Representation Theory

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lecture 1

1 Introduction

What is Representation Theory?

The study of how $\underbrace{\text{symmetry}}_{\text{groups}}$ $\underbrace{\text{occurs}}_{\text{act linearly on}}$ in $\underbrace{\text{nature}}_{\text{finite dimensional vector spaces}}$.

Main goal: Understand for a given group G all the ways it can act linearly on a finite dimensional vector space, i.e. classify them. Subproblem: What does it mean for two such to be the "same"? How do they break into smaller pieces?

Secondary goal: Use representations to understand groups, e.g. give a proof that no finite simple group has order with precisely two prime factors.

1.1 Linear Algebra Revision

By vector space we will always mean finite dimensional vector space (unless we say not) over a field k. k will usually be algebraically closed and characteristic zero, for example $\mathbb{C}$, but that is because it is an easy first case, but theories are normally more general and sometimes we'll look at these.

Given a vector space V we define the general linear group of V

$$\mathrm{GL}(V) = \mathrm{Aut}(V) = \{\alpha : V \hookrightarrow V \mid \alpha \text{ is } k\text{-linear and invertible}\}$$

This is a group under composition of linear maps.

Since V is finite dimensional, there is a (linear) isomorphism $k^d \simeq V$ for some $d \geq 0$ called the dimension. The choice of isomorphism determines a basis $e_1, \dots, e_d$ of B where e_1 is the image of the i-th standard basis vector under the isomorphism.

Then

$$\mathrm{GL}(B) \simeq \underbrace{\{A \in \mathrm{Mat}_d(k) \mid \det A \neq 0\}}_{\text{group under matrix multiplication}}$$

This isomorphism sends a linear map α to the matrix A_{ij} such that

$$\alpha(e_i) = \sum_{j=1}^{d} A_{ji} e_j$$

Exercise: check that this does define an isomorphism of groups.

The choice of isomorphism also gives a decomposition of V as a direct sum of 1 dimensional subspaces

$$V = \bigoplus_{i=1}^{d} k e_i$$

This decomposition is not unique unless $d = 1$, but the number of summands is always $\dim V$.

1.2 Group Representation – Definitions and Examples

Recall that an action of a group G on a set X is a function $\cdot : G \times X \to X$, $(g, x) \mapsto g \cdot x$ such that

(i) $e \cdot x = x \ \forall x \in X$;

(ii) $g \cdot (h \cdot x) = (gh) \cdot x \ \forall g, h \in G, x \in X$.

Recall also that to define an action is equivalent to defining a group homomorphism $\rho : G \to S(X)$ where $S(X)$ is the symmetric group of X i.e. the set of bijections $X \to X$ with operation composition of functions via $\rho(g)(x) = g \cdot x$ for all $g \in G, x \in X$.

Definition (Representation)**.** A *representation* of a group G on a vector space V is a group homomorphism $\rho : G \to \mathrm{GL}(V)$.

Notation. By abuse of notation, we'll sometimes call the representation ρ, sometimes (ρ, V) and sometimes just V.

Defining a representation of G in V corresponds to assigning a linear map $\rho(g) : V \to V$ to each $g \in G$ such that

(i) $\rho(e) = \mathrm{id}_V$

(ii) $\rho(gh) = \rho(g)\rho(h)$

(iii) $\rho(g^{-1}) = \rho(g)^{-1}$

Exercise: Show that if condition (ii) holds then (i) is equivalent to (iii). Moreover, both can be replaced by $\rho(g) \in \mathrm{GL}(V) \ \forall g \in G$.

Given a basis for V a representation can be viewed as an assignment of matrix $\rho(g)$ in $\mathrm{Mat}_{\dim V}(k)$ to each $g \in G$ such that (i), (ii) and (iii) hold.

Definition (Degree of representation). The *degree* of ρ or *dimension* of ρ is $\dim V$.

Definition (Faithful representation). ρ is *faithful* if $\ker \rho = \{e\}$.

Examples

(1) Let G be any group and $V = k$. Then $\rho : G \to \mathrm{GL}(k)$, $g \mapsto \mathrm{id}$ is called the *trivial representation*.

(2) Let $G = C_2 = (\{\pm 1\}, \cdot)$, $V = \mathbb{R}^2$ then

$$\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho(-1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

defines a representation of G on V since $\rho(-1)^2 = \rho(1)$.

(3) Let $G = (\mathbb{Z}, +)$, V a vector space and ρ a representation of G on V. Necessarily $\rho(0) = \mathrm{id}_V$, $\rho(1) : V \hookrightarrow V$ is an invertible linear map α, say $\rho(1+1) = \rho(1) = \alpha^2$. By induction $\rho(n) = \alpha^n$ for all $n \geq 0$, and for $n < 0$, $\rho(n) = \rho(-n)^{-1} = (\alpha^{-n})^{-1} = \alpha^n$ so $\rho(n) = \alpha^n$ for all $n \in \mathbb{Z}$.

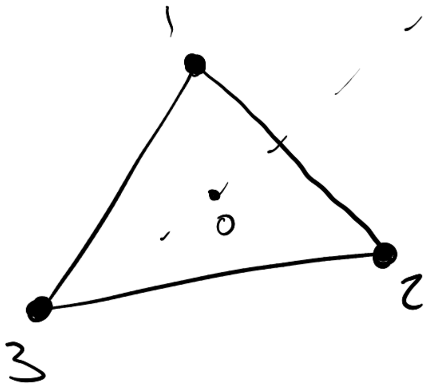
Notice conversely for any $\alpha \in \mathrm{GL}(V)$, $\rho(n) = \alpha^n$ $\forall n \in \mathbb{Z}$ defines a representation of G on V. So

$$\begin{aligned} \{\text{representations of } G \text{ on } V\} &\overset{1-1}{\leftrightarrow} \mathrm{GL}(V) \\ \rho &\mapsto \rho(1) \end{aligned}$$

(4) Let $G = (\mathbb{Z}/N\mathbb{Z}, +)$ and $\rho : G \to \mathrm{GL}(V)$ a representation. As before $\rho(n + N\mathbb{Z}) = \rho(1 + N\mathbb{Z})^n$ for all $n \in \mathbb{Z}$. But now $\rho(N + N\mathbb{Z}) = \rho(0 + N\mathbb{Z}) = \mathrm{id}_V$ so $\rho(1 + N\mathbb{Z})^N = \mathrm{id}_V$. So

$$\begin{aligned} \{\text{representations of } (\mathbb{Z}/N\mathbb{Z}, +) \text{ on } V\} &\overset{1-1}{\leftrightarrow} \{\alpha \in \mathrm{GL}(V) \mid \alpha^N = \mathrm{id}\} \\ \rho &\mapsto \rho(1 + N\mathbb{Z}) \end{aligned}$$

(5) $G = S_3 = S(\{1, 2, 3\})$ and $V = \mathbb{R}^2$. Take an equilateral triangle in $\mathbb{R}^2$ centred at the origin and labelled vertices $1, 2, 3$.



Then G acts on the triangle by permuting vertices. Each such symmetry induces a linear transformation of $\mathbb{R}^2 = V$. For example $g = (12)$ induces reflection in the line through the origin and 3, and $g = (123)$ induces a rotation by $\frac{2\pi}{3}$.

Exercise: Choose a basis for $\mathbb{R}^2$. Write down the coordinates of the vertices of triangle. For $g \in S_3$ write down the matrix of the induced linear map. Check it defines a representation. Would a different basis have made the calculation easier?

Start of lecture 2

(6) Given a finite set X we can form the vector space

$$kX := \{f : X \to k\}$$

with pointwise operations. This has a basis $\langle \delta_x : x \in X \rangle$ where $\delta_x(y) = \delta xy$ for $y \in X$. If $f \in kX$ then $f = \sum_{x \in X} f(x)\delta_x$.

If a group G acts on X, we can define

$$\rho : G \to \operatorname{Aut}(kX)\rho(g)(f)(x) \qquad = f(g^{-1}x) \qquad \forall f \in kX, g \in G, x \in X$$

If is easy to check $\rho(g)$ is linear for all $g \in G$ and $\rho(e) = \mathrm{id}_{kX}$. So it suffices to show $\rho(gh) = \rho(g)\rho(h)$ $\forall g, h \in G$. To show this, note that for all $g, h \in G$, $f \in kX$, $x \in X$, we have

$$\begin{aligned}\rho(gh)(f)(x) &= f(h^{-1}g^{-1}x) \\ &= \rho(h)(f)(g^{-1}x) \\ &= \rho(g)\rho(h)(f)(x)\end{aligned}$$

as required. Note that for $g \in G$, $x, y \in G$,

$$(\rho(g)\delta_x)(y) = \delta_x(g^{-1}y) = \delta_{x,g^{-1}y} = \delta_{gx,y} = \delta_{gx}(y)$$

So by linearity $\rho(g)\left(\sum f(x)\delta_x\right) = \sum_{x \in X} f(x)\delta_{gx}$.

(7) In particular if G is finite then G acts on itself by left multiplication $G \times G \to G$, $(g, h) \mapsto gh$. This induces a representation of G on kG, called the *regular representation*. If $g \in G$ then $\rho(g)(\delta_e) = \delta_g$ so $\rho(g) = e \iff g = e$. So the regular representation is always faithful.

(8) If (ρ, V) is a representation of G we can define a representation ρ^* of G on V^* as follows

$$\rho^*(g)(\theta)(v) = \theta(g^{-1}v) \qquad \forall g \in G, \theta \in V^*, v \in V$$

$\rho^*(g)$ can be viewed as the adjoint of $\rho(g)^{-1}$ and recall that with respect to a pair of dual bases for V and V^*, the matrix of the adjoint of a linear map is the transpose of the matrix of the map. So if $V = k^d$ so $\rho : G \to \mathrm{GL}_d(k)$ then $\rho^*(g) = (\rho(g)^{-1})^\top$. This is a homomorphism because $\mathrm{GL}_d(k) \to \mathrm{GL}_d(k)$, $A \mapsto (A^{-1})^\top$ is a homomorphism.

(9) More generally, if (ρ, V) and (σ, W) are two representations of G then $(\tau, \mathrm{Hom}_k(V, W))$ is a representation of G as follows

$$\tau(g)(\alpha) = \sigma(g) \circ \alpha \circ \rho(g)^{-1} \qquad \forall g \in G, \alpha \in \mathrm{Hom}_k(V, W)$$

Exercise: check the details (this is on Example Sheet 1).

Note that if $W = k$ is the trivial representation then we recover the previous example. Moreover if $V = k^n$, $W = k^m$ with the standard bases (so $\mathrm{Hom}_k(V, W) = \mathrm{Mat}_{m,n}(k)$) then $\tau(g)(A) = \sigma(g)A\rho(g)^{-1}$ for all $g \in G$, $A \in \mathrm{Mat}_{m,n}(k)$.

