f(y))^2 \\
&\geq 0
\end{aligned}
$$

Corollary 2.4. For any graph G, L_G is a positive semi definite matrix with eigenvector 1, which has eigenvalue 1.

Proof. Since $Q_L(f) = \sum_{x\sim y}(f(x) - f(y))^2$, we see that $Q_L(f) \geq 0$, with equality when f is constant. □

Proposition 2.5. $\lambda_2(L_g) > 0$ if and only if G is connected. $\lambda_k(L_G) = 0$ if and only if G has at least k connected components.

Proof.

$$\lambda_2 = \min_{\substack{f\perp 1\\ f\neq 0}} \frac{\sum_{x\sim y}(f(x) - f(y))^2}{\langle f, f\rangle} \geq 0$$

Equality happens if and only if $Q_L(f) = 0$, which happens if and only if f is constant on connected components. The dimension of $\{f : \text{constant on connected components}\}$ is the number of connected components of G.

TODO? $\square$

Lecture 3 TODO

Lecture 4 TODO

2.1 Irregular graphs

A, D, $L = D - A$, $Q_L(f) = \sum_{x\sim y}(f(x) - f(y))^2$, $q_L(f) = \frac{\sum_{x\sim y}(f(x)-f(y))^2}{\sum_x f(x)^2}$.

$$\begin{aligned} Q_L(f) &= \langle f, Lf\rangle \\ &= \langle, (D - A)f\rangle \\ &= \langle f, (2D - (D + A))f\rangle \\ &= \langle f, 2Df\rangle - \langle f, (D + A)f\rangle \end{aligned}$$

When G is d-regular,

$$\tilde{L}_G = \frac{1}{d}L_G = I - \frac{1}{d}A_G,$$

$\lambda_i(\tilde{L}_G \in [0, 2]$.

$$q_{\tilde{L}}(f) = \frac{\sum_{x\sim y}(f(x) - f(y))^2}{\sum_x d(x)f(x)^2} = 2 - \frac{\sum_{x\sim y}(f(x) + f(y))^2}{\sum_x d(x)f(x)^2}.$$

Want M such that $q_M(f)$ equals the expression above. Recall $q_M(f) = \frac{\langle f,Mf\rangle}{\langle f,f\rangle}$. But the above expression is $\frac{\langle f,Mf\rangle}{\langle f,Df\rangle}$.

Let $D^{\frac{1}{2}}(x, y) = \mathbb{1}_{x=y}\sqrt{d(x)}$. Assume $d(x) \geq 1$ for all $x \in G$. Note

$$q_M(D^{\frac{1}{2}}f) = \frac{\langle D^{\frac{1}{2}}f, MD^{\frac{1}{2}}f\rangle}{\langle D^{\frac{1}{2}}f, D^{\frac{1}{2}}f\rangle} = \frac{\langle f, D^{\frac{1}{2}}MD^{\frac{1}{2}}f\rangle}{\langle f, Df\rangle}.$$

Want $D^{\frac{1}{2}}MD^{\frac{1}{2}} = L = D - A$. Define

$$\tilde{L}_G = D^{-\frac{1}{2}}(D - A)D^{-\frac{1}{2}} = I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}}.$$

So

$$\tilde{L}_G = \begin{cases} 1 & \text{if } x = y \\ -\frac{1}{\sqrt{d(x)d(y)}} & \text{if } x \neq y \text{ and } x \sim y \\ 0 & \text{if } x \neq y \text{ and } x \not\sim y \end{cases}$$

Also,

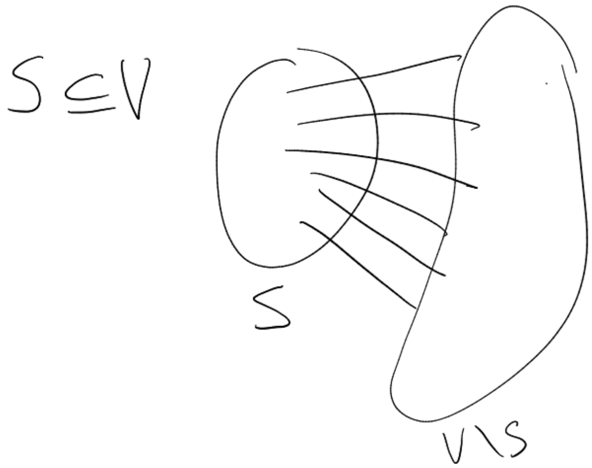
$$q_{\tilde{L}_G}(D^{\frac{1}{2}}f) = \frac{\sum_{x\sim y}(f(x) - y(y))^2}{\sum_x d(x)f(x)^2}.$$

We have

$$\lambda_k(\tilde{L}_G) = \min_{\dim W=K} \max_{\substack{f\in W\\ f\neq 0}} q_{\tilde{L}_G}(D^{\frac{1}{2}}f).$$

3 Expansion and Cheeger inequality

Assume G is d-regular. Write $G = (V, E)$.



Definition 3.1 (Expansion)**.** Given a d-regular graph G and $S \subseteq V$, the *expansion of* S is

$$\Phi(S) := \frac{e(S, V \setminus S)}{d|S|}.$$

Note that $0 \leq \Phi \leq 1$, for example because

$$\Phi(S) = \mathbb{P}_{\substack{x \sim U(S) \\ y \sim N(x)}}(y \notin S).$$

Definition 3.2 (Edge expansion)**.** The *(edge) expansion* of a cut $(S, V \setminus S)$ is defined as

$$\Phi(S, V \setminus S) := \max\{\Phi(S), \Phi(V \setminus S)\} = \frac{e(S, V \setminus S)}{d \min\{|S|, |V \setminus|\}}.$$

Definition 3.3 (Edge expansion of a graph)**.** The *edge expansion of a graph* is

$$\Phi(G) := \min_{\substack{S \subseteq V \\ \emptyset \neq S \neq V}} \Phi(S, V \setminus S) = \min_{\substack{S \subseteq V \\ 0 < |S| < |V|/2}} \Phi(S).$$

Theorem 3.4 (Cheeger's inequality)**.** Assuming that:

- G be d-regular

Then
$$\frac{\lambda_2(\tilde{L}_G)}{2} \leq \Phi(G) \leq \sqrt{2\lambda_2(\tilde{L}_G)}.$$

Consider $\mathbb{1}_S : V \to \mathbb{R}$, where $\mathbb{1}_S(x) = 1$ if $x \in S$ and $\mathbb{1}_S(x) = 0$ otherwise. Then
$$Q_L(\mathbb{1}_S) = \sum_{x \sim y} (\mathbb{1}_S(x) - \mathbb{1}_S(y))^2 = e(S, V \setminus S)$$
$$q_{\tilde{L}}(\mathbb{1}_S) = \frac{e(S, V \setminus S)}{d|S|} = \Phi(S)$$

Recall
$$\lambda_2(\tilde{L}_G) = \min_{\dim W = 2} \max_{\substack{f \in W \\ f \neq 0}} q_{\tilde{L}_G}(f).$$

We pick $W = \operatorname{span}\{\mathbb{1}_S, \mathbb{1}_{V \setminus S}\}$. Note
$$\begin{aligned}\lambda_2(\tilde{L}_G) &\leq \max_{\substack{\alpha, \beta \\ (\alpha,\beta) \neq (0,0)}} q_{\tilde{L}_G}(\alpha \mathbb{1}_S + \beta \mathbb{1}_{V \setminus S}) \\ &\leq \max_{\alpha,\beta} 2 \max\{q_{\tilde{L}(G)}(\alpha \mathbb{1}_S), q_{\tilde{L}_G}(\beta \mathbb{1}_{V \setminus S})\}\end{aligned}$$

Lemma 3.5. Assuming that:

- M is symmetric positive semi-definite
- $\langle f, g \rangle = 0$

Then
$$q_M(f+g) \leq 2 \max\{q_M(f), q_M(g)\}.$$

Proof. Let λ_i, φ_i such that $f = \sum_i \lambda_i \varphi_i$ and $g = \sum_i \beta_i \varphi_i$. Then
$$q_M(f+g) = \frac{\sum_i \lambda_i (\alpha_i + \beta_i)^2}{\|f+g\|^2} \leq \frac{\sum_i \lambda_i (2\alpha_i^2 + 2\beta_i^2)}{\|f\|^2 + \|g\|^2}.$$

Then
$$\begin{aligned}q_M(f+g) &\leq 2\left(\frac{\sum_i \lambda_i \alpha_i^2 + \sum_i \lambda_i \beta_i^2}{\|f\|^2 + \|g\|^2}\right) \\ &= 2\left(\frac{q_M(f)\|f\|^2 + q_M(g)\|g\|^2}{\|f\|^2 + \|g\|^2}\right) \\ &\leq 2 \max\{q_M(f), q_M(g)\}\end{aligned}$$
□

Proof of left inequality in Cheeger's inequality. $\lambda_2 \leq 2\max\{\Phi(S), \Phi(V \setminus S)\} = 2\Phi(S, V \setminus S)$. Then minimise over all S, to get $\lambda_2 \leq 2\Phi(G)$. □

Lecture 5

Recall that

$$\Phi(S) = q_{\tilde{L}_G}(\mathbb{1}_S)$$

and

$$\Phi(G) = \min_{\substack{S \subseteq V \\ 1 \leq |S| \leq |V|/2}} q_{\tilde{L}_G}(\mathbb{1}_S).$$

Fiedler's Algorithm

Input: $G, \varphi : V \to \mathbb{R}$.

- Sort vertices $x_1, \dots, x_n$ such that $\varphi(x_1) \leq \cdots \leq \varphi(x_n)$.
- Find cut K that minimises $\Phi(\{x_2, \dots, x_k\}, \{x_{k+1}, \dots, x_n\})$.

Output: The cut.

Running time: $O(|V| \log |V| + |E|)$.