(10) If $\rho : G \to \mathrm{GL}(V)$ is a representation (of G) and $\theta : H \to G$ is a group homomorphism then $\rho\theta : H \to \mathrm{GL}(V)$ is a representation off H. If $H \leq G$ and θ is the inclusion map then we call this the *restriction of* ρ *to* H.

1.3 The Category of Representations

If $\rho : G \to \mathrm{GL}(V)$ is a representation and $\varphi : V \to W$ is a homomorphism of vector spaces then $\sigma : G \to \mathrm{GL}(W)$ defined by $\sigma(g) = \varphi \circ \rho(g) \circ \varphi^{-1}$ for all $g \in G$.

Definition (Isomorphic Representations)**.** We say that $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are *isomorphic representations* if there exists $\varphi : V \to W$ a k-linear isomorphism such that

$$\sigma(g) = \varphi \circ \rho(g) \circ \varphi^{-1} \qquad \forall g \in G$$

We say φ *intertwines* ρ and σ.

Note that:

(1) id_V intertwines ρ and ρ.

(2) If φ intertwines ρ and σ then φ^{-1} intertwines σ and ρ.

(3) If φ intertwines ρ and σ and φ' intertwines σ and τ, then $\varphi' \circ \varphi$ intertwines ρ and τ.

Therefore this definition of isomorphism is an equivalence relation.

Since every vector space is isomorphic to k^d for some $d \geq 0$, every representation is isomorphic to a matrix representation.

If $\rho, \sigma : G \to \mathrm{GL}_d(k)$ are matrix representations of the same degree then an intertwining map from ρ to σ is an invertible matrix $P \in \mathrm{GL}_d(k)$ such that

$$\sigma(g) = P\rho(g)P^{-1} \qquad \forall g \in G$$

Thus matrix representations are isomorphic precisely if they represent the same family of maps with respect to different bases.

Example.

(1) If $G = \{e\}$ then (ρ, V) and (σ, W) are isomorphic if and only if $\dim V = \dim W$.

(2) If $G = (\mathbb{Z}, +)$, then (ρ, V) and (σ, W) are isomorphic if and only if there are bases for V and W such that $\rho(1)$ and $\sigma(1)$ are the same matrix. So

$$\{\text{representations of } (\mathbb{Z}, +)\}/\sim \leftrightarrow \{\text{conjugacy classes of invertible matrices}\}$$

If $k = \mathbb{C}$ the RHS is classified by Jordan Normal Form (more generally rational canonical form).

(3) If $G = C_2 = (\{\pm 1\}, \cdot)$ then

$$\{\text{representations of } C_2\}/\sim \leftrightarrow \{\text{conjugacy classes of matrices } A \text{ such that } A^2 = I\}$$

Since the minimal polynomial of A in RHS divides $X^2 - 1 = (X-1)(X+1)$ (which has distinct roots if characteristic of k is not 2), every such matrix is conjugate to a diagonal matrix and all diagonal entries are 1 or -1.

Exercise: Show that there are precisely $n+1$ isomorphism classes of representations of C_2 of degree n (for any field of characteristic not equal to 2).

(4) If G acts on sets X and Y and there is a bijection $f : X \to Y$ such that $g \cdot (f(x)) = f(g \cdot x)$ for all $g \in G$, $x \in X$, then f induces an isomorphism of representations $\tilde{f} : kX \to kY$, $\tilde{f}(\theta)(y) = \theta(f^{-1}y)$.

Exercise: check this.

Start of lecture 3

Definition (Subrepresentation). Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation and $W \le V$ is a k-linear subspace. Then we say W is *G-invariant* if $\rho(g)(W) \subset W$ for all $g \in G$.

In this case we can define a representation (ρ_W, W) via

$$\rho_W(g)(w) = \rho(g)(w) \qquad \forall g \in G, w \in W.$$

We call (ρ_W, W) a *subrepresentation* of W.

Definition (Proper Subrepresentation). If $W \neq 0$ and $W \neq V$ we say W is a *proper* subrepresentation.

Definition (Irreducible Representation). We say $V \neq 0$ is *irreducible* or *simple* if V has no proper subrepresentations.

Examples

(1) Any 1-dimensional representation of any group is irreducible.

(2) If $G = C_2$, $\rho : G \to \mathrm{GL}_2(k)$, with

$$\rho(-1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(char $k \neq 2$). Then (ρ, k^2) has exactly 2 proper subrepresentations, namely

$$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

Proof. It is easy to see that two given subrepresentations are G-invariant, since the vectors are $\rho(-1)$-eigenvectors. Conversely, any proper subrepresentation must have dimension 1, so is spanned by an eigenvector of $\rho(-1)$ and the eigenspaces of $\rho(-1)$ are those described above. □

(3) If $G = C_2$ then any simple representation of G has dimension 1.

Proof. Suppose $\rho : G \to \mathrm{GL}(V)$ is an irreducible representation of G. The minimal polynomial of $\rho(-1)$ divides $X^2 - 1 = (X-1)(X+1)$ since $\rho(-1)^2 = \mathrm{id}_V$. So it has a linear factor and $\rho(-1)$ has a nonzero eigenvector v. Then $\rho(-1)\langle v\rangle \subseteq \langle v\rangle$ but also $\rho(1)\langle v\rangle \subseteq \langle v\rangle$. So $\langle v\rangle$ is a G-invariant subspace of V. As V is irreducible and $\langle v\rangle \neq 0$, $\langle V\rangle = V$ has dimension 1. □

Note we see that if $\mathrm{char}\, k \neq 2$ there are precisely 2 simple representations of C_2 up to isomorphism and only 1 if $\mathrm{char}\, k = 2$.

(4) If $G = D_6$, then every complex irreducible representation has degree ≤ 2.

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation of G. Let r be a non-trivial rotation in G and s a reflection in G so $r^3 = e = s^2$, $srs = r^{-1}$ and $\{r, s\}$ generate G.

Since $\rho(r)^3 = \mathrm{id}_V$, the minimal polynomial of $\rho(r)$ divides $X^3 - 1$, so $\rho(r)$ has an eigenvector v with eigenvalue λ such that $\lambda^3 = 1$.

Consider $W := \langle v, \rho(s)v\rangle \leq V$, so $\dim W \leq 2$. Now $\rho(s)\rho(s)v = \rho(s^2)v = v$, and $\rho(r)\rho(s)v = \rho(s)\rho(r)^{-1}v = \lambda^{-1}\rho(s)v$, so $\rho(s)W \leq W$ and $\rho(R)W \leq W$. Since r and s generate G, it follows W is a G-invariant subspace of V.

So if V is irreducible, must have $W = V$, so $\dim V \leq 2$ as required. □

Exercise: Show that there are precisely 3 irreducible representations of D_6 up to isomorphism: two of degree 1 and one of degree 2. (Hint: consider proof above and split into cases for each value of λ).

Definition (Quotient Representation)**.** If (ρ, V) is a representation of G and $W \leq V$ is a G-invariant subrepresentation then we can define a *quotient representation* $(\rho_{V/W}, V/W)$ via

$$\rho_{V/W}(g)(v + W) = \rho(g)(v) + W.$$

(This is well-defined since $\rho(g)(W) \subset W \forall g \in G$ means that the choice of coset representative doesn't matter).

We're going to start dropping the ρ now where it doesn't cause confusion.

Definition (G-linear map)**.** If (ρ, V) and (σ, W) are two representations of G we say a k-linear map $\varphi : V \to W$ is *G-linear* if $\varphi \circ g = g \circ \varphi$ for all $g \in G$, i.e. $\varphi \circ \rho(g) = \sigma(g) \circ \varphi$. We write

$$\mathrm{Hom}_G(V, W) = \{\varphi \in \mathrm{Hom}_k(V, W) \mid \varphi \text{ is } G\text{-linear}\}$$

a k-vector subspace of $\mathrm{Hom}_k(V, W)$.

(1) $\varphi \in \mathrm{Hom}_k(V, W)$ is an intertwining map if and only if φ is a bijection and $\varphi \in \mathrm{Hom}_G(V, W)$, since $\varphi \circ \rho(g) = \sigma(g) \circ \varphi \iff \varphi \circ \rho(g) \circ \varphi^{-1} = \sigma(g)$.

(2) If $W \leq V$ is a G-subrepresentation then the natural inclusion map

$$\iota : W \to V, w \mapsto w \in \mathrm{Hom}_G(W, V)$$

and the natural projection map

$$\pi : V \to V/W, v \mapsto v + W \in \mathrm{Hom}_G(V, V/W).$$

(3) Recall that $\mathrm{Hom}_k(V, W)$ is a representation of G via $g \cdot \varphi = g \circ \varphi \circ g^{-1}$ for $g \in G$, $\varphi \in \mathrm{Hom}_k(V, W)$ and $\varphi \in \mathrm{Hom}_G(V, W)$ if and only if $g \cdot \varphi = \varphi$ for all $g \in G$.

Lemma. If U, V and W are representations of a group G with $\varphi_1 \in \mathrm{Hom}_k(V, W)$, $\varphi_2 \in \mathrm{Hom}_k(U, V)$ then

$$g \cdot (\varphi_1 \circ \varphi_2) = (g \circ \varphi_1) \circ (g \circ \varphi_2) \forall g \in G.$$

In particular, if:

- $\varphi_1 \in \mathrm{Hom}_G(V, W)$ then $g \cdot (\varphi_1 \circ \varphi_2) = \varphi_1 \circ (g \cdot \varphi_2)$ for all $g \in G$
- $\varphi_2 \in \mathrm{Hom}_G(U, V)$ then $g \cdot (\varphi_1 \circ \varphi_2) = (g \cdot \varphi_1) \circ \varphi_2$ for all $g \in G$
- $\varphi_1 \in \mathrm{Hom}_G(V, W)$ and $\varphi_2 \in \mathrm{Hom}_G(U, V)$

Then $\varphi_1 \circ \varphi_2 \in \mathrm{Hom}_G(U, W)$.

Proof. With the notation in the statement,

$$(g \cdot \varphi_1) \circ (g \circ \varphi_2) = (g \circ \varphi_1 \circ g^{-1}) \circ (g \circ \varphi_2 \circ g^{-1}) = g \circ (\varphi_1 \circ \varphi_2) \circ g^{-1} = g \cdot (\varphi_1 \circ \varphi_2)$$

Everything else follows immediately. □

Lemma (First Isomorphism for Representations)**.** Suppose V and W are two representations of G and $\varphi \in \mathrm{Hom}_G(V, W)$. Then

(i) $\ker \varphi$ is a subrepresentation of V;

(ii) $\mathrm{im}\, \varphi$ is a subrepresentation of W;

(iii) the linear isomorphism $\overline{\varphi} : V/\ker \varphi \to \mathrm{im}\, \varphi$ given by the first isomorphism theorem for vector spaces is an intertwining map. Thus $V/\ker \varphi$ and $\mathrm{im}\, \varphi$ are isomorphic as representations.

Proof. (i) If $v \in \ker \varphi$ and $g \in G$ then $\varphi(g \cdot v) = g \cdot \varphi(v) = g \cdot 0 = 0$. So $g \cdot v \in \ker \varphi$.

(ii) If $w = \varphi(v) \in \mathrm{im}\, \varphi$ and $g \in G$ then $g \cdot w = g \cdot \varphi(v) = \varphi(g \cdot v) \in \mathrm{im}\, \varphi$.

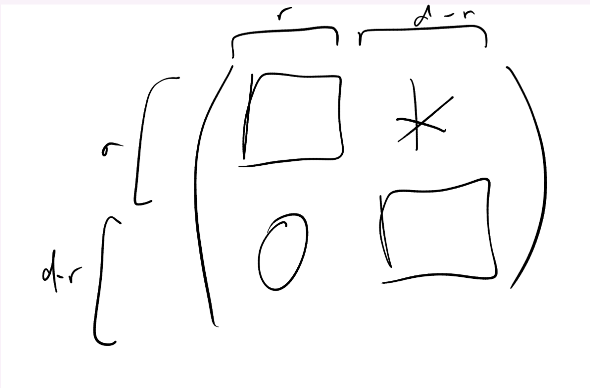
(iii) We know $\overline{\varphi}$ is given by the formula $\overline{\varphi}(v + \ker \varphi) = \varphi(v)$. Then $g \circ \overline{\varphi}(v + \ker \varphi) = g \cdot \varphi(v) = \varphi(g \cdot v) = \overline{\varphi}(g(v + \ker \varphi))$.

□

Proposition. Suppose $\rho : G \to \mathrm{GL}(V)$ and $W \leq V$ is a subspace. Then the following are equivalent

(i) W is a subrepresentation of V.

(ii) There exists a basis of V such that $v_1, \ldots, v_r$ is a basis of W and each $\rho(g)$ with respect to this basis is block upper triangular:



(iii) For every basis $v_1, \ldots, v_d$ of V such that $v_1, \ldots, v_r$ is a basis of W each $\varphi(g)$ with respect to the basis is block upper triangular as in (ii).

Proof. Linear Algebra Example Sheet last year. □

Start of lecture 4

2 Complete Reducibility and Maschke's Theorem

Question: What can a representation V of a group G be decomposed as a direct sum of simple subrepresentations?

Example.

(1) If $G = \{e\}$ the answer is always as seen in lecture 1 since a simple subrepresentation is precisely a 1 dimensional subspace.

(2) If $G = C_2$ and $V = \mathbb{R}^2$,

$$\rho(-1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

we've seen that the proper subrepresentations of V are

$$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

and

$$\mathbb{R}^2 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \oplus \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

is the only such decomposition.

(3) If $G = (\mathbb{Z}, +)$ and $\rho : G \to \mathrm{GL}_2(k)$ is determined by

$$\rho(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then there is precisely one proper subrepresentation

$$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$$

as any such must be spanned by an eigenvector. So this cannot be decomposed in this way.

Definition (direct sum of representations)**.** We say a representation V is a *direct sum of* $(V_i)_{i=1}^k$ if each V_i is a subrepresentation of V and $V = \bigoplus_{i=1}^k V_i$ as vector spaces (recall direct sum notation from Linear Algebra). Given a family of representations $(\rho_i, V_i)_{i=1}^k$ of G, we may define the (external) direct sum to be the representation of G on the vector space

$$V := \bigoplus_{i=1}^k V_i := \{(v_i)_{i=1}^k \mid v_i \in V_i\}$$

with pointwise operations via

$$\rho(g)((v_i)) = (\rho_i(g)v_i)$$

We write

$$(\rho, V) := \bigoplus_{i=1}^k (\rho_i, V_i) = \bigoplus_{i=1}^k \rho_i = \bigoplus_{i=1}^k V_i$$

Examples

(1) Suppose G acts on a finite set X and $X = X_1 \cup X_2$ with $X_1 \cap X_2 = \emptyset$ and X_1, X_2 both G stable ($g \cdot x \in X_i$ if $x \in X_i$, $g \in G$). Then $kX \simeq kX_i \oplus kX_2$ under

$$f \mapsto (f|_{X_1}, f|_{X_2})$$

Internally

$$kX = \{f \mid f(x) = 0 \ \forall x \in X_2\} \oplus \{f \mid f(x) = 0 \ \forall x \in X_1\}$$

More generally if the G-action decomposes into orbit $X = \bigcup_{i=1}^r O_i$, then

$$kX = \bigoplus_{i=1}^r \mathbb{1}_{O_i}(kX) \simeq \bigoplus_{i=1}^r kO_i$$

where $\mathbb{1}_{O_i} : kX \to kX$ given by

$$\mathbb{1}_{O_i}(f)(x) = \begin{cases} f(x) & x \in O_i \\ 0 & \text{otherwise} \end{cases}$$

(2) If G acts transitively on a finite set X then

$$U := \left\{ f \in kX \;\middle|\; \sum_{x \in X} f(x) = 0 \right\} \qquad \text{and} \qquad W := \{f \in kX \mid f \text{ is constant}\}$$

are subrepresentations if $|X| > 1$.

Proof. If $f \in U$ and $g \in G$

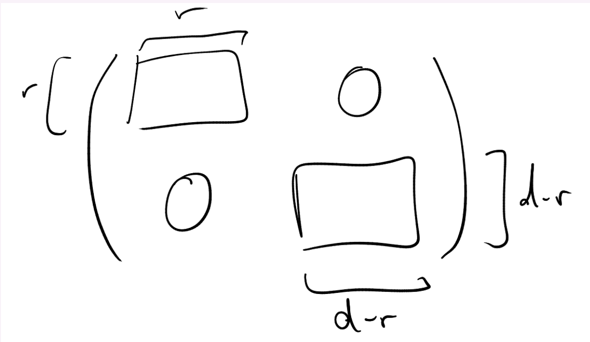
$$\sum_{x \in X} (g \cdot f)(x) = \sum_{x \in X} f(g^{-1}x) = \sum_{x \in X} f(x) = 0$$

since $x \mapsto g^{-1}x$ defines a permutation of X. So $g \cdot f \in U$ and U is G-invariant. Similarly if $f \in W$ then there exists $\lambda \in k$ such that for all $x \in X$, $f(x) = \lambda$ and $(g \cdot f)(x) = f(g^{-1}x) = \lambda$. If $\operatorname{char} k = 0$ then $kX = U \oplus W$ is a direct sum of representations. What happens if $\operatorname{char} k = p > 0$? □

Proposition. Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation and $V = U \oplus W$ as vector spaces. Then the following are equivalent:

(i) $V = U \oplus W$ as representations

(ii) There is a basis $v_1, \ldots, v_d$ of V such that $v_1, \ldots, v_r$ is a basis of U and $v_{r+1}, \ldots, v_d$ is a basis of W and the corresponding matrices $\rho(g)$ are block diagonal



(iii) For every basis $v_1, \ldots, v_d$ of V such that $v_1, \ldots, v_r$ is a basis of U and $v_{r+1}, \ldots, v_d$ is a basis of W, the corresponding matrices $\rho(g)$ are all block diagonal as in (ii).

Proof. Think about it! □

Warning. $\rho : C_2 \to \mathrm{GL}_2(\mathbb{R})$, $\rho(-1) = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$ defines a representation of C_2 (check). The representation on $\mathbb{R}^2$ decomposes as a direct sum of subrepresentations $\langle e_1 \rangle$ and $\langle e_1 + e_2 \rangle$ even though $\rho(-1)$ is not diagonal.

Definition (completely reducible)**.** We say a representation V of a group G is *completely reducible* if

$$V \simeq \bigoplus_{i=1}^{r} V_i$$

for some irreducible representation V_i.

We've seen that $(\mathbb{Z}, +)$ has representations that are not completely reducible, given by

$$\rho(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Lemma. Suppose (ρ, V) is a representation such that for every pair of G-invariant subspaces $W_1, W_2 \le V$ such that $W_1 \le W_2$, there is a G-invariant complement to W_1 in W_2.

Then V is completely reducible.

Proof. By induction on $\dim V$. If $V = 0$ or V is irreducible the result is clear. Otherwise V has a proper G-invariant subspace W. Then by assumption W has a G-invariant complement U in V so $V = U \oplus W$ as representations.

Now $\dim U, \dim W < \dim V$ and U and W inherit the condition on V. So by induction hypothesis,

$$U \cong \bigoplus_{i=1}^{r} U_i \qquad \text{and} \qquad W \cong \bigoplus_{j=1}^{s} W_j$$

for some simple representations $U_1, \dots, U_r$ and $W_1, \dots, W_s$. Then

$$V \simeq \bigoplus_{i=1}^{r} U_i \oplus \bigoplus_{j=1}^{s} W_j$$

as required. □

Recall, if V is a $\mathbb{C}$-vector space then a *Hermitian inner product* is a positive definite Hermitian sesquilinear form, i.e. $(\bullet, \bullet) : V \times V \to \mathbb{C}$ such that

(i) Sesquilinear:

$$(ax + by, z) = \overline{a}(x, z) + \overline{b}(y, z) \qquad \forall x, y, z \in V, a, b \in \mathbb{C}$$

$$(x, ay + bz) = a(x, y) + b(x, z) \qquad \forall x, y, z \in V, a, b \in \mathbb{C}$$

(ii) $(x,y) = \overline{(y,x)}$ for all $x, y \in V$ (Hermitian)

(iii) $(x,x) > 0$ $\forall x \in V \setminus \{0\}$ (positive definite)

The standard inner product on $\mathbb{C}^n$ is given by

$$(x,y) = \sum_{i=1}^{n} \overline{x_i} y_i$$

Recall the *unitary group* $U(n)$ is the subgroup of $\mathrm{GL}_n(\mathbb{C})$ given by

$$\begin{aligned} U(n) &= \{A \in \mathrm{GL}_n(\mathbb{C}) \mid \overline{A^\top} A = I\} \\ &= \{A \in \mathrm{GL}_n(\mathbb{C}) \mid (Ax, Ay) = (x,y) \ \forall x, y \in \mathbb{C}^n\} \end{aligned}$$

Definition (unitary representation)**.** A $\mathbb{C}$ representation of a group G is *unitary* if there exists a basis $e_1, \dots, e_n$ of V so that the corresponding matrix representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ has image contained in $U(n)$.