Lemma 3.6. Assuming that:

- $\psi : V \to \mathbb{R}$
- $\langle \psi, 1 \rangle = 0$
- let $(S, V \setminus S) = \text{Fiedler}(G, \psi)$

Then

$$\Phi(S, V \setminus S) \leq \sqrt{2q_{\tilde{L}_G}(\psi)}.$$

If $\psi : V \to \mathbb{R}$, call a cut $(\{x : \psi(x) \geq \tau)\}, \{x : \psi(x) < \tau\})$ a threshold cut for ψ.

Lemma 3.7. Assuming that:

- $\varphi : V \to \mathbb{R}$
- $\langle \varphi, 1 \rangle = 0$

Then there is $\psi : V \to \mathbb{R}_{\geq 0}$ such that $q_{\tilde{L}_G}(\psi) \leq q_{\tilde{L}_G}(\psi)$, $|\operatorname{supp}\psi| = |\{x : \psi(x) > 0\}| \leq \frac{|V|}{Q}$ and any threshold cut for ψ is a threshold cut for φ.

Lemma 3.8. Assuming that:

- $\psi : V \to \mathbb{R}_{\geq 0}$

Then there is $0 < t \leq \|\psi\|_\infty$ such that

$$\Phi(\{x : \psi(x) \geq t\}) \leq \sqrt{2q_{\tilde{L}_G}(\psi)}.$$

Proof of Lemma 3.7. If $\langle \varphi, 1 \rangle = 0$ then

$$\begin{aligned} q_{\tilde{L}_G}(\varphi + \alpha 1) &= \frac{Q_{\tilde{L}_G}(\varphi + \alpha 1)}{\|\varphi + \alpha 1\|^2} \\ &= \frac{Q_{\tilde{L}_G}(\varphi)}{\|\varphi\|^2 + \alpha^2} \\ &\leq \frac{Q_{\tilde{L}_G}(\varphi)}{\|\varphi\|^2} \\ &= q_{\tilde{L}_G}(\varphi) \end{aligned}$$

Let $m \in \mathbb{R}$ be the median of φ.

$$|\{x \in V : \varphi(x) > m\}| \leq \frac{|V|}{2}$$
$$|\{x \in V : \varphi(x) < m\}| \leq \frac{|V|}{2}$$

$\overline{\varphi} = \varphi - m1$, $q_{\tilde{L}_G}(\overline{\varphi}) \leq q_{\tilde{L}_G}(\varphi)$. Let $\overline{\varphi} = \overline{\varphi}^+ - \overline{\varphi}^-$, where $\overline{\varphi}^+, \overline{\varphi}^- : V \to \mathbb{R}_{\geq 0}$. So

$$\overline{\varphi}(x) = \begin{cases} \overline{\varphi}^+(x) & \varphi(x) > m \\ -\overline{\varphi}^-(x) & \varphi(x) < m \\ 0 & \varphi(x) = m \end{cases}.$$

Note $\langle \overline{\varphi}^-, \overline{\varphi}^+ \rangle = 0$.

Claim: either $\overline{\varphi}^+$ or $\overline{\varphi}^-$ suffices.

$$\begin{aligned} q_{\tilde{L}_G}(\varphi) &\geq q_{\tilde{L}_G}(\overline{\varphi}) \\ &= q_{\tilde{L}_G}(\overline{\varphi}^+ - \overline{\varphi}^-) \\ &= \frac{\sum_{x \sim y} (\overline{\varphi}^+(x) - \overline{\varphi}^-(x) - \overline{\varphi}^+(y) + \overline{\varphi}^-(y))^2}{\|\overline{\varphi}^+ - \overline{\varphi}^-\|^2} \\ &= \frac{\sum_{x \sim y} ((\overline{\varphi}^+(x) - \overline{\varphi}^+(y)) - (\overline{\varphi}^-(x) - \overline{\varphi}^-(x))^2}{\|\overline{\varphi}^+\|^2 + \|\overline{\varphi}^-\|^2} \\ &\geq \frac{\sum_{x \sim y} (\overline{\varphi}^+(x) - \overline{\varphi}^+(y))^2 + (\overline{\varphi}^-(x) - \overline{\varphi}^-(y))^2}{\|\overline{\varphi}^+\|^2 + \|\overline{\varphi}^-\|^2} \\ &= \frac{q_{\tilde{L}_G}(\overline{\varphi}^+)\|\overline{\varphi}\|^2 + q_{\tilde{L}_G}(\overline{\varphi}^-)\|\overline{\varphi}^-\|^2}{\|\overline{\varphi}\|^2 + \|\overline{\varphi}^-\|^2} \end{aligned}$$

□

Proof of Lemma 3.8. Assume $\|\psi\|_\infty = 1$. We find $0 < t \le 1$. Choose t at random, such that $t^2 \sim \mathrm{Unif}([0,1])$. Let

$$S_t = \{x \in V : \psi(x) > t\}.$$

Then

$$\mathbb{E}|S_t| = \sum_x \mathbb{P}(x \in S_t) = \sum_x \mathbb{P}(\psi(x)^2 > t^2) = \sum_x \psi(x)^2.$$

Have

$$\Phi(S_t) = \frac{e(S_t, V \setminus S_t)}{d|S_t|}.$$

Also, can calculate:

$$\begin{aligned}
\mathbb{E}d|S_t| &= d\sum_x \psi(x)^2 \\
\mathbb{E}e(S_t, V \setminus S_t) &= \sum_{x\sim y} \mathbb{P}(xy \text{ is cut by } (S_t, V \setminus S_t)) \\
&= \sum_{x\sim y} |\psi(x)^2 - \psi(y)^2| \\
&= \sum_{x\sim y} |\psi(x) - \psi(y)|(\psi(x) + \psi(y)) \\
&\le \sqrt{\sum_{x\sim y}(\psi(x) - \psi(y))^2} \cdot \sqrt{\sum_{x\sim y}(\psi(x) + \psi(y))^2}
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\mathbb{E}e(S_t, V \setminus S_t)}{\mathbb{E}d|S_t|} &\le \sqrt{q_{\tilde{L}_G}(\psi)} \cdot \sqrt{\frac{\sum_{x\sim y}(\psi(x) + \psi(y))^2}{d\sum_x \psi(x)^2}} \\
&\le \sqrt{2q_{\tilde{L}_G}(\psi)}
\end{aligned}$$

Now use the following fact to finish:

Fact: If X and Y are random variables with $\mathbb{P}(Y > 0) = 1$, then $\mathbb{P}\left(\frac{X}{Y} \le \frac{\mathbb{E}X}{\mathbb{E}Y}\right)$.

(Proof: let $R = \frac{\mathbb{E}X}{\mathbb{E}Y}$. Then $\mathbb{E}(X - RY) = 0$, so $\mathbb{P}(X - RY \le 0)) > 0$, hence $\mathbb{P}(\frac{X}{Y} \le R) > 0$). □

Lecture 6

Example. C_N. This has $\lambda_2(\tilde{L}_{C_N} = \theta\left(\frac{1}{N^2}\right)$.
For $S \subseteq C_N$ with $|1 \le S| \le \frac{1}{2}|C_N|$, we have $e(S, V \setminus S) \ge 2$. So

$$\Phi(S) = \min_{\substack{S\subseteq V \\ 0<|S|<|V|/2}} \frac{e(S, V \setminus S)}{d|S|} = \min_{1\le s\le N/2} \frac{2}{2s} \simeq \frac{2}{N}.$$

Compare with Cheeger's inequality:

$$\frac{1}{N^2} < \frac{1}{N} \leq \sqrt{\frac{1}{N^2}}.$$

Example. $G = Q_n$, $N = 2^n$. $G = \mathrm{Cay}((\mathbb{Z}/2\mathbb{Z})^n, \{e_1, \dots, e_n\})$. We index eigenfunctions by sets $T \subseteq [n]$. $\chi_T(x) = (-1)^{\sum_{i \in T} x_i}$, $\lambda_T = \frac{2|T|}{n}$.
$\lambda_2^(\tilde{L}_{Q_n}) = \frac{2}{n} = \frac{2}{\log N}$.
$\Phi(Q_n) \geq \frac{2}{2n} = \frac{1}{n}$. If $S \subseteq Q_n$, $|S| \leq N/2$, then

$$\frac{e(S, V \setminus S)}{n|S|} \geq \frac{1}{n} \quad \Longrightarrow \quad e(S, V \setminus S) \geq |S|.$$

Harper gives a better bound:

$$e(S, V \setminus S) \geq |S| \log_2 \left(\frac{2^n}{|S|} \right).$$

By considering S being half of the cube, we get

$$\Phi(Q_n) = \frac{1}{n} = \frac{\lambda_2(\tilde{K}_{Q_n})}{2}.$$

Fiedler's algorithm: Let $f = \sum_{i=1}^n \chi_{\{i\}}$, $\tilde{L}_{Q_n} f = \frac{2}{n} f$. $f(x) = \sum_{i=1}^n (-1)^{x_i} = n - 2|x|$.

$$\Phi_k = \frac{\binom{n}{k}(n-k)}{n \sum_{j=0}^k \binom{n}{j}}.$$

$k = \frac{n}{2}$,

$$\frac{\binom{n}{n/2} \frac{n}{2}}{n 2^{n-1}} = \frac{\binom{n}{n/2}}{2^n} \approx \frac{1}{\sqrt{n}}.$$

4 Loewner order

Definition 4.1 (Loewner order)**.** For A, B matrices, write $A \preccurlyeq B$ if $B - A$ is positive semidefinite. In particular, $A \succcurlyeq 0$ if and only if A is positive semidefinite.

$q_A(f) = \frac{\langle f, Af \rangle}{\langle f, f \rangle} = 0$.

$A \preccurlyeq B$ if and only if $\forall f \neq 0$,

$$\frac{\langle f, Af \rangle}{\langle f, f \rangle} \leq \frac{\langle f, Bf \rangle}{\langle f, f \rangle}.$$

$\langle f, Af \rangle \leq \langle f, Bf \rangle$.