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Definition (G-invariant inner product)**.** A Hermitian inner product on a representation V of G is G-invariant if $(gx, gy) = (x,y)$ for all $g \in G$, $x, y \in V$.

Equivalently if $(gx, gx) = (x,x)$ $\forall g \in G, x \in V$.

Proposition. A representation (ρ, V) of G is unitary if and only if V has a G-invariant inner product.

Proof. If (ρ, V) is contrary let $e_1, \dots, e_n$ be a basis with respect to which each $\rho(g) \in U(n)$. Now,

$$\left(\sum_i \lambda_i e_i, \sum_j \mu_j e_j \right) = \sum_i \overline{\lambda_i} \mu_i$$

is a G-invariant inner product on V.

Conversely, if V has a G-invariant inner product$(\bullet, \bullet)$, we can find an orthonormal basis $v_1, \dots, v_n$ of V with respect to $(\bullet, \bullet)$. Then

$$\left(\sum_i \lambda_i v_i, \sum_j \mu_j v_j \right) = \sum_i \overline{\lambda_i} \mu_i \qquad \forall \lambda, \mu \in \mathbb{C}^n$$

i.e. $(\bullet, \bullet)$ is the standard inner product with respect to $v_1, \dots, v_n$, so since it is G-invariant each $\rho(g)$ is unitary with respect to this basis. $\square$

Note subrepresentations of unitary representations are thus unitary since we can restrict a G-invariant inner product.

Theorem. Suppose (ρ, V) is a unitary representation of a group G. Then every subrepresentation W of V has a G-invariant complement. Thus V is completely reducible.

Proof. Since V is unitary it has a G-invariant inner product $(\bullet, \bullet)$. If W is a subrepresentation then

$$W^\perp = \{v \in V \mid (v, w) = 0 \ \forall w \in W\}$$

is a vector space complement to W in V by standard linear algebra. Moreover, if $g \in G$, $v \in W^\perp$ and $w \in W$ then

$$(gv, w) = (v, g^{-1}w) = 0$$

since $g^{-1}w \in W$, so $gv \in W^\perp$ and $W^\perp$ is a subrepresentation as required.

The last part follows from lemma proved last time. $\square$

Theorem (Maschke's Theorem)**.** Let G be a finite group and (ρ, V) is a representation of G over k, a field of characteristic zero. Suppose $W \leq V$ is a subrepresentation. Then W has a G-invariantcomplement in V. In particular, V is completely reducible.

Key idea: If (ρ, V) is a representation of a finite group G over a field k then for all $v \in V$,

$$\sum_{g \in G} g \cdot v \in V^G = \{v \in V \mid g \cdot v = v \ \forall g \in G\} \leq V$$

Proof of Key idea. If $h \in G$,

$$h \left(\sum_{g \in G} g \cdot v \right) = \sum_{g \in G} (hg) \cdot v = \sum_{g' \in G} g' \cdot v$$

since $G \to G$, $g \mapsto hg$ is a permutation of G and $h : V \to V$ is linear. $\square$

Proposition (Weyl's unitary trick)**.** If V is a $\mathbb{C}$-representation of a finite group G then V has a G-invariant inner product. Thus Maschke's Theorem is true over $\mathbb{C}$.

Proof. Pick any Hermitian inner product $\langle \bullet, \bullet \rangle$ on V. Then we can define a new inner product on V via

$$(x, y) = \sum_{g \in G} \langle gx, gy \rangle$$

It is easy to see that $(\bullet, \bullet)$ is a Hermitian inner product because $\langle \bullet, \bullet \rangle$ is since, for example, if $a, b \in \mathbb{C}$ and $x, y, z \in V$ then

$$\begin{aligned}
(x, ay + bz) &= \sum_{g \in G} \langle gx, g(ay + bz) \rangle \\
&= \sum_{g \in G} \langle gx, ag(y) + bg(z) \rangle \\
&= \sum_{g \in G} (a \langle gx, gy \rangle + b \langle gx, gz \rangle) \\
&= a(x, y) + b(x, z)
\end{aligned}$$

But now if $h \in G$ and $x, y \in V$,

$$(hx, hy) = \sum_{g \in G} \langle ghx, ghy \rangle = \sum_{g' \in G} \langle g'x, g'y \rangle = (x, y)$$

since $g \mapsto gh$ is a permutation of G. □

Remark. This proof can be phrased as follows:

(i) $\mathrm{Herm}(V) = \{\text{Hermitian sesquilinear forms}\}$ is naturally an $\mathbb{R}$-vector space.

(ii) $G \to \mathrm{Aut}(\mathrm{Herm}(V))$, $g \cdot (\bullet, \bullet)|_{(x,y)} = (g^{-1}x, g^{-1}y)$ defines an $\mathbb{R}$-linear representation of G.

(iii) All $\mathbb{R}^{>0}$-linear combinations of positive definite elements of $\mathrm{Herm}(V)$ are positive definite.

(iv) The key idea transforms an inner product into a G-invariant one.

It follows that studying $\mathbb{C}$-representations of a finite group is the same as studying unitary representations of the group.

Corollary. Every finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ is conjugate to a subgroup of $U(n)$.

Proof. If $G \leq \mathrm{GL}_n(\mathbb{C})$ then the inclusion map $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ is a representation. By the unitary trick, there exists a basis for $\mathbb{C}^n$ with respect to which each $\rho(g) \in U(n)$, i.e. $\exists P \in \mathrm{GL}_n(\mathbb{C})$ such that $\forall g \in G$, $P\rho(g)P^{-1} \in U(n)$. □

We now generalise to all char 0 fields.

Proof of Maschke's Theorem. Idea: if $\pi : V \to V$ is a projection, i.e. $\pi^2 = \pi$, then $V = \operatorname{Im}\pi \oplus \ker\pi$ as vector spaces. If π is G-linear then $\ker\pi$ and $\operatorname{Im}\pi$ are subrepresentations. So $\operatorname{Im}\pi$ has a G-invariant complement. So we pick a projection onto W and average it.

Let $\pi : V \to V$ be any k-linear projection onto W (so $\pi(w) = w$ for all $w \in W$ and $\operatorname{Im}\pi = W$). Recall that $\mathrm{Hom}_k(V, V)$ is a representation of G via $g \cdot \varphi = g \circ \varphi \circ g^{-1}$. Let $\pi^G = \frac{1}{|G|}\sum_{g\in G}(g \circ \pi) \in \mathrm{Hom}_G(V, V)$ by the key idea.

Moreover, $\operatorname{Im}\pi^G \leq W$ since $g \circ \pi g^{-1}(v) \in W$ for all $v \in V, g \in G$ and

$$\pi^G(v) = \sum_{g\in G} \frac{1}{|G|} g \circ \pi \circ g^{-1}(v)$$

and if $w \in W$ then

$$\pi^G(w) = \frac{1}{|G|}\sum_{g\in G} g \circ \pi \circ g^{-1}(w) = \frac{1}{|G|}\sum_{g\in G} g \circ g^{-1}(w) = w$$

since $g^{-1}(w) \in W$ $\forall g \in G, w \in W$. So π^G is a G-invariant linear projection onto W and $\ker\pi$ is a G-invariant complement to W in V. □

Remark.

(1) We can explicitly compute π^G and $\ker \pi^G$ via formula

$$\pi^G = \frac{1}{|G|} \sum_{g \in G} g \cdot \pi$$

(2) Notice we only used $\mathrm{char}(k) = 0$ when we divided by $|G|$. So in fact the result holds whenever $\mathrm{char}(k) \nmid |G|$.

(3) As an extension of our key idea, for any G-representation V (and $\mathrm{char}\, k \nmid |G|$),

$$\pi : v \mapsto \frac{1}{|G|} \sum_{g \in G} gv$$

is a projection in $\mathrm{Hom}_G(V, V)$ onto V^G. Notice $\dim V^G = \mathrm{Tr}\,\pi = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(g)$ since Tr is linear.

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Question: Suppose

$$V \simeq \bigoplus_{i \in I} V_i \simeq \bigoplus_{j \in J} W_j$$

with V_i, W_j are irreducible representations of G. Can these decompositions be different?

3 Schur's Lemma

Recall that if V is a vector space then $\mathrm{Aut}(V) = \mathrm{GL}_{\dim V}(k)$.

Theorem (Schur's Lemma)**.** Suppose V and W are irreducible representations of a group G. Then

(i) Every $\varphi \in \mathrm{Hom}_G(V, W)$ is either 0 or an isomorphism;

(ii) if k is algebraically closed then $\dim \mathrm{Hom}_G(V, W)$ is 0 or 1.

Proof.

(i) Note if $\varphi \in \mathrm{Hom}_G(V, W) \setminus \{0\}$ then $\ker \varphi \lneq V$ is a G-invariant subspace. So as V is irreducible, $\ker \varphi = 0$. Similarly $0 \neq \mathrm{Im}\, \varphi \leq W$ is a G-invariant subspace so $\mathrm{Im}\, \varphi = W$. Then by First Isomorphism for Representations, φ is an isomorphism.

(ii) Suppose $\varphi_1, \varphi_2 \in \mathrm{Hom}_G(V, W) \setminus \{0\}$. By (i), φ_1 is an isomorphism such that $\varphi_1^{-1} \circ \varphi_2 \in \mathrm{Hom}_G(V, V)$. But then as k is algebraically closed, every element of $\mathrm{Hom}_k(V, V)$ has an eigenvalue. In particular, $\exists \lambda \in k$ such that $\ker(\varphi_1^{-1} \circ \varphi_2 - \lambda \mathrm{id}_V) \neq 0$. But $\varphi_1^{-1} \circ \varphi_2 - \lambda \mathrm{id}_V \in \mathrm{Hom}_G(V, V)$. So $\ker(\varphi_1^{-1} \circ \varphi_2 - \lambda \mathrm{id}_V)$ is G-invariant so is equal to V (since V irreducible), i.e. $\varphi_1^{-1} \circ \varphi_2 = \lambda \mathrm{id}_V$ and $\varphi_2 = \lambda \varphi_1$, i.e. $\mathrm{Hom}_G(V, W) = k\varphi_1$. □

This says that in particular, $\mathrm{Hom}_G(V, V) = k$ (such that $k = \overline{k}$), so an irreducible representation is rigid in the same sense that a 1-dimensional vector space is rigid since their automorphism groups are the same.

Proposition. If V, V_1, V_2 are representations of G then

(a) $\mathrm{Hom}_G(V, V_1 \oplus V_2) \simeq \mathrm{Hom}_G(V, V_1) \oplus \mathrm{Hom}_G(V, V_2)$

(b) $\mathrm{Hom}_G(V_1 \oplus V_2, V) \simeq \mathrm{Hom}_G(V_1, V) \oplus \mathrm{Hom}_G(V_2, V)$

Proof.