This is indeed an order: $A \preccurlyeq B \preccurlyeq C$ implies $A \preccurlyeq C$

$$\langle f, Af \rangle \leq \langle f, Bf \rangle \leq \langle f, Cf \rangle.$$

If $A \preccurlyeq B$, then $\lambda_k(A) \leq \lambda_k(B)$.

$$\lambda_k(A) = \min_{\dim W = k} \max_{\substack{f \in W \\ f \neq 0}} \frac{\langle f, Af \rangle}{\langle f, f \rangle}.$$

$A \preccurlyeq B$ if and only if $A + C \preccurlyeq B + C$.

If G is a graph, then $L_G \preccurlyeq 0$.

Definition 4.2 (eps-approximation)**.** G is an *ε-approximation of H* if

$$(1 - \varepsilon) L_H \preccurlyeq L_G \preccurlyeq (1 + \varepsilon) L_H.$$

Lemma 4.3. Given the definition $\|M\| = \max_{f \neq 0} \frac{\|Mf\|}{\|f\|}$ (for M symmetric), we have $\|M\| = \max\{|\lambda_k(M)|\}$.

Proof. $f = \sum_i \alpha_i \varphi_i$, $M\varphi_i = \lambda_i \varphi_i$, $\|f\|^2 = \sum_i \alpha_i^2$,

$$\|Mf\|^2 = \left\| \sum_i \alpha_i \lambda_i \varphi_i \right\|^2 = \sum_i \alpha_i^2 \lambda_i^2.$$

$$\frac{\|Mf\|}{\|f\|} = \sqrt{\frac{\sum_i \alpha_i^2 \lambda_i^2}{\sum_i \alpha_i^2}} \leq \max_k |\lambda_k|.$$

□

Lemma 4.4. Assuming: - G is an ε-approximation of H Then: $\|L_G - L_H\| \leq \varepsilon$.

Proof. $-\varepsilon \tilde{L}_H \preccurlyeq \tilde{L}_G - \tilde{L}_H \preccurlyeq \varepsilon \tilde{L}_H$. $\lambda_k(\tilde{L}_G - \tilde{L}_H) \le \lambda_k(\tilde{L}_H) \le 2\varepsilon$. □

Definition ((d,eps)-expander)**.** $G = (V, E)$, $|V| = n$ is a (d, ε)-*expander* if G is an ε-approximation of $\frac{d}{n} K_n$.

Equivalent:

$$(1-\varepsilon)\frac{d}{n} L_{K_n} \preccurlyeq L_G \preccurlyeq (1+\varepsilon)\frac{d}{n} L_{K_n}.$$

$L_{K_n} = (n-1)I - A_{K_n} = nI - J$, where J is the all ones matrix ($J(x,y) = 1$). If $f \perp q$, then $Jf = \langle f, 1 \rangle f = 0$. In this case, $L_{K_n} f = nIf - Jf = nf$.

So $\lambda_k(L_{K_n}) = n$ for $k \ge 0$.

$$(1-\varepsilon)\frac{d}{n}(nI - J) \preccurlyeq dI - A_G \preccurlyeq (1+\varepsilon)\frac{d}{n}(nI - J)$$

$$-\varepsilon \frac{2}{n}(nI - J) \preccurlyeq dI - A_G - \frac{d}{n}(nI - J) \preccurlyeq \varepsilon \frac{d}{n}(nI - J)$$

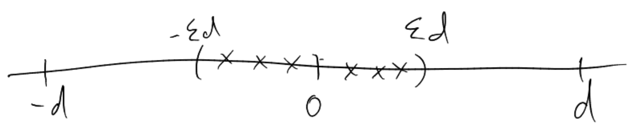
$$-\varepsilon \left(dI - \frac{d}{n} J \right) \preccurlyeq \frac{d}{n} J - A_G \preccurlyeq \varepsilon \left(dI - \frac{d}{n} J \right)$$

For $f \perp 1$, $-\varepsilon d \langle f, f \rangle \le -\langle f, Af \rangle \le \varepsilon d \langle f, f \rangle$.

So G is a (d, ε)-expander if and only if

$$|\lambda_k(A_G)| \le \varepsilon d$$

for all $1 \le k \le n - 1$.



Lemma 4.5 (Expander Mixing Lemma)**.** Assuming that:

- G is d-regular
- G is a (d, ε)-expander

Then $\forall S, T \subseteq V$,

$$\left| e(S,T) - \frac{d}{n}|S||T| \right| \le \frac{\varepsilon d}{n} \sqrt{|S||T||S^c||T^c|}.$$

Lecture 7

Proposition 4.6. Assuming that:

- G a d-regular graph on n vertices
- $\lambda, \varepsilon > 0$, $\varepsilon d = \lambda$

Then the following are equivalent:

(i) G is a (n, d, λ)-graph

(ii) $\lambda_k(A_G) \ni [-\lambda, \lambda]$, for $1 \le k \le n-1$

(iii) $\lambda_k(L_G) \in [d-\lambda, d+\lambda]$ for $2 \le k \le n$

(iv) $\lambda_k(\tilde{L}_G) \in \left[1 - \frac{\lambda}{d}, 1 + \frac{\lambda}{d}\right] = [1-\varepsilon, 1+\varepsilon]$ for $1 \le k \le n$

(v) $(1-\varepsilon)\frac{d}{n} L_{K_n} \preccurlyeq L_G \preccurlyeq (1+\varepsilon)\frac{d}{n} L_{K_n}$

(vi) G is (d, ε)-expander (also $(d, \frac{\lambda}{d})$-expander)

(vii) $\left\|L_G - \frac{d}{n} L_{K_n}\right\| \le \varepsilon d = \lambda$

(viii) $\left\|A_G - \frac{d}{n} J\right\| \le \varepsilon d = \lambda$

Lemma 4.7 (Expander Mixing Lemma)**.** Assuming that:

- $G = (V, E)$ an (n, d, λ)-graph
- $S, T \subseteq V$ (and define $e(S, T) = \sum_{x \in S} \sum_{y \in T} \mathbb{1}_{xy \in E}$)

Then

$$\begin{aligned} \left| e(S,T) - \frac{d}{n}|S||T| \right| &\le \frac{\lambda}{n}\sqrt{|S||S^c||T||T^c|} \\ &\le \lambda\sqrt{|S||T|} \end{aligned}$$

Proof. $\langle \mathbb{1}_S, L_G \mathbb{1}_T \rangle = \langle \mathbb{1}_S, (dI - A_G)\mathbb{1}_T \rangle$, $\langle \mathbb{1}_S, dI\mathbb{1}_T \rangle = d|S \cap T|$.

$$\begin{aligned} \langle \mathbb{1}_S, A_G \mathbb{1}_T \rangle &= \sum_x \sum_y \mathbb{1}_S(x) A_G(x,y) \mathbb{1}_T(y) \\ &= \sum_x \sum_y \mathbb{1}_{xy \in E} \\ &= e(S,T) \end{aligned}$$

$\frac{d}{n} L_{K_n} = dI - \frac{d}{n} J$.

$$\langle \mathbb{1}_S, \frac{d}{n} J \mathbb{1}_T \rangle = \frac{d}{n} \langle \mathbb{1}_S, J\mathbb{1}_T \rangle = \frac{d}{n} \sum_x \sum_y J(x,y) = \frac{d}{n}|S||T|.$$

$$
\begin{aligned}
\left|e(S,T) - \frac{d}{n}|S||T|\right| &= \left|\left\langle \mathbb{1}_S, \left(\frac{d}{n}L_{K_n} - LG\right)\mathbb{1}_T\right\rangle\right| \\
&\le \|\mathbb{1}_S\| \left\|\left(\frac{d}{n}L_{K_n} - L_G\right)\mathbb{1}_T\right\| \\
&\le \|\mathbb{1}_S\| \left\|\left(\frac{d}{n}L_{K_n} - L_G\right)\right\| \|\mathbb{1}_T\| \\
&\le \lambda\|\mathbb{1}_S\|\|\mathbb{1}_T\| \\
&= \lambda\sqrt{|S||T|}
\end{aligned}
$$

To get the better bound, we should consider functions which are perpendicular to 1: balanced function $f_S = \mathbb{1}_S - \frac{|S|}{n}1$.

$L_G f_S = L_G(\mathbb{1}_S - \alpha) = L\mathbb{1}_S$.

$$
\begin{aligned}
\left|e(S,T) - \frac{d}{n}|S||T|\right| &= \left|\left\langle f_S, \left(\frac{d}{n}L_{K_n} - L_G\right) f_T\right\rangle\right| \\
&\le \lambda\|f_S\|\|f_T\|
\end{aligned}
$$

$$
\begin{aligned}
\|f_S\|^2 &= |S|\left(1 - \frac{|S|}{n}\right)^2 + (n - |S|)\left(-\frac{|S|}{n}\right)^2 \\
&= |S|\left(1 - \frac{2|S|}{n} + \frac{|S|^2}{n^2}\right) + (n - |S|)\frac{|S|^2}{n^2} \\
&= |S| - \frac{2|S|^2}{n} + \frac{|S|^3}{n^2} + \frac{|S|^2}{n} - \frac{|S|^3}{n^2} \\
&= |S| - \frac{|S|^2}{n} \\
&= \frac{n|S| - |S|^2}{n} \\
&= \frac{|S||S^c|}{n}
\end{aligned}
$$

So

$$
\left|e(S,T) - \frac{d}{n}|S||T|\right| \le \sqrt{|S||S^c||T||T^c|}. \qquad \square
$$

If G is an (n,d,λ)-graph and $I \subseteq V$ an independent set, then

$$
0 = e(I,I) \ge \frac{d}{n}|I|^2 - \frac{\lambda}{n}|I||I^c|.
$$

$\lambda|I||I^c| \ge d|I|^3$.

$$
|I| \le \frac{\lambda}{d}|I^c| = \frac{\lambda}{d}(n - |I|).
$$

$$
\left(1 + \frac{\lambda}{d}\right)|I| \le \frac{\lambda}{d}n
$$

$$|I| \leq \frac{\lambda}{d\left(1+\frac{\lambda}{d}\right)} n = \frac{\lambda}{d+\lambda} n.$$

Hoffman bound: $\alpha(G) \leq \frac{\lambda}{d+\lambda} n$.