(a) There are natural G-linear inclusion maps $\iota_i : V_i \to V_1 \oplus V_2$ for $i = 1, 2$. These induce by post-composition G-linear maps $\mathrm{Hom}_k(V, V_i) \to \mathrm{Hom}_k(V, V_1 \oplus V_2)$, $f \mapsto \iota_i \circ f$. Together these give a linear isomorphism

$$\mathrm{Hom}_k(V, V_1) \oplus \mathrm{Hom}_k(V, V_2) \to \mathrm{Hom}_k(V, V_1 \oplus V_2) \qquad (f_1, f_2) \mapsto \iota_1 f_1 + \iota_2 f_2$$

Since ι_1, ι_2 are G-linear, this is an intertwining map

$$g \cdot (\iota_1 f_1 + \iota_2 f_2) = \iota_1 \circ (g \cdot f_1) + \iota_2 \circ (g \cdot f_2)$$

Since in general if $\varphi : U \to W$ is an intertwining map between representations of G it induces an isomorphism on G-fixed points since $g \cdot (\varphi(u)) = \varphi(u) \iff g \cdot u = u$ (φ injective). It follows that there is an induced isomorphism as in (a).

(b) Since the natural projection maps $\pi_i : V_1 \oplus V_2 \to V_i$, $(v_1, v_2) \mapsto v_i$ for $i = 1, 2$ are also G-linear and induce a G-linearisomorphism

$$\mathrm{Hom}_k(V_1, V) \oplus \mathrm{Hom}_k(V_2, V) \to \mathrm{Hom}_k(V_1 \oplus V_2, V), \qquad (f_1, f_2) \mapsto f_1 \circ \pi_1 + f_2 \circ \pi_2$$

and again taking G-invariants gives the result. □

Corollary. If $V \simeq \bigoplus_{i \in I} V_i$ and $W \simeq \bigoplus_{j \in J} W_j$ then

$$\mathrm{Hom}_G(V, W) = \bigoplus_{i \in I} \bigoplus_{j \in J} \mathrm{Hom}_G(V_i, W_j)$$

Proof. This follows from the previous proposition and a simple induction argument. □

Corollary. If $k = \overline{k}$ and $V \simeq \bigoplus_{i=1}^r V_i$ is a decomposition of V as a direct sum of simple representations then for each simple representation W of G

$$|\{i : V_i \simeq W\}| = \dim \mathrm{Hom}_G(V, W) = \dim \mathrm{Hom}_G(W, V).$$

and does not depend on the choice of decomposition.

Proof. By the last result $\dim \mathrm{Hom}_G(V, W) = \bigoplus_{i=1}^r \mathrm{Hom}_G(V_i, W)$ so $\dim \mathrm{Hom}_G(V, W) = \sum_{i=1}^r \dim \mathrm{Hom}_G(V_i, W)$ and similarly $\dim \mathrm{Hom}_G(W, V) = \sum_{i=1}^r \dim \mathrm{Hom}_G(W, V_i)$. Thus it suffices to show

$$\dim \mathrm{Hom}_G(V_i, W) = \dim \mathrm{Hom}_G(W, V_i) = \begin{cases} 1 & W \simeq V_i \\ 0 & W \not\simeq V_i \end{cases}$$

This is part of the statement of Schur's Lemma when $k = \overline{k}$. □

Exercise for enthusiasts: Give a version of this corollary when $k \neq \overline{k}$.

Important question: How can we compute these numbers $\dim \operatorname{Hom}_G(V, W)$? Note our final remark last lecture may help us at least when $\operatorname{char}(k) = 0$ since it said $\dim V^G = \frac{1}{|G|}\sum_{g\in G} \operatorname{Tr} \rho(g)$. So we need to understand these traces for the representation $\operatorname{Hom}_k(V, W)$.

Corollary. If G is an abelian group then every irreducible $\mathbb{C}$-representation of G has degree 1.

Proof. Let (ρ, V) be a complex irreducible representation of G. For each $g \in G$, $\rho(g)\rho(h) = \rho(h)\rho(g)$ for all $h \in G$. So $\rho(g) \in \operatorname{Hom}_G(V, V) = \mathbb{C}\mathrm{id}_V$ by Schur's Lemma. Now if $v \in V \setminus \{0\}$ then $\rho(g)\langle v\rangle \le \langle v\rangle\ \forall g \in G$ so $\langle v\rangle$ is a G-invariant subspace of V, so $\langle v\rangle = V$ since V is irreducible. □

Corollary. If G is a finite group with a faithful irreducible representation over an algebraically closed field k, then $Z(G)$ is cyclic.

Proof. Let (ρ, V) be a faithful irreducible representation of G and $z \in Z(G)$. Then $\rho(z)\rho(g) = \rho(g)\rho(z)$ for all $g \in G$, i.e. $\varphi(z) \in \operatorname{Hom}_G(V, V) = k\mathrm{id}_V$ by Schur's Lemma. We can write $\rho(z) = \lambda_z \mathrm{id}_V$ for $\lambda_z \in k$, and $\lambda_{z_1 z_2} = \lambda_{z_1} \cdot \lambda_{z_2}$ for all $z_1, z_2 \in Z(G)$, and $\lambda_e = 1$. So $Z(G) \to k^\times$, $z \mapsto \lambda_z$ is a faithful representation of $Z(G)$ since V is faithful, i.e. $Z(G)$ is isomorphic to a finite subgroup of $k^\times$, and any such subgroup is cyclic. □

Example. $G = C_4 = \langle x\rangle$. 1-dimensional $\mathbb{C}$-representations of G are given by

1	x	x^2	x^3
1	1	1	1
1	i	-1	$-i$
1	-1	1	-1
1	$-i$	-1	i

Example. $C_2 \times C_2 = \langle x \rangle \times \langle y \rangle$. 1-dimensional $\mathbb{C}$-representations of G are given by

1	x	y	xy
1	1	1	1
1	1	-1	-1
1	-1	1	-1
1	-1	-1	1

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Proposition. If G is a finite abelian group then G has exactly $|G|$ irreducible $\mathbb{C}$-representations.

Proof. We saw last lecture that all irreducible representations of an abelian group have degree 1.

$$S = \{\text{simple representations of } G\}/\sim \leftrightarrow \mathrm{Hom}(G, \mathbb{C}^\times)$$

Moreover, if $G = H \times K$, then

$$\mathrm{Hom}(G, \mathbb{C}^\times) \leftrightarrow \mathrm{Hom}(H, \mathbb{C}^\times) \times \mathrm{Hom}(K, \mathbb{C}^\times) \qquad \varphi \mapsto (\varphi|_H, \varphi|_K)$$

(need $\mathbb{C}^\times$ is abelian to get surjectivity). Now by structure theorem for finite abelian groups, $G \simeq C_{n_1} \times \cdots \times C_{n_r}$ for some $n_1, \dots n_r \in \mathbb{N}$. So by a simple induction argument we can reduce to the case where G is cyclic, $G \cong C_n = \langle x \rangle$ say.

Then $\rho \in \mathrm{Hom}(G, \mathbb{C}^\times)$ is determined by $\rho(x)$ and $\rho(x)^n = 1$, i.e. $\rho(x)$ is an n-th root of unity. Moreover, for $j = 0, \dots, n-1$, $x^k \mapsto e^{2\pi ijk/n}$ defines a 1-dimensional representation of G. □

Lemma. If (ρ_1, V_1) and (ρ_2, V_2) are non-isomorphic 1-dimensional representations of a finite group G, then

$$\sum_{g \in G} \rho_1(g^{-1}) \rho_2(g) = 0$$

(Note $\rho_1(g^{-1}) = \rho_1(g)^{-1}) = \overline{\rho_1(g)}$ since $\rho_1(g)^j = 1$ for some j if $k = \mathbb{C}$).

Proof. We've seen that $\mathrm{Hom}_k(V_1, V_2)$ is a representation of G via $g \cdot \varphi = \rho_2(g) \circ \varphi \circ \varphi_1(g^{-1})$. Moreover,

$$\sum_{g \in G} g \cdot \varphi \in \mathrm{Hom}_G(V_1, V_2) = 0$$

by our key idea from lecture 5 and Schur's Lemma. Pick an isomorphic $\varphi \in \mathrm{Hom}_k(V_1, V_2)$ and then

$$0 = \sum_{g \in G} \rho_2(g)\varphi\rho_1(g^{-1}) = \left(\sum_{g \in G} \rho_1(g^{-1})\rho_2(g)\right)\varphi$$

So as φ is injective, we're done. □

Definition (isotypic component)**.** If V is a completely reducible representation of a group G and W is any simple representation of G, the *W-isotypic component* of V is the smallest subrepresentation of V containing all subrepresentations of V isomorphic to W.

This exists since if $(V_i)_{i \in I}$ are subrepresentations of V containing all subrepresentations of V isomorphic to W then $\bigcap_{i \in I} V_i$ is another (or we can simply take the vector space sum of all subrepresentations isomorphic to W).

Definition (unique isotypical decomposition)**.** We say V has a *unique isotypical decomposition* if V is the direct sum of its W-isotypic component as W goes over all simple representations of G (up to isomorphism).

Corollary. If G is a finite abelian group, then every $\mathbb{C}$-representation of V has a unique isotypical decomposition.

Proof. For each homomorphism $\theta_i : G \to \mathbb{C}^\times$, $i = 1, \ldots, |G|$ (i.e. each simple representation of G) we can define

$$W_i = \{v \in V \mid g \cdot v = \theta_i(g)v \ \forall g \in G\}$$

the θ_i-isotypic component of V. Since V is completely reducible, $V = \sum_{i=1}^{|G|} W_i$. We need to show that if $\sum_{i=1}^{|G|} w_i = 0$ with $w_i \in W_i$ for each i, then $w_i = 0$ for each i. But

$$\sum_{i=1}^{|G|} w_i = 0 \implies 0 = g \cdot \sum_{i=1}^{|G|} w_i = \sum_{i=1}^{|G|} \theta_i(g)w_i \ \forall g \in G$$

Then for each j,

$$\sum_{i=1}^{|G|} \left(\sum_{g \in G} \theta_j(g^{-1})\theta_i(g)\right) w_i = 0$$

But by the previous Lemma,

$$\text{LHS} = \sum_{g \in G} \theta_j(g^{-1})\theta_j(g) w_j = |G| w_j$$

Thus $w_j = 0$ as required. $\square$

This proof also works when $\mathbb{C}$ is replace by any other algebraically closed field with characteristic 0.