Fix d. How small can λ be such that there is an infinite family G_n of (n, d, λ)-graphs?

Note A_G^2 has d in each entry of the diagonal, so

$$\operatorname{Tr} A_G^2 = dn = \sum_i \lambda_i(A_G^2) = \sum_i (\lambda_i(A_G))^2.$$

So $dn \leq d^2 + (n-1)\lambda^2$.

So $(n-1)\lambda^2 \geq dn - d^2 = d(n-d)$, so

$$\lambda^2 \geq \frac{d(n-d)}{n-1} = d\left(\frac{n-1-(d-1)}{n-1}\right) = d\left(1 - \frac{d-1}{n-1}\right).$$

$\lambda \geq \sqrt{d}(1 - o(1))$ as $n \to \infty$.

Alon-Boppana Theorem: $\lambda \geq 2\sqrt{d-1} - o(1)$.

Claim: There exist families of (n, d, λ)-graphs with $\lambda = 2\sqrt{d-1}$. They are called Ramanujan graphs. We will probably not prove existence of these.

Call ε-Ramanujan if $\lambda \geq 2\sqrt{d-1} + \varepsilon$.

Theorem 4.8 (Friedman)**.** Assuming: - $\varepsilon > 0$, $n \to \infty$ Then:

$$\mathbb{P}(\text{random } d\text{-regular graph on } n \text{ vertices is } \varepsilon\text{-Ramanujan}) \to 1.$$

Maxcut in (n, d, λ)-graph:

$$\begin{aligned} e(S, S^c) &\leq \frac{d}{n}|S||S^c| + \frac{\lambda}{n}|S||S^c| \\ &\leq \left(\frac{d}{n} + \frac{\lambda}{n}\right)\frac{n^2}{4} \\ &= \frac{dn}{4} + \frac{\lambda n}{4} \\ &\leq \frac{e(G)}{2} + \frac{\lambda n}{4} \end{aligned}$$

Diameter, vertex expansion. $S \subseteq V$, $\partial S = \{x \in S^c : x \sim S\}$.

$$\frac{|\partial S|}{|S|} \geq \frac{e(S, S^c)}{d|S|} = \Phi(S) \geq \frac{\lambda_2(\tilde{L}_G)}{2} \geq \frac{\left(1 - \frac{\lambda}{d}\right)}{2}$$

$|\partial S| \geq \left(\frac{1-\lambda/d}{2}\right)|S|$.

Lecture 8 Exercise: Diameter $O(\log n)$.

Vertex expansion

(n, d, λ)-graph $G = (V, E)$. Have for all $S, T \subseteq V$,

$$\left|e(S,T) - \frac{d}{n}|S||T|\right| \leq \frac{\lambda}{n}\sqrt{|S||S^c||T||T^c|}.$$

$\Phi_v(S) = \frac{|\partial S|}{|S|}$.

$$\begin{aligned}
|\partial S| &\geq \frac{e(S, S^c)}{d} \\
\Phi_v(S) &\geq \frac{e(S, S^c)}{d|S|} \\
&= \Phi_e(S)
\end{aligned}$$

If $S \subseteq V$, $|S| \leq n/2$, then

$$\begin{aligned}
\Phi_V(S) &\geq \Phi_e(G) \\
&\geq \frac{\lambda_2(\tilde{L}_G)}{2} \\
\Phi_v(S) &\geq \frac{\left(1 - \frac{\lambda}{d}\right)}{2} \\
|S \cup \partial S| &\geq \left(1 + \frac{1}{2} - \frac{\lambda}{2d}\right)
\end{aligned}$$

$n \to \infty$, d, λ fixed. $\lambda \leq d - \delta$, $\delta > 0$. $|S \cup \partial S| \geq (1+\kappa)|S|$ for some $\kappa > 0$. Hence

$$|B(x, r)| \geq (1+\kappa)^r$$

if $|B(x, r-1)| \leq n/2$.

$\operatorname{diam} G = O_{\frac{\lambda}{d}}(\log n)$ (by considering ball around start and end).

Why aren't Cayley graphs of abelian groups expanders?

Let Γ be an abelian group and $S \subseteq \Gamma$ a set of generators of size d. Let $|\Gamma| = n \to \infty$. Let $G = \mathrm{Cay}(\Gamma, S)$. Then

$$|B(x, r)| \leq (2r+1)^d.$$

This is not exponential in r, so the Cayley graph can't be an expander.

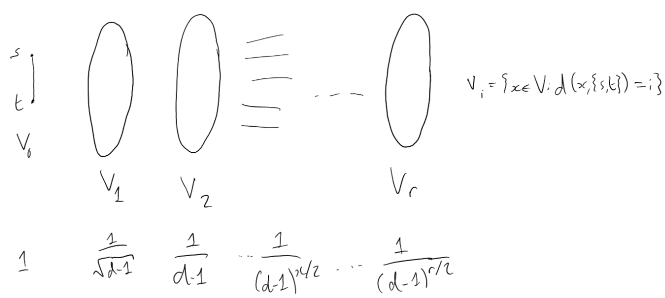
Theorem 4.9 (Alon-Boppana)**.** Assuming that:

- $G = (V, E)$ an (n, d, λ)-graph

Then as $n \to \infty$

$$\lambda \geq 2\sqrt{d-1} - O(1).$$

Proof 1. Pick edge st, pick $r \in \mathbb{Z}_{\geq 0}$.



$\varphi : V \to \mathbb{R}$,

$$\varphi(x) = \begin{cases} (d-1)^{-i/2} & \text{if } x \in V_i,\ i \leq r \\ 0 & \text{if } x \in V_i,\ i > r \end{cases}$$

$$q_L(\varphi) = \frac{\langle \varphi, L\varphi \rangle}{\langle \varphi, \varphi \rangle} \qquad \langle \varphi, \varphi \rangle = \sum_{i=0}^{r} \frac{|V_i|}{(d-1)^i}.$$

$$\begin{aligned} \langle \varphi, L\varphi \rangle &= \sum_{x \sim y} (\varphi(x) - \varphi(y))^2 \\ &= \sum_{i=0}^{r-1} e(V_i, V_{i+1}) \left(\frac{1}{(d-1)^{i/2}} - \frac{1}{(d-1)^{(i+1)/2}} \right)^2 + \frac{e(V_r, V_{r+1})}{(d-1)^r} \\ &= \sum_{i=0}^{r-1} \frac{e(V_i, V_{i+1})}{(d-1)^i} \left(1 - \frac{1}{\sqrt{d-1}} \right)^2 + \frac{e(V_r, V_{r+1})}{(d-1)^r} \end{aligned}$$

$e(V_i, V_{i+1}) \le (d-1)|V_i|$.

$$\langle \varphi, L\varphi \rangle \le \sum_{i=0}^{r-1} \frac{|V_i|}{(d-1)^i}(\sqrt{d-1}-1)^2 + \frac{|V_r|}{(d-1)^r}(d-1)$$

$$(\sqrt{d-1}-1)^2 = (d-1) - 2\sqrt{d-1} + 1 = d - 2\sqrt{d-1}$$

$$\langle \varphi, L\varphi \rangle \le (d-2\sqrt{d-1}) \sum_{i=0}^{r-1} \frac{|V_i|}{(d-1)^i} + (d-1)\frac{|V_r|}{(d-1)^r}$$

$$|V_r| \le (d-1)|V_{r-1}| \le \cdots \le (d-1)^{r-i}|V_i|$$

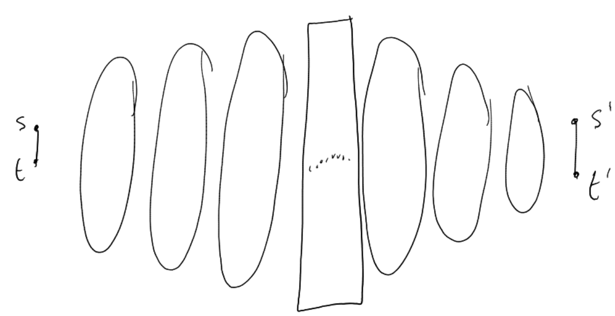
$$\implies \frac{|V_r|}{(d-1)^r} \le \frac{1}{r} \sum_{i=0}^{r} \frac{|V_i|}{(d-1)^i}$$

$$\langle \varphi, L\varphi \rangle \le (d-2\sqrt{d-1}) \sum_{i=0}^{r} \frac{|V_i|}{(d-1)^i} + (2\sqrt{d-1})\frac{|V_r|}{(d-1)^r}$$

$$\le \left(d - 2\sqrt{d-1} + \frac{2\sqrt{d-1}-1}{r+1} \right) \langle \varphi, \varphi \rangle$$

$$\implies \lambda_2(L) = \min_{\dim W = 2} \max_{\substack{f \in W \\ f \ne 0}} \frac{\langle f, Lf \rangle}{\langle f, f \rangle}$$

Suppose G has 2 deges at distance $> 2r+2$.



f, f' as above. $\langle f, f' \rangle = 0$.

$$Q_L(\alpha f + \beta f') = Q_L(\alpha f) + Q_L(\beta f').$$

Let $W = \operatorname{span}\{f, f'\}$.

$$\begin{aligned}\lambda_2(L) &\le \max_{(\alpha,\beta)\neq(0,0)} \frac{Q_L(\alpha f + \beta f')}{\|\alpha f + \beta f'\|} \\ &\le \frac{\alpha^2 Q_L(f) + \beta^2 Q_L(f')}{\alpha^2\|f\| + \beta^2\|f'\|} \\ &\le d - 2\sqrt{d-1} + \frac{2\sqrt{d-1}-1}{r+1}\end{aligned}$$

□

Corollary 4.10. For all d, there are finitely many (n, d, λ)-graphs with $\lambda < 2\sqrt{d-1}$.

$r = c \log n$.

$$\lambda_{n-1}(A_G) \ge 2\sqrt{d-1} - O_d\left(\frac{1}{\log n}\right).$$

For Alon-Boppana:

Proof 2. $\operatorname{Tr} A^{2k} = \sum_x A^{2k}(x,x)$. Note that

$$\#\{\text{closed walks of length } 2k \text{ in } G \text{ starting from } x\}.$$

is at least

$$\#\{\text{closed walks of length } 2k \text{ in } \prod_d \text{ starting from } 0\}$$

($\prod_d$ is an infinite d-regular tree). The latter is at least

$$(d-1)^k \frac{1}{k+1}\binom{2k}{k} \approx (2\sqrt{d-1})^{2k+o(1)} 2^{2k}.$$

$$\le \sum_{i=1}^{n} \lambda_i^{2k} \le d^{2k} + (n-1)\lambda^{2k}.$$

Exercise: finish details. □

Lecture 9 $\min\{|\lambda_1(A)|\lambda_{n-1}(A)\} \ge \cdots$

4.1 One-sided expanders

Goal is to find: G_n graphs with n vertices, d-regular such that $\lambda_{n-1}(A_G) \le \lambda < d$.

Reminder: (Friedman) If $G \sim G_{n,d}$ uniform random d-regular graph then

$$\mathbb{P}(G_n \text{ is } (n, d, 2\sqrt{d-1}+\varepsilon)\text{-graph}) \to 1.$$

$$\Phi(G_n) \geq \frac{\lambda_2(\tilde{L}_G)}{2} = \frac{\lambda_2(L_G)}{2d} = \frac{(d - \lambda_{n-1}(A_G))}{2d} \geq \frac{1}{2} - \frac{\varepsilon}{2} > 0.$$

Theorem 4.11. Assuming that:

- $d = 2l$ large enough

Then there is $\varepsilon = \varepsilon_d > 0$ such that there are d-regular graphs (or multi-graphs) G_n on n vertices with $\Phi(G_n) \geq \varepsilon$ for all sufficiently large n.

Graphs G_n are as follows:

- $V(G_n) = [n]$, $d = 2l$.
- Take $\pi_1, \ldots, \pi_l : [n] \to [n]$ uniform independent permutations.
- Set
$$E(G_n) = \{\{x, \pi_i(x)\} : x \in [n], i \in [l]\}.$$

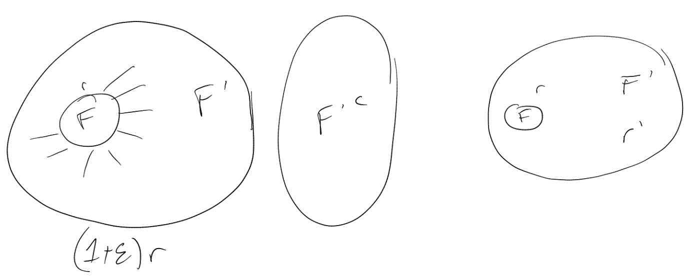
Lemma 4.12. There is $c = c_d > 0$ such that

$$\mathbb{P}(G_n \text{ is } d\text{-regular, i.e. } G_n \text{ is simple}) \geq c - o(1).$$

Lemma 4.13. If G_n is d-regular, then there exists $\varepsilon = \varepsilon_d > 0$ such that

$$\mathbb{P}(\Phi(G_n) < \varepsilon) \to 0.$$

Proof. If $\Phi(G_n) < \varepsilon$ then there exists $F \subseteq [n]$, $r = |F| \leq \frac{n}{2}$, there exists F', $F \subseteq F'$ and $e(F, F') = d|F|$ and $|F'| = r + r'$, $r' = \lfloor \varepsilon r \rfloor$.



$$\mathbb{P}(\Phi(G_n) < \varepsilon) < \sum_{1 \leq r \leq \frac{n}{2}} \sum_{\substack{F \subseteq F' \subseteq [n] \\ |F| = r \\ |F'| = r + r'}} \mathbb{P}(\pi_i(F) \subseteq F' \text{ for all } i \in [l]).$$

$\mathbb{P}(\pi_1(F) \subseteq F') = \frac{\binom{r+r'}{r}}{\binom{n}{r}}$.

$$\mathbb{P} \leq \sum_{1 \leq r \leq \frac{n}{2}} \frac{n!}{r!r'!(n-r-r')!} \left(\frac{\binom{r+r'}{r}}{\binom{n}{r}} \right)^l .$$

Fact 1: $\frac{\binom{a}{k}}{\binom{b}{k}} \leq \left(\frac{a}{b}\right)^k$ if $a \leq b$.

Proof: It is equivalent to

$$ba \cdot b(a-1) \cdots b(a-k+1) \leq ab \cdot a(b-1) \cdots a(b-k+1).$$

Compare the product term by term.

Fact 2: $n! \geq \left(\frac{n}{e}\right)^n$. Proof:

$$e^n = \sum_{k \geq 0} \frac{n^k}{k!} \geq \frac{n^n}{n!}.$$

Using these:

$$\frac{n!}{r!r'!(n-r-r')!} \leq \frac{n^{r+r'}}{r!r'!} = \frac{n^{r+r'}}{(r+r')!} \binom{r+r'}{r} \leq (2e)^{r+r'} \left(\frac{n}{e+e'} \right)^{r+r'} .$$

So

$$\begin{aligned} \mathbb{P} &\leq \sum_{1 \leq r \leq \frac{n}{2}} (2e)^{r+r'} \left(\frac{n}{r+r'} \right)^{r+r'} \left(\frac{r+r'}{n} \right)^{lr} \\ &\leq \sum_{1 \leq r \leq \frac{n}{2}} (2e)^{2r} \left(\frac{(1+\varepsilon)r}{n} \right)^{r(l-1-\varepsilon)} \end{aligned}$$

Decompose as $\sum_{1 \leq r \leq n/2} = \sum_{1 \leq r \leq K} + \sum_{K < r \leq n/2} = S_1 + S_2$. Choose $\varepsilon > 0$ small so that $\gamma = \frac{1+\varepsilon}{2}$ is small. Choose l large so that $\gamma^{l-1-\varepsilon} < \frac{1}{2(2e)^2}$.

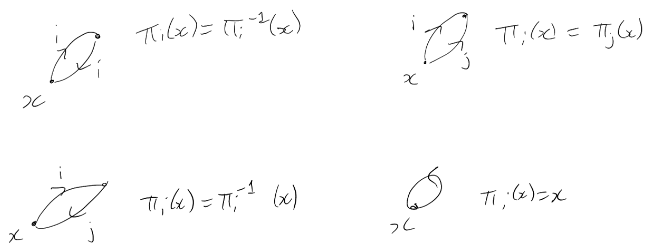
$$\begin{aligned} S_2 &\leq \sum_{K < r \leq \frac{n}{2}} (2e)^{2r} \left(\frac{(1+\varepsilon)}{2} \right)^{r(l-1-\varepsilon)} \\ &\leq \sum_{K < r \leq \frac{n}{2}} \left(\frac{1}{2} \right)^r \\ &\leq \frac{1}{2^K} \end{aligned}$$

Now S_1.

$$\begin{aligned} S_1 &\leq \sum_{1\leq r\leq K} (2e)^{2r} \left(\frac{(1+\varepsilon)r}{n}\right)^{r(l-1-\varepsilon)} \\ &\leq \left(\frac{2K}{n}\right)^{l-2} \sum_{1\leq r\leq K} (2e)^{2r} \\ &\leq c^K \left(\frac{K}{n}\right)^l \end{aligned}$$

Now let $K = \log\log\log n$, so this is $\leq O((\log\log n)n^{-l}(\log\log\log n)^l) \to 0$. Also, $S_1 \leq \frac{1}{2^K}$ term goes to 0. $\square$

For the lemma about $\mathbb{P}(d\text{-regular}) \geq c - o(1)$:



These are the bad things. Use Bonferroni inequalities (the partial sum of inclusion exclusion principle
Lecture 10 inequalities).

(n, d, λ)-graph

n: number of vertices, d-regular, $\lambda_k(A) \in [-\lambda, \lambda]$, $k \neq n$.

Ramanujan graph: $(n, d, 2\sqrt{d-1})$-graph.

- Petersen $(10, 3, 2\sqrt{2})$-graph.
- Complete (bipartite), $d = n - 1$ big :(.
- Paley $\mathrm{Cay}(\mathbb{Z}/p\mathbb{Z}, \{x^2 : x \in \mathbb{Z}/p\mathbb{Z}, x \neq 0\})$, $p \equiv 1 \pmod 4$. $d = \frac{p-1}{2}$ big :(.

Alon-Boppana: For every $\varepsilon > 0$ fixed $d \geq 3$, there are finitely many $(n, d, 2\sqrt{d-1} - \varepsilon)$-graphs.

Bipartite: (n, d, λ)-graph, n vertices, d-regular, $\lambda_k(A) \in [-\lambda, \lambda]$ for $k \neq n, 1$.

Theorem 4.14 (Lubotsky-Phillips-Sanak / Margoulis)**.** Assuming that:

- p prime
- $d = p + 1$
- n arbitrarily large

Then there exists a $(n, d, 2\sqrt{d-1})$-graph.