You will extend this to all finite groups on Example Sheet 2.

4 Characters

Summary so far: We want to classify all representations of a given (finite) group G. We've seen when G is finite and $\operatorname{char} k = 0$ then every representation decomposes as $V \cong \bigoplus_{i=1}^r n_i V_i$ with $V_1, \dots, V_r$ simple and pairwise non-isomorphic and $n_i \geq 0$.

Moreover, if $k = \overline{k}$ then $n_i = \dim \operatorname{Hom}_G(V_i, V)$. Next we want to discuss how to classify irreducible ($\mathbb{C}$-)representations of a finite group and understand how to *compute* the n_i given V. We'll do both by character theory.

4.1 Definitions

Definition (Character)**.** Given a representation $\rho : G \to \mathrm{GL}(V)$, the *character* of ρ is the function $G \to k$, $\chi = \chi_\rho = \chi_V : G \to k$, $g \mapsto \operatorname{Tr} \rho(g)$.

Since for matrices $\operatorname{Tr}(BA) = \operatorname{Tr}(AB)$, the character does not depend on a choice of basis

$$\operatorname{Tr}(PAP^{-1}) = \operatorname{Tr}(AP^{-1}P) = \operatorname{Tr}(A)$$

Similarly, isomorphic representations have the same character.

Example. Let $G = D_6 = \langle s, t : s^2 = t^3 = e, sts = t^{-1} \rangle$, the dihedral group of order 6. Let $G \to \mathrm{GL}_2(\mathbb{R})$ be the action of G by symmetries of a triangle. To compute χ_ρ we just need to know the eigenvalues of each matrix $\rho(g)$. Each reflection (element st^i) has eigenvalue $1, -1$ so $\chi(st^i) = 0$ for all i. The eigenvalues of the non-trivial rotation must be non-trivial cube roots of 1 and sum to be a real number. Thus $\chi(t) = \chi(t^2) = e^{2\pi i/3} + e^{4\pi i/3} = -1$. Also, $\chi(e) = 2$.

Proposition. Let (ρ, V) be a representation of G. Then

(i) $\chi_V(e) = \dim V$

(ii) $\chi_V(g) = \chi_V(hgh^{-1})$ $\forall g, h \in G$

(iii) If W is another representation, then $\chi_{V \oplus W} = \chi_V + \chi_W$

(iv) If V is unitary then $\chi_V(g^{-1}) = \overline{\chi_V(g)} \forall g \in G$

Proof.

(i) $\chi(e) = \operatorname{Tr} \mathrm{id}_V = \dim V$

(ii) $\rho(hgh^{-1}) = \rho(h)\rho(g)\rho(h^{-1})$ thus $\rho(g)$ and $\rho(hgh^{-1})$ are conjugate in $\mathrm{GL}(V)$ so have the same trace.

(iii) Clear

(iv) By choosing a basis we may view ρ as a homomorphism $G \to U(n)$. Then $\rho(g^{-1}) = \rho(g)^{-1} = \overline{\rho(g)}^\top$. So $\operatorname{Tr}\rho(g^{-1}) = \overline{\operatorname{Tr}(\rho(g))}$ since Tr is transpose invariant. □

The characters contain very little data: an element of k for each conjugacy class in G. But when G is finite and $k = \mathbb{C}$, it contains all we need to reconstruct V up to isomorphism.

Definition (Class function)**.** A function $f : G \to k$ is a *class function* if $f(hgh^{-1}) = f(g)\ \forall g, h \in G$. We'll write $\mathcal{C}_G$ for the k-vector space of class functions.

Notice that if $O_1, \ldots O_r$ are the conjugacy classes in G then the indicator functions

$$\mathbb{1}_{O_i} : G \to k, \qquad g \mapsto \begin{cases} 1 & g \in O_i \\ 0 & g \notin O_i \end{cases}$$

form a basis for $\mathcal{C}_G$. So $\dim \mathcal{C}_G = \#$conjugacy classes.

Start of lecture 8

4.2 Orthogonality of characters

Assume G is always a finite group and $k = \mathbb{C}$.

Recall

$$\mathcal{C}_G := \{f : G \to \mathbb{C} : f(hgh^{-1}) = f(g)\ \forall g, h \in G\} \leq \mathbb{C}G$$

and if $O_1, \ldots, O_r$ are the conjugacy classes then the indicator functions $\mathbb{1}_{O_1}, \ldots, \mathbb{1}_{O_r}$ are a basis.

We can define a Hermitian inner product in $\mathcal{C}_G$ (restricted from one on $\mathbb{C}G$) via

$$\langle f_1, f_2 \rangle_G := \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g)$$

The indicator functions $\mathbb{1}_{O_i}$ are pairwise orthogonal with respect to $\langle \bullet, \bullet \rangle_G$ and moreover,

$$\langle \mathbb{1}_{O_i}, \mathbb{1}_{O_i} \rangle_G = \frac{1}{|G|}|O_i| = \frac{1}{|C_G(x_i)|}$$

for any $x_i \in O_i$. Thus if $x_1, \ldots, x_r$ are representatives of $O_1, \ldots, O_r$, then

$$\langle f_1, f_2 \rangle_G = \sum_{i=1}^{r} \frac{1}{|C_G(x_i)|} \overline{f_1(x_i)} f_2(x_i)$$

for $f_1, f_2 \in \mathcal{C}_G$.

Example. If $G = D_6 = \langle s, t \mid s^2 = t^3 = e, sts^{-1} = t^{-1} \rangle$ then

$$\langle f_1, f_2 \rangle_{D_6} = \frac{1}{6}\overline{f_1}(e) f_2(e) + \frac{1}{2}\overline{f_1(s)} f_2(s) + \frac{1}{3}\overline{f_1(t)} f_2(t).$$

In particular if V is the natural 2-dimensional representation of D_6 and $\mathbb{C}$ is the trivial representation then

$$\chi_{\mathbb{C}} = \mathbb{1}_G$$

$$\chi_V(e) = 2, \chi_V(s) = 0, \chi_V(t) = -1$$

$$\langle \chi_{\mathbb{C}}, \chi_{\mathbb{C}} \rangle_{D_6} = \frac{1}{6} \cdot 1 + \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 = 1$$

$$\langle \chi_V, \chi_V \rangle_{D_6} = \frac{1}{6} 2^2 + \frac{1}{2} 0^2 + \frac{1}{3}(-1)^2 = 1$$

$$\langle \chi_{\mathbb{C}}, \chi_V \rangle_{D_6} = \frac{1}{6} 2 + \frac{1}{2} 0 + \frac{1}{3}(-1) = 0$$

Lemma. If V and W are (unitary) representations of G then

$$\chi_{\mathrm{Hom}_k(V,W)}(g) = \overline{\chi_V}(g) \chi_W(g) \qquad \forall g \in G$$

Proof. Given $g \in G$ we can choose bases $v_1, \ldots, v_n$ of V and $w_1, \ldots, w_m$ of W such that $g \cdot v_i = \lambda_i v_i$ and $g \cdot w_j = \mu_j w_j$ for some $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n \in \mathbb{C}$. Then the functions $\alpha_{ij}(v_k) = \delta_{jk} w_i$ extend to linear maps $\alpha_{ij} \in \mathrm{Hom}_k(V, W)$ that form a basis (with respect to given basis α_{ij} as represented by a matrix with 0s everywhere except a single 1 in the i, j position).

$$\begin{aligned} (g \cdot \alpha_{ij})(v_k) &= g(\alpha_{ij}(g^{-1} v_k)) \\ &= g(\alpha_{ij}(\lambda_k^{-1} v_k)) \\ &= \lambda_k^{-1}(g(\delta_{jk} w_i)) \\ &= \lambda_k^{-1} \mu_i \delta_{jk} w_i \end{aligned}$$

i.e. $g \cdot \alpha_{ij} = \lambda_j^{-1} \mu_i \alpha_{ij}$. So $\chi_{\mathrm{Hom}_k(V,W)}(g) = \sum_{i,j} \lambda_j^{-1} \mu_i = \chi_V(g^{-1}) \chi_W(g) = \overline{\chi_V(g)} \chi_W(g)$. $\square$

Lemma. If U is a representation of G then

$$\dim U^G = \frac{1}{|G|}\sum_{g\in G}\chi_V(g) = \langle \mathbb{1}_G, \chi_V\rangle_G.$$

Proof. We've seen before that $\pi : U \to U_j$, $\pi(u) = \frac{1}{|G|}\sum_{g\in G} g\cdot u$ is a projection of U onto U^G and so $\dim U^G = \operatorname{Tr}\pi = \frac{1}{|G|}\sum_{g\in G}\operatorname{Tr}(g) = \frac{1}{|G|}\sum_{g\in G}\chi_U(g)$. □

Proposition. If V and W are any representations of G then

$$\dim \operatorname{Hom}_G(V,W) = \langle \chi_V, \chi_W\rangle_G$$

Proof.

$$\begin{aligned}\dim \operatorname{Hom}_G(V,W) &= \dim(\operatorname{Hom}_k(V,W))^G \\ &= \langle \mathbb{1}_G, \chi_{\operatorname{Hom}_k(V,W)}\rangle_G \\ &= \frac{1}{|G|}\sum_{g\in G}\overline{\chi_V}(g)\chi_W(g) \\ &= \langle \chi_V, \chi_W\rangle_G\end{aligned}$$

□

Theorem (Orthogonality of characters)**.** If V and W are irreducible $\mathbb{C}$-representations of G, then

$$\langle \chi_V, \chi_W\rangle = \begin{cases} 1 & V \simeq W \\ 0 & V \not\simeq W\end{cases}$$

Proof. Use the fact that $\dim \operatorname{Hom}_G(V,W) = \langle \chi_V, \chi_W\rangle_G$ and Schur's Lemma. If $\chi_V = \chi_W$ with V and W irreducible then $\langle \chi_V, \chi_W\rangle_G = \langle \chi_V, \chi_V\rangle_G > 0$ since $\chi_V \neq 0$ so $\dim \operatorname{Hom}_G(V,W) > 0$ and $V \simeq W$ by Schur's Lemma. □

Corollary. If (ρ, V) is a representation of G then

$$V \simeq \bigoplus_{W \text{ irreducible representations of } G/\simeq} \langle \chi_W, \chi_\rho\rangle_G W$$

In particular if σ is a another representation with $\chi_\rho = \chi_\sigma$ then $\sigma \simeq \rho$.

Proof. By Maschke's Theorem there are $n_w \geq 0$ such that

$$V \simeq \bigoplus_{W \text{ irreducible}} n_w W$$

Moreover, we've seen before that $n_w = \dim \operatorname{Hom}_G(W, V) = \langle \chi_W, \chi_\rho \rangle_G$ by the previous Proposition. So the first part follows.

Since

$$\bigoplus_{W \text{ irreducible}} \langle \chi_W, \chi_\rho \rangle_G W$$

depends only on χ_ρ (up to isomorphism), the second part follows. □

Notice this proof depends on Maschke's Theorem / completely reducible as well as orthogonality of characters. For example, if we have the two representations of $(\mathbb{Z}, +)$ determined by

$$\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

they are not isomorphism but have the same characters. $\rho(n) = \sigma(n) = 2\ \forall n \in \mathbb{Z}$. Indeed both have trivial subrepresentations $\langle e_1 \rangle$ with trivial quotients. Slogan: "characters cannot see gluing data".

Corollary. If ρ is a $\mathbb{C}$-representation of G with character χ then

$$\rho \text{ is irreducible} \iff \langle \chi, \chi \rangle_G = 1.$$

Proof.

$\Rightarrow$ Is clear from orthogonality of characters.

$\Leftarrow$ ρ decomposes as $\rho \simeq \bigoplus n_w W$ with $n_w \geq 0$. Then $\chi = \sum n_w \chi_W$ but

$$\langle \chi, \chi \rangle_G = \sum n_w^2$$

so $\langle \chi, \chi \rangle_G = 1 \implies \chi = \chi_w$ for some W. □

This is a good way to prove irreducibility.

Example. If V is the natural 2-dimensional representation of D_6 then $\langle \chi_V, \chi_V \rangle_{D_6} = 1$ and so V is irreducible.

Theorem (The character table is square)**.** The irreducible characters of G form an orthonormal basis of $\mathcal{C}_G$ with respect to $\langle \bullet, \bullet \rangle_G$.

Proof. We've already seen that the irreducible characters form an orthonormal set so it remains to prove that they span. Let $I = \langle \chi_1, \ldots, \chi_s \rangle$ be their $\mathbb{C}$-linear span.

It suffices to show that

$$I^\perp := \{g \in \mathcal{C}_G : \langle f, \chi_i \rangle_G = 0 \text{ for } i = 1, \ldots, s\} = 0.$$

For $f \in \mathcal{C}_G$ and (ρ, V) a representation of G show

$$\varphi_{f,V} = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} \rho(g) \in \mathrm{Hom}_G(V, V)$$

and use Schur's Lemma to show if $f \in I^\perp$ then $\varphi_{f,v} = 0$. Finally show $0 = \varphi_{f,\mathbb{C}G} \delta_e = \frac{1}{|G|} \overline{f}$ so $f = 0$.

$$I = \langle \chi_1, \ldots, \chi_s \rangle$$

where $\chi_1, \ldots, \chi_s$ are the irreducible characters. For $f \in \mathcal{C}_G$ and a representation (ρ, V) we can define

$$\varphi_{f,V} = \varphi = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} \rho(g) \in \mathrm{Hom}_{\mathbb{C}}(V, V)$$

If $h \in G$ then

$$\begin{aligned} \rho(h)^{-1} \varphi \rho(h) &= \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} \rho(h^{-1} g h) \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{f(hgh^{-1})} \rho(g) && \text{since } g \mapsto hgh^{-1} \text{ is a bijection} \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} \rho(g) && \text{since } f \in \mathcal{C}_G \\ &= \varphi \end{aligned}$$

So $\varphi \rho(h) = \rho(h) \varphi \ \forall h \in G$ and $\varphi \in \mathrm{Hom}_G(V, V)$. If in particular, (ρ, V) is irreducible, then $\exists \lambda \in \mathbb{C}$ such that $\varphi_{f,V} = \lambda \mathrm{id}_{\mathbb{C}}$ since $\mathbb{C}$ is algebraically closed. Then $\langle f, \chi_\rho \rangle_G = \mathrm{Tr}\, \varphi_{f,V} = \lambda \dim V$. So if $f \in I^\perp$ then $\lambda = 0$ and $\varphi_{f,V} = 0$. But in general if (ρ, V) is any representation, then $V \simeq \bigoplus V_i$ for some irreducible representations V_i (Maschke's Theorem) and $\rho = \bigoplus \rho_i$ and $\varphi_{f,V} = \bigoplus \varphi_{f,V_i}$. So again if $f \in I^\perp$, then $\varphi_{f,V} = 0$. In particular if $V = \mathbb{C}G$ is the regular representation then

$$0 = \varphi_{f,\mathbb{C}G} \delta_e = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)} \delta_g = \frac{1}{|G|} \overline{f}$$

So $f = 0$ and $I^\perp = 0$. $\square$

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Corollary. The number of irreducible $\mathbb{C}$-representations of G is the number of conjugacy classes in G.

Notation. For $g \in G$ we'll write $[g]_G = \{xgx^{-1} \mid x \in G\}$ for the conjugacy class containing g.

Corollary. For $g \in G$, $\chi(g) \in \mathbb{R}$ for every irreducible character χ if and only if $[g]_G = [g^{-1}]_G$.

Proof. Since $\chi(g^{-1}) = \overline{\chi(g)}$, $\chi(g) \in \mathbb{R} \iff \chi(g) = \chi(g^{-1})$. So we can rephrase the statement as

$$\chi(g) = \chi(g^{-1}) \text{ for every character } \chi \iff [g]_G = [g^{-1}]_G$$

Since the (irreducible) character span $\mathcal{C}_G$,

$$\chi(g) = \chi(g^{-1}) \text{ for every (irreducible) character } \iff f(g) = f(g^{-1}) \text{ for every } f \in \mathcal{C}_G$$

Since $\mathbb{1}_{[g]_G}$ is a class function $f(g) = f(g^{-1})$ for every $f \in \mathcal{C}_G \iff [g]_G = [g^{-1}]_G$. $\square$

4.3 Character Tables

Definition (Character table)**.** The *character table* of a finite group is defined as follows. We list the conjugacy classes of G, $[g_1]_G, \dots, [g_r]_G$ (by convention $g_1 = e$ always). We list the irreducible character of G (over $\mathbb{C}$) $\chi_1, \dots, \chi_r$ (by convention $\chi_1 = \mathbb{1}_G$ the character of trivial representation). Then we write the matrix

	e	g_2	g_3	$\cdots$	g_j	$\cdots$	g_r
χ_1	1	1	1	$\cdots$	1	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
χ_i	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\chi_i(g_j)$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
χ_i	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

We sometimes write $|[g_i]_G|$ above g_i and sometimes $|\mathcal{C}_G(g_i)|$ (recall $|[g_i]_G||\mathcal{C}_G(g_i)| = |G|$ by Orbit-Stabiliser Theorem).

Examples

(1) $C_3 = \langle x \rangle$. Let $\omega = e^{2\pi i/3}$. So $\omega^2 = \omega^{-1} = \overline{\omega}$.

	e	x	x^2
χ_1	1	1	1
χ_2	1	ω	$\overline{\omega}$
χ_3	1	$\overline{\omega}$	ω

Note the rows are indeed orthogonal with respect to $\langle \bullet, \bullet \rangle_G$ and the columns are orthogonal with respect to standard Hermitian inner product.

(2) S_3. Conjugacy classes are $\{e\}$, $\{(12), (13), (23)\}$, $\{(123), (132)\}$. So there must be 3 irreducible representations / character. $\chi_1 = \mathbb{1}_G$ is the character of the trivial representation. $\chi_2 = \varepsilon : S_3 \to \{\pm 1\} \subset \mathbb{C}^\times$ (where ε is the sign of a permutation) is a homomorphism and so a 1-dimensional representation and so a character. To compute χ_3 we can use orthogonality of characters. Let $\chi_3(e) = a$, $\chi_3((12)) = b$, $\chi_3((123)) = c$. Since every g in S_3 is conjugate to g^{-1} in S_3, $a, b, c \in \mathbb{R}$. Then

$$0 = \langle \mathbb{1}, \chi_3 \rangle_G = \frac{1}{6}(a + 3b + 2c)$$

$$0 = \langle \varepsilon, \mathbb{1} \rangle_G = \frac{1}{6}(a - 3b + 2c)$$

which can be solved to give $b = 0$, $a = -2c$. Then

$$1 = \langle \chi_3, \chi_3 \rangle_G = \frac{1}{6}(a^2 + 3b^2 + 2c^2)$$

So $c^2 = 1$. But a is the dimension of the representation with character χ_3 so $a \geq 1$. So $a = 2$, $c = -1$.

$\lvert C_{S_3}(g_i) \rvert$	6	2	3
g_i	e	(12)	(123)
$\mathbb{1}$	1	1	1
ε	1	-1	1
χ_3	2	0	-1

In fact we already know about χ_3 as the character of the representation of S_3 ($\cong D_6$) on $\mathbb{R}^2$ induced from the symmetries of a triangle. Once again the columns are orthogonal and their lengths are $1^2 + 1^2 + 2^2 = 6 = |C_{S_3}(e)|$, $1^2 + (-1)^2 + 0^2 = 2 = |C_{S_3}((12))|$, $1^2 + 1^2 + (-1)^2 = 3 = |C_{S_3}((123))|$.

Proposition (Column orthogonality)**.** If G is a finite group and χ_1, ..., χ_r is a complete list of its irreducible characters then for $g, h \in G$,

$$\sum_{i=1}^{r} \overline{\chi_i(g)}\chi_i(h) = \begin{cases} 0 & \text{if } [g]_G \neq [h]_G \\ |C_G(g)| & \text{if } [g]_G = [h]_G \end{cases}$$

In particular

$$\sum_{i=1}^{(} \dim V_i)^2 = \sum_{i=1}^{r} \chi_i(e)^2 = |G|$$

where V_i is a representation such that $\chi_{V_i} = \chi_i$.

Proof. Let X be the character table viewed as a matrix $X_{ij} = \chi_i(g_j)$ and D be the diagonal (real) matrix with $D_{ii} = |C_G(g_i)|$. Orthogonality of characters gives

$$\begin{aligned} \delta_{ij} &= \langle \chi_i, \chi_j \rangle_G \\ &= \sum_k \frac{1}{|C_G(g_k)|} \overline{\chi_i(g_k)} \chi_j(g_k) \\ &= \sum_k \frac{1}{D_{kk}} \overline{\chi_{ik}} \chi_{jk} \\ &= (\overline{X} D^{-1} X^\top)_{ij} \end{aligned}$$

So $\overline{X} D^{-1} X^\top I$ since X is square, X is invertible and $\overline{D^{-1} X^\top} = X^{-1}$ so $D = \overline{X^\top} X$ since D is real, i.e. $\sum_k \overline{\chi_k(g_i)} \chi_k(g_j) = D_{ij} = \delta_{ij} |C_G(g_i)|$. □

4.4 Permutation Representations

If recall that if a group G acts on a finite set X, $\mathbb{C}X = \{f : X \to \mathbb{C}\}$ is a representation via

$$(g \cdot f)(x) = f(g^{-1}x)$$

or equivalently $g \cdot \delta_x = \delta_g x \ \forall g \in G, x \in X$.