Goal:

Theorem 4.15 (Marcus, Spielman, Srivastava)**.** Assuming that:

- $d \geq 3$

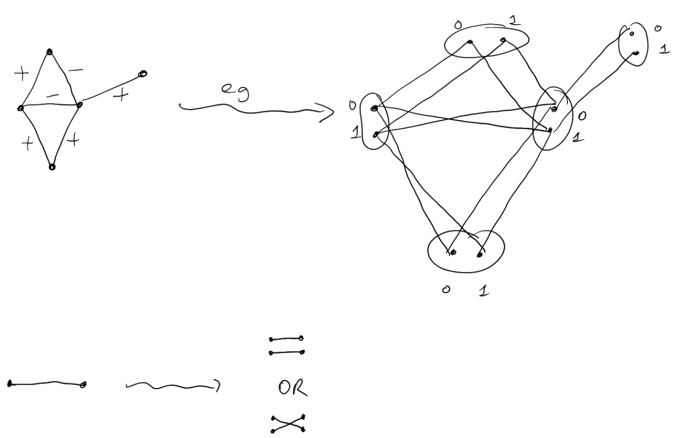
Then there are $(n, d, 2\sqrt{d-1})$-bipartite graphs for arbitrarily large n.

Strategy: Bilu and Linial. Lifts of graphs:

Definition 4.16 (2-lift)**.** A *2-lift* of a graph $G = (V, E)$ is a graph $\hat{G} = (\hat{V}, \hat{E})$ with

- $x \in V \implies x_0, x_1 \in \hat{V}$.
- $xy \in E \implies$ either $x_0y_0, x_1y_1 \in \hat{E}$ or $x_0y_1, x_1y_0 \in \hat{E}$.

(and no other vertices or edges).



Definition 4.17 (Signing)**.** $S : V \times V \to \mathbb{R}$ is a *signing* of G if

$$S(x,y) = \begin{cases} \pm 1 & \text{if } A(x,y) = 1 \\ 0 & \text{if } A(x,y) = 0 \end{cases}$$

and $S(x,y) = S(y,x)$ (symmetric). So can think of S as a function $E \to \{\pm 1\}$.

$A_S^+(x,y) = A(x,y)\mathbb{1}_{S(x,y)=1}$. $A_S^-(x,y) = A(x,y)\mathbb{1}_{S(x,y)=-1}$. Have $A = A_S^+ + A_S^-$, $S = A_S^+ - A_S^-$.

Lemma 4.18. The eigenvalues of $\hat{G}$ are the eigenvalues of A_G (old) together with the eigenvalues of S (new).

Proof.

$$A_{\hat{G}_S} = \begin{pmatrix} A_S^+ & AS_S^- \\ A_S^- & AS_S^+ \end{pmatrix}$$

Let $A\varphi = \lambda\varphi$. Then

$$A_{\hat{G}_S}\begin{pmatrix}\varphi \\ \varphi\end{pmatrix} = \begin{pmatrix} A_S^+\varphi + A_S^-\varphi \\ A_S^-\varphi + A_S^+\varphi \end{pmatrix} = \begin{pmatrix} A\varphi \\ A\varphi \end{pmatrix} = \begin{pmatrix} \lambda\varphi \\ \lambda\varphi \end{pmatrix} = \lambda \begin{pmatrix}\varphi \\ \varphi\end{pmatrix}.$$

Let $S\eta = \mu\eta$. Now

$$A_{\hat{G}_S}\begin{pmatrix}\eta \\ -\eta\end{pmatrix} = \begin{pmatrix} A_S^+\eta - A_S^-\eta \\ A_S^-\eta - A_S^+\eta \end{pmatrix} = \begin{pmatrix} S\eta \\ -S\eta \end{pmatrix} = \begin{pmatrix} \mu\eta \\ -\mu\eta \end{pmatrix} = \mu \begin{pmatrix}\eta \\ -\eta\end{pmatrix}.$$ □

Conjecture 4.19 (Bilu, Linial)**.** If G is d-regular, then there exists a signing $S : E(G) \to \{\pm 1\}$ whose eigenvalues are in $[-2\sqrt{d-1}, 2\sqrt{d-1}]$.

Theorem 4.20 (Bilu, Linial)**.** Can find signings S with eigenvalues λ satisfying $|\lambda| = O(\sqrt{d(\log d)^3})$.

Theorem 4.21 (Marcusm, Spielman, Srivastava)**.** Assuming that:

- G is a d-regular graph

Then there exists a signing with eigenvalues λ with $\lambda \le 2\sqrt{d-1}$.

Theorem 4.21 implies Theorem 4.15.

- Start with $K_{d,d}$.

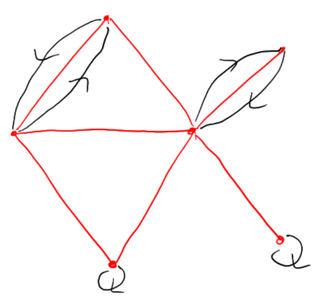
- Keep applying Theorem 4.21 to find signing.
- Build lift with that signing.
- 2-lift of bipartite graph is bipartite.
- Spectrum of adjacency matrix of bipartite graph is symmetric around 0. □

Notation. For $\pi \in \mathrm{Sym}(X)$, let $|\pi|$ be the number of inversions.

Theorem 4.21: $S \sim \mathrm{Unif}(\{\pm 1\}^{E(G)})$.

$$\begin{aligned}\mathbb{E}_S \det(xI - S) &= \mathbb{E}_S \left(\sum_{\pi \in \mathrm{Sym}(X)} (-1)^{|\pi|} \prod_{y \in V} (xI - S)(y, \pi(y)) \right) \\ &= \sum_{k=0}^{n} x^{n-k} (-1)^k \sum_{\substack{T \subseteq V \\ |T| = k}} \sum_{\pi \in \mathrm{Sym}(T)} \mathbb{E}_S \left((-1)^{|\pi|} \prod_{y \in T} (xI - S)(y, \pi(y)) \right)\end{aligned}$$

For $xy \in E$: $\mathbb{E}S(x,y)^{2k+1} = 0$, $\mathbb{E}S(x,y)^{2k} = 1$.



$$\begin{aligned}&= \sum_{\substack{k=0 \\ k \text{ even}}}^{n} x^{n-k} (-1)^k \sum_{\substack{M \text{ matching} \\ \text{of size} \le \frac{k}{2}}} (-1)^{k/2} \dots \\ &= \sum_{k \ge 0} x^{n-2k} (-1)^k M_k(G) \\ &= \mu_G(x)\end{aligned}$$

where $M_k(G)$ is the number of matchings of size k in G.

$\mu_G(x)$ is the matching polynomial of G.

Heilman-Lieb-Godsil:

- $\mu_G(x)$ is real rooted for all G.
- If G has degree $\leq d$, then $\mu_G(x)$ has roots in $[-2\sqrt{d-1}, 2\sqrt{d-1}]$.

Lecture 11

Last time, when working on $\det(xI - A_s)$, we showed:

Theorem 4.22 (Godsil-Gutman, 80s)**.**

$$\mathbb{E}_{S\sim\{\pm 1\}} \det(xI - A_s) = \mu_G(x),$$

where

$$\mu_G(x) = \sum_{k\geq 0} x^{n-2k}(-1)^k m_k(G),$$

where $m_k(G)$ is the number of matchings in G with k edges.

Fact: $\mu_G(x)$ is real rooted.

Theorem 4.23 (Heilman-Lieb, 72)**.** Assuming that:

- G is d-regular

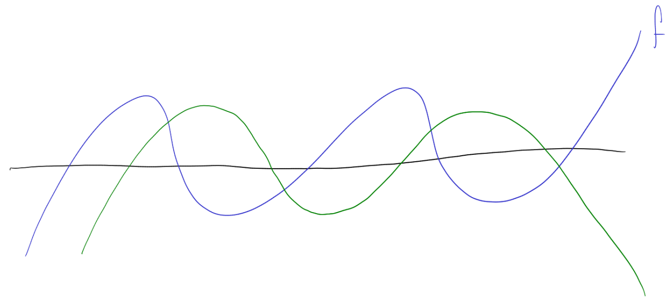
Then

$$\operatorname{maxroot} \mu_G(x) \leq 2\sqrt{d-1}.$$

If only we could say that $\operatorname{maxroot} \mathbb{E}_S \det(xI - A_s)$ is an average of $\operatorname{maxroot} \det(xI - A_h)$, $h \in \{\pm 1\}^E$. Hopelessly false:

Definition 4.24 (Interlacing)**.** Let f be a real rooted polynomial of degree n with roots $\alpha_1, \ldots, \alpha_n$ and g a real rooted polynomial of degree $n-1$ with roots $\beta_1, \ldots, \beta_{n-1}$. We say g *interlaces* f if

$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \cdots \leq \beta_{n-2} \leq \alpha_{n-1} \leq \beta_{n-1} \leq \alpha_n.$$



Example. If f is real rooted, then so is f', and also f' interlaces f.

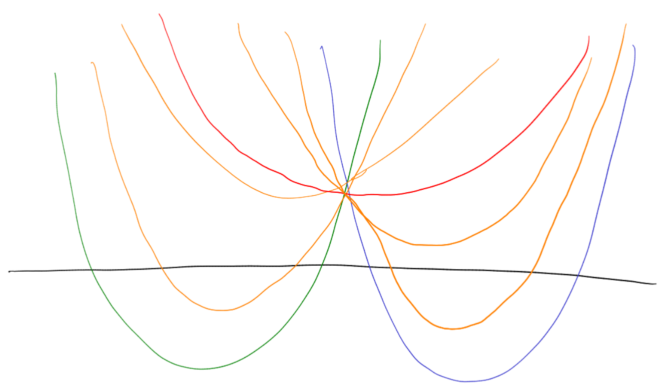
Definition 4.25 (Common interlacement)**.** We say that real rooted polynomials f, g of degree n have a *common interlacement* if there is a polynomial h of degree $n+1$ such that f and g both interlace h. In other words, if the roots of f and g are $\alpha_1 \leq \cdots \alpha_n$ and $\beta_1 \leq \cdots \leq \beta_n$ respectively, then they have a common interlacement if and only if there are some $\gamma_0 \leq \cdots \leq \gamma_n$ such that

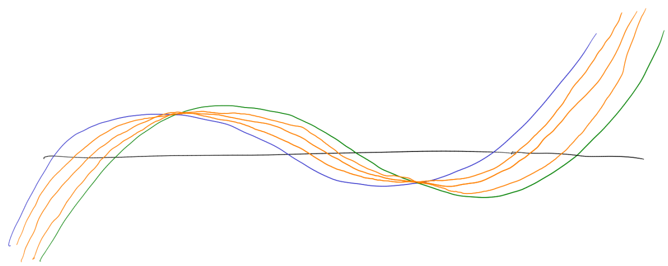
$$\gamma_0 \leq \alpha_1, \beta_1 \leq \gamma_1 \leq \alpha_2, \beta_2 \leq \gamma_2 \leq \cdots \leq \gamma_{n-1} \leq \alpha_n, \beta_n \leq \gamma_n.$$

Theorem 4.26 (Fell 80)**.** Assuming that:

- f, g are real rooted
- both degree n, monic

Then f and g have a common interlacing if and only if every convex combination of f and g are real rooted.





Proof. Assume f and g without repeated or common roots (general case requires more work and details checking).

$\Leftarrow$ Let $h_t(x) = tf(x) + (t-1)g(x)$ for $t \in (0,1)$. We know these are real rooted, and waht to show that f, g have a common interlacement. Let $\lambda_i(t)$ be the i-th root of $h_t(x)$. $\lambda_i(t)$ includes $(\lambda_i(0), \lambda_i(1)) \subseteq \mathbb{R}$.

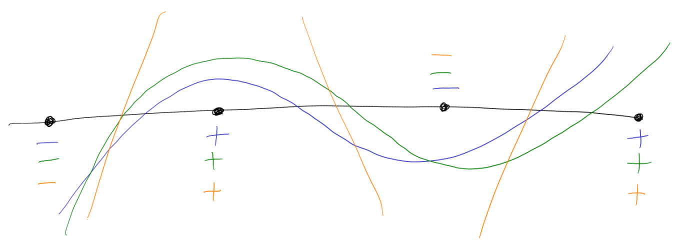
Claim: $(\lambda_I(0), \lambda_I(1))$ are disjoint.

Suppose not. Let λ_k be a root of f and $\lambda_k \in (\lambda_I(0), \lambda_I(1))$. So there exists $t \in (0,1)$ such that $\lambda_I(t) = \lambda_k$.

$$h_t(\lambda_k) = 0 = tf(\lambda_k) + (1-t)g(\lambda_k),$$

so $g(\lambda_k) = 0$, contradiction as we assumed no common roots.

$\Rightarrow$



The dots represent the roots of the polynomial that interlaces them. By monicness, we get the blue and green +s and −s. Then get the orange ones, and use intermediate value theorem. ☐

Corollary 4.27. Assuming that:

- f, g are real rooted of degree n
- have a common interlacing

Then for all $t \in [0,1]$

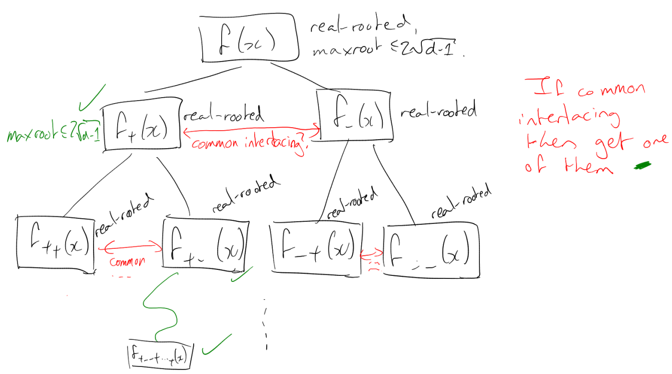
$$\min\{\text{maxroot}(f), \text{maxroot}(g)\} \le \text{maxroot}(tf(x) + (1-t)g(x)).$$

Approach to Theorem 4.21:

$f(x) = \mathbb{E}_{S\in\{\pm1\}^E} \det(xI - A_s)$, $|E| = m$. For $h_1, \dots, h_k \in \{\pm 1\} = \{+,-\}$, then

$$f_{h_1,\dots,h_k}(x) = \mathbb{E}_{s_{k+1},\dots,s_m\in\{\pm1\}}(\det(xI - A_s) \mid s_1 = h_1, \dots, s_k = h_k).$$

"Method of interlacing families of polynomials"



Theorem 4.28 (Marcus, Spielman, Srivastava (real rooted))**.** For every $\rho_1, \dots, \rho_m \in [-1,1]$ the following polynomial is real rooted

$$\chi_{\rho_1,\dots,\rho_m}(x) = \sum_{S\in\{\pm1\}^m} \det(xI - A_s) \prod_{J:s_J=1} \left(\frac{1+\rho_J}{2}\right) \prod_{J:S_J=-1} \left(\frac{1-\rho_J}{2}\right).$$

Theorem 4.28 implies Theorem 4.21:

$$f_{h_1,\dots,h_k}(x) = \chi_{h_1,\dots,h_k,0,0,\dots,0}(x).$$

$$\mu_G(x) = \chi_{0,\dots,0}(x).$$

$$tf_{h_1,\dots,h_k,1}(x) + (1-t)f_{h_1,\dots,h_k,-1}(x) = \chi_{h_1,\dots,h_k,2t-1,0,\dots,0}(x).$$

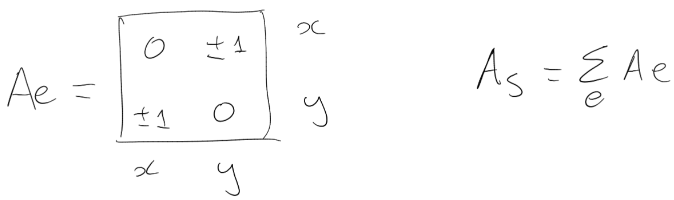
Theorem 4.29 (Marcus, Spielman, Srivastava (matrix))**.** Assuming that:

- $r_1, \dots, r_m \in \mathbb{R}^n$ are independent random vectors, whose support has ≤ 2 points

> Then
> $$\mathbb{E}\det\left(xI - \sum_{j=1}^{m} r_j r_j^\top\right)$$
> is real rooted.

Theorem 4.29 implies Theorem 4.28:

$\chi_{\rho_1,\ldots,\rho_m}(x)$ is
$$\mathbb{E}\det(xI - A_s).$$



$\mathbb{E}\det(xI - A_s)$ is real rooted if and only if $\mathbb{E}\det((x+d)I - A_s)$ is real rooted. This equals
$$\mathbb{E}\det(xI + (dI - A_s)).$$

Note
$$dI - A_S = \sum_{x \sim y} (e_x + S(x,y)e_y)(e_x + S(x,y)e_y)^\top.$$

Lecture 12 Done.

> **Definition 4.30** (Real stable)**.** A polynomial $p(z_1,\ldots,z_n) \in \mathbb{R}[z_1,\ldots,z_n]$ is *real stable* if $p(z_1,\ldots,z_n) \neq 0$ for all $(z_1,\ldots,z_n) \in \mathcal{H}^n$, $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$.

Notice that for a single variable, $p(z)$ is real stable if and only if real rooted.

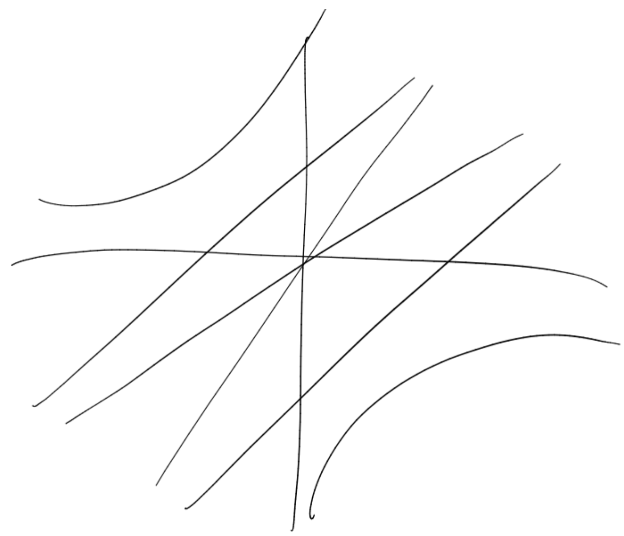
> **Example.** $1 - z_1 z_2$.

> **Proposition 4.31** (Stable iff real rooted)**.** $p(z_1,\ldots,z_n)$ is real stable if and only if for all $\mathbb{R}^n_{\geq 0}$, $b \in \mathbb{R}^n$, the polynomial $t \mapsto p(at+b)$ is real rooted.

Proof.

$\Rightarrow$ If p not stable, then there exists $z_0,\ldots,z_n \in \mathcal{H}$ with $p(z_1,\ldots,z_n) = 0$, $z_j = b_j + ia_j$, $a_j > 0$. Then $p(ai+b) = 0$, $t = i$.

$\Leftarrow$ Suppose $\exists a \in \mathbb{R}^n_{\geq 0}$, $b \in \mathbb{R}^n$, $t \in \mathbb{C} \setminus \mathbb{R}$ with $p(at + b) = 0$. Assume $\operatorname{Im} t > 0$. $z_j + a_j t + b_j \in \mathcal{H}$. $p(z_1, \ldots, z_n) = p(at + b) = 0$.



□

Proposition 4.32 (Stability of real stable)**.** Assuming that:

- $p(z_1, \ldots, z_n)$ is real stable

Then the following are also:

- $p(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$, $\sigma \in S_n$.
- $p(az_1, z_2, \ldots, z_n)$, $a \in \mathbb{R}_{\geq 0}$.
- $p(z_2, z_2, z_3, \ldots, z_n)$.
- $p(c, z_2, \ldots, z_n)$, $c \in \mathcal{H} \cup \mathbb{R}$.
- $z_1^{d_1} p\left(-\frac{1}{z_1}, z_2, \ldots, z_n\right)$.
- $\partial_{z_1} p(z_1, \ldots, z_n)$.
- $\operatorname{MAP}(p(z_1, \ldots, z_n))$. $\operatorname{MAP}(z_1 z_2 + z_2 z_3^2 + 2z_1 z_4 + z_2 z_3^2) = z_1 z_2 + 2z_1 z_4$.
- If p, q are real stable, then so is pq.