Lemma. If χ is the character of $\mathbb{C}X$ then $\chi(g) = |\{x \in X \mid gx = x\}|$.

Proof. If $X = \{x_1, \ldots, x_d\}$ then with respect to the basis $\delta_{x_1}, \ldots, \delta_{x_d}$ the matrix of g has i-th column with a 1 in entry j and 0 elsewhere if $g \cdot x_i = x_j$. So i-th column contributes 1 to the trace if $i = j$ and 0 otherwise. □

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Theorem. If $V_1, \dots, V_r$ is a complete list of irreducible representations of a finite group $G/\mathbb{C}$, then the regular representation $\mathbb{C}G$ decomposes as

$$\bigoplus_{i=1}^{r} (\dim V_i) V_i.$$

In particular, $|G| = \sum_{i=1}^{r} (\dim V_i)^2$.

Proof. We just need to show

$$\dim \operatorname{Hom}_G(\mathbb{C}G, V_i) = \dim V_i \ \forall i$$

But

$$\begin{aligned} \dim \operatorname{Hom}_G(\mathbb{C}G, V_i) &= \langle \chi_{\mathbb{C}G}, \chi_{V_i} \rangle_G \\ &= \frac{1}{|G|} \sum_{g \in G} |\{h \in G \mid gh = h\}| \chi_{V_i}(g) \\ &= \frac{1}{|G|} |G| \chi_{V_i}(e) \\ &= \chi_{V_i}(e) \end{aligned}$$

as required. □

Proposition (Burnside's Lemma)**.** Let G be a finite group and X a finite set with G-action. Then

$$\langle \mathbb{1}, \chi_{\mathbb{C}X} \rangle_G = \#\text{orbits of } G \text{ on } X.$$

Proof.

$$\begin{aligned} |G| \langle \mathbb{1}, \chi_{\mathbb{C}X} \rangle_G &= \sum_{g \in G} \chi_{\mathbb{C}X}(g) \\ &= \sum_{g \in G} |\{x \in X \mid gx = x\}| \\ &= |\{(g, x) \in G \times X \mid gx = x\}| \\ &= \sum_{x \in X} |\{g \in G \mid gx = x\}| \end{aligned}$$

So

$$\begin{aligned}
\langle \mathbb{1}, \chi_{\mathbb{C}X} \rangle_G &= \sum_{x \in X} \frac{|\operatorname{Stab}_G(x)|}{|G|} \\
&= \sum_{x \in X} \frac{1}{|\operatorname{Orb}_G(X)|} \\
&= \sum_{\text{orbits } O_i} \left(\sum_{x \in O_i} \frac{1}{|O_i|} \right) \\
&= \#\text{orbits}
\end{aligned}$$

□

Note that if $X = \bigcup O_i$ is the orbit decomposition then we've seen before $\mathbb{C}X = \bigoplus \mathbb{C}O_i$, so Burnside's Lemma says each $\mathbb{C}O_i$ contains precisely one copy of the trivial representation $\mathbb{C}$ – the constant functions on O_i. This is not so hard to prove directly (exercise).

If X, Y are two sets with G-actions then $X \times Y$ is a set with G-action via $g \cdot (x, y) = (g \cdot x, g \cdot y)$ for $(x, y) \in X \times Y$, $g \in G$.

Lemma. If X, Y are finite, then $\chi_{\mathbb{C}X \times Y} = \chi_{\mathbb{C}X} \cdot \chi_{\mathbb{C}Y}$.

Proof. If $g \in G$,

$$\begin{aligned}
\chi_{\mathbb{C}X \times Y}(g) &= |\{(x, y) \in X \times Y \mid (gx, gy) = (x, y)\}| \\
&= |\{x \in X \mid gx = x\}| \cdot |\{y \in Y \mid gy = y\}| \\
&= \chi_{\mathbb{C}X}(g)\chi_{\mathbb{C}Y}(g)
\end{aligned}$$

□

Corollary. If G is a finite group and X, Y are finite sets with G-actions, then

$$\langle \chi_{\mathbb{C}X}, \chi_{\mathbb{C}Y} \rangle_G = \#G\text{-orbits in } X \times Y$$

Proof.

$$\begin{aligned}
\langle \chi_{\mathbb{C}X}, \chi_{\mathbb{C}Y} \rangle_G &= \frac{1}{|G|} \sum_{g \in G} \chi_{\mathbb{C}X}(g)\chi_{\mathbb{C}Y}(g) \\
&= \frac{1}{|G|} \sum_{g \in G} 1\chi_{\mathbb{C}X \times Y} \\
&= \langle \mathbb{1}, \chi_{\mathbb{C}X \times Y} \rangle_G \\
&= \#G\text{-orbits on } X \times Y && \text{Burnside's Lemma}
\end{aligned}$$

□

Remark. If X is any set with G-action and at least 2-elements then $\{(x,x) \mid x \in X\} \subset X \times X$ is G-stable and non-empty and its complement $\{(x,y) \mid x,y \in X, x \neq\}$ is also non-empty and G-stable.

Definition (2-transitive action). We say G acts 2-*transitively* on X if for all x_1, x_2, y_1, y_2 with $x_1 \neq y_1$, $x_2 \neq y_2$, there exists $g \in G$ such that $gx_1 = x_2$ and $gy_1 = y_2$ (i.e. $g \cdot (x_1, y_1) = (x_2, y_2)$). Equivalently if the G-action on $X \times X$ has precisely two orbits.

Example. S_n acts 2-transitively on $\{1, \dots, n\}$ for all $n \geq 2$. If g acts 2-transitively on $X \times X$ then by the last corollary,

$$\langle \chi_{\mathbb{C}X}, \chi_{\mathbb{C}X} \rangle_G = 2$$

So if $\chi_{\mathbb{C}X} = \bigoplus_{i=1}^r n_i V_i$, V_i irreducible and pairwise non-isomorphic then $\sum n_i^2 = 2$. That is, $\mathbb{C}X$ has two non-isomorphic irreducible summands, namely the constant functions and

$$V = \{f \in \mathbb{C}X \mid \sum_{x \in X} f(x) = 0\}$$

Then χ_V is an irreducible character so

$$\chi_V(g) = \chi_{\mathbb{C}X}(g) - \mathbb{1}_G(g) = (\#\text{ fixed points of } g \text{ on } X) - 1$$

Moreover, if V is irreducible then the action must be 2-transitive.

Exercise: If $G = \mathrm{GL}_2(\mathbb{F}_p)$ then decompose the permutation representation of G coming from action of G on $\mathbb{F}_p \cup \{\infty\}$ by Möbius maps:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

if $z \in \mathbb{F}_p \setminus \{-d/c\}$ etc.

Examples

(1) $G = S_4$. The character table of S_4 is

$\lvert [g_i]_G \rvert$	1	3	8	6	6
g_i	e	$(12)(34)$	(123)	(12)	(1234)
$\mathbb{1}$	1	1	1	1	1
$\sum$	1	1	1	1	1
χ_3	3	-1	0	1	-1
χ_4	3	-1	0	-1	1
χ_5	2	2	-1	0	0

Proof. $\mathbb{1}, \sum$ are constructed as for S_3. By our discussion above, $\chi_{\mathbb{C}\{1,2,3,4\}} = \mathbb{1} + \chi_V$ for some irreducible representation V of degree 3 and we can let $\chi_3 = \chi_V$ such that $\chi_3(g) = \#\text{fixed points of } g - 1$. We saw on Example Sheet 1, Question 2 that if θ is a 1-dimensional representation and ρ is any irreducible representation then $(\rho \otimes \theta)(g) := \theta(g)\rho(g)$ is an irreducible representation of G and $\chi_{\rho\otimes\theta}(g) = \chi_\rho(g)\theta(g)$. Thus we can set $\chi_4 = \sum \cdot \chi_3$. We can compute χ_5 via column orthogonality

$$1^2 + 1^2 + 3^2 + 3^2 + \chi_5(e)^2 = 24$$

so $\chi_5(e) = 2$, and $\sum_{i=1}^{5} \chi_i(e)\chi_i(e) = 0 \ \forall g \in S_4 \setminus \{e\}$. □

(In fact there is a homomorphism $S_4 \to S_3$ giving a 2-transitive action of S_4 on $\{1, 2, 3\}$ and $\chi_5 = \chi_{\mathbb{C}\{1,2,3\}} - \mathbb{1}$).

(2) $G = A_4$. Every irreducible representation of S_4 may be restricted to A_4 and its character values won't change. In this way we get 3 characters of A_4

$$\mathbb{1} = \mathbb{1}_{s_4}|_{A_4}, \quad \psi_2 = \chi_3|_{A_4} = \chi_3|_{A_4}, \quad \psi_3 = \chi_5|_{A_4}$$

It is irreducible since it has dimension 1:

$$\langle \psi_2, \psi_2 \rangle_{A_4} = \frac{1}{12}(13^2 + 3(-1)^2 + 8 \cdot 0^2) = 1$$

so ψ_2 is irreducible. However,

$$\langle \psi_3, \psi_3 \rangle_{A_4} = \frac{1}{12}(12^2 + 3(2)^2 + 8(-1)^2) = 2$$

so ψ_3 decomposes into two 1-dimensional non-isomorphic pieces.

Exercise: Use this to construct the character table of A_4. Recall $[(123)]_{S_4}$ is a union of two classes in A_4.

5 The Character Ring

We've already seen that the algebraic structure on $\mathcal{C}_G$ for a finite group G has representation theoretic meaning, e.g. if V_1, V_2 are representations then

$$\begin{aligned} \chi_{V_1 \oplus V_2} &= \chi_{V_1} + \chi_{V_2} \\ \chi_0 &= 0 \\ \chi_k &= \mathbb{1}_G \\ \langle \chi_{V_1}, \chi_{V_2} \rangle_G &= \dim \operatorname{Hom}_G(V_1, V_2) \end{aligned}$$

We've also seen $\chi_{\mathbb{C}X \times Y} = \chi_{\mathbb{C}X} \cdot \chi_{\mathbb{C}Y}$ and if θ, ρ are representations such that θ is 1-dimensional then $\chi_{\theta \oplus \rho} = \chi_\theta \cdot \chi_\rho (= \theta \chi_\rho)$. We want to generalise this to any pair of representations so $\chi_{\sigma \oplus \rho} = \chi_\sigma \cdot \chi_\rho$.

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$\chi_0 = 0$ chere 0 is the k-vector space of dimension