Proposition 4.33 (Mother of all stable polynomials)**.** Assuming that:

- $B \in \mathbb{R}^{d \times d}$ symmetric matrix

- $A_1, \ldots, A_m \in \mathbb{R}^{d \times d}$ positive semi definite

Then $\det(B + z_1 A_1 + \cdots + z_n A_n)$ is real stable.

Proof. Let $a \in \mathbb{R}^n_{\geq 0}$, $b \in \mathbb{R}^n$. Assume positive definite instead of positive semi definite.

$$\det\left(B + \sum_{j=1}^{n}(a_j t + b_j)A_j\right) = \det\left(\underbrace{\left(\sum_{j=1}^{n} a_j A_j\right)}_{=:A} t + \underbrace{\left(\sum_{j=1}^{n} b_j A_j + B\right)}_{=:C}\right)$$

A is positive definite, and C is symmetric. So

$$\begin{aligned}
&= \det(At + C) \\
&= \det(A^{\frac{1}{2}}(tI + A^{-\frac{1}{2}}CA^{-\frac{1}{2}})A^{\frac{1}{2}}) \\
&= \det A \det(tI + A^{-\frac{1}{2}}CA^{-\frac{1}{2}})
\end{aligned}$$

The roots are eigenvalues of a real symmetric matrix, hence real. □

Proof of Theorem 4.29. Say r_j equals r_j^+ with probability p_j^+, and equals r_j^- with probability p_j^-. We will use Cauchy Binet:

$$\det(AB) = \sum \det(A_{S \times [n]}) \det(B_{[n] \times C}) TODO$$

$$\begin{aligned}
p(z_1, \ldots, z_n) &= \mathbb{E} \det\left(\sum_{j=1}^{n} z_j r_j r_j^\top\right) \\
&= \mathbb{E} \sum_{\substack{S \subseteq [m] \\ |S| = n}} \det\left(\sum_{j \in S} z_j r_j r_j^\top\right) \\
&= \mathbb{E} \sum_{\substack{S \subseteq [m] \\ |S| = n}} \left(\prod_{j \in S} z_j\right) \det\left(\sum_{j \in S} r_j r_j^\top\right)
\end{aligned}$$

$$\det\left(\mathbb{E} \sum_{j=1}^{n} z_j r_j r_j^\top\right) = \det\left(\sum_{j=1}^{n} z_j \mathbb{E} r_j r_j^\top\right)$$

is real stable since $\mathbb{E} r_j r_j^\top \geq 0$ is a covariance matrix.

$$= \det\left(\sum_{j=1}^m z_j (p_j^+ r_j^+ (r_j^+)^\top + p_j^- r_j^- (r_j^-)^\top)\right)$$

$$= \sum_{\substack{S \subseteq [m] \\ |S| = n}} \left(\prod_{j \in S} z_j\right) \sum_{n \in \{\pm 1\}^S} \det(p_j^{h(j)} r_j^{h(j)} (r_j^{h(j)})^\top) + \text{"terms with squares"}$$

So the earlier thing is real stable. So

$$\mathbb{E} \det\left(\sum_{j=1}^m z_j r_j r_j^\top\right)$$

is real stable. So

$$\mathbb{E} \det\left(\sum_{j=1}^m z_j r_j r_j^\top + \sum_{j=1}^m x_j e_j e_j^\top\right)$$

is real stable. Take $z_j = -1$, $x_j = x$. Then

$$\mathbb{E} \det\left(xI - \sum_{j=1}^m r_j r_j^\top\right)$$

is real stable. □

Theorem 4.34 (Godsil)**.** Assuming that:

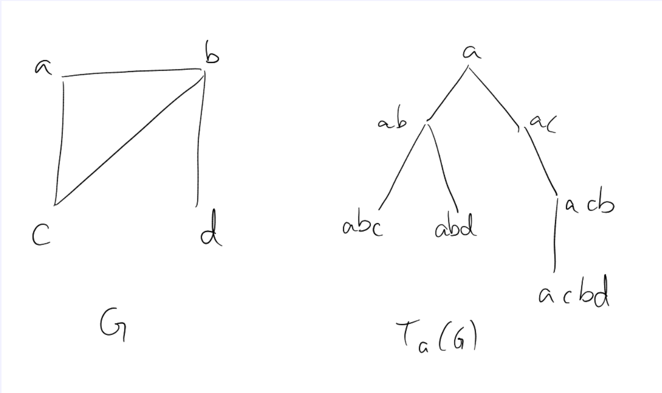
- G is d-regular
- $\lambda_1 \leq \cdots \leq \lambda_n$ are the roots of $\mu_G(x)$

Then

$$\sum_{k=1}^n \lambda_k^l = \sum_a W_a^l(G),$$

where $W_a^l(G)$ is the number of closed walks of length l from a in the path tree $T_a(G)$ of G.

Definition 4.35 (Path tree)**.** Example: $T_a(G)$



Proof of **??** *implies Theorem 4.23.* $W_a^l(G) \leq W_a^l(\prod_d) \leq (2\sqrt{d-1} + o_l(1))^l$.

$\lambda_n^l \leq \sum \lambda_k^l = \sum_a W_a^l(G) \leq n(2\sqrt{d-1} + o_l(1))^l$.

$\lambda_n \leq n^{\frac{1}{l}}(2\sqrt{d-1} + o_l(1))$.

Take $l \to \infty$. □

Proof of **??**. $\mu'_G(x) = \sum_a \mu_{G-a}(x)$ Hence

$$\begin{aligned}\sum_a \sum_{k\geq 0} x^{n-1-2k}(-1)^k m_k(G-a) &= \sum_{k\geq 0}(n-2k)x^{n-1-2k}(-1)^k m_k(G) \\ &= \mu'_G(x)\end{aligned}$$

$$
\begin{aligned}
\mu_G(x) &= x\mu_{G-a}(x) - \sum_{b\sim a} \mu_{G-a-b}(x) \\
m_k(G) &= m_k(G-a) + \sum_{b\sim a} m_{k-1}(G-a-b) \\
x^{n-2k}(-1)^k m_k(G) &= x \cdot x^{n-1-2k}(-1)^k m_k(G-a) - \sum_{b\sim a} x^{n-2-2(k-1)}(-1)^{k-1} m_{k-1}(G-a-b)
\end{aligned}
$$

$$
\begin{aligned}
\sum_a \frac{\mu_{G-a}(x)}{\mu_G(x)} &= \frac{\mu'_G(x)}{\mu_G(x)} \\
&= \sum_{j=1}^n \frac{1}{x-\lambda_j} \\
&= \frac{1}{x}\sum_{j=1}^n \frac{1}{1-\frac{\lambda_j}{x}} \\
&= \frac{1}{x}\sum_{j=1}^n \sum_{l\geq 0} \frac{\lambda_j^l}{x^l} \\
&= \frac{1}{x}\sum_{l\geq 0} \frac{\sum_j \lambda_j^l}{x^l}
\end{aligned}
$$

Claim: $\frac{\mu_{G-a}(x)}{\mu_G(x)} = \frac{1}{x}\sum_{l\geq 0} \frac{W_a^l(G)}{x^l}$.

$$
\begin{aligned}
\frac{\mu_{G-a}(x)}{\mu_G(x)} &= \frac{\mu_{G-a}(x)}{x\mu_{G-a}(x) - \sum_{b\sim a}\mu_{G-a-b}(x)} \\
&= \frac{1}{x}\frac{1}{1-\frac{1}{x}\sum_{b\sim a}\frac{\mu_{G-a-b}(x)}{\mu_{G-a}(x)}} \\
&= \frac{1}{x}\cdot\sum_{k\geq 0}\left(\frac{1}{x}\sum_{b\sim a}\frac{\mu_{G-a-b}(x)}{\mu_{G-a}(x)}\right)^k \\
&= \frac{1}{x}\sum_{k\geq 0}\frac{1}{x^k}\left(\sum_{b\sim a}\frac{1}{x}\sum_{l\geq 0}\frac{W_b^l(G-a)}{x^l}\right)^k \\
&= \frac{1}{x}\sum_{k\geq 0}\frac{1}{x^{2k}}\left(\sum_{b_1\sim a}\sum_{l_1\geq 0}\frac{W_G^{l_1}(G-a)}{x^{l_1}}\right)\cdots\left(\sum_{b_k\sim a}\sum_{l_k\geq 0}\frac{W_{b_k}^{l_k}(G-a)}{x^{l_k}}\right) \\
&= \frac{1}{x}\sum_{l\geq 0}\frac{W_a^l(G)}{x^l}
\end{aligned}
$$

Why? A tree-like walk in $W_a^l(G)$ that visits a exactly k times is determined by:

- A sequence $b_1, \ldots, b_k$ of neighbours of a.

- A sequence $\gamma_1, \ldots, \gamma_k$ of walks in $T_{b_i}^{l_i}(G - a)$ where $2k + l_1 + \cdots + l_k = l$.

$\square$

Index

adjm 7, 9, 18, 19, 21, 25, 30

degm 7

(d, ε)-expander 18

edgeexp 11, 12

ε-approximation 17, 18

graphexp 11, 12, 13, 15, 16, 25, 26

Hermitian 2, 3

lapm 7, 8, 9, 12, 17, 18, 19, 20, 23, 24, 25

lord 17, 18

(n, d, λ)-graph 18, 20, 21, 22, 23, 25, 28

q 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 23, 24

setexp 11, 12, 13, 15, 21, 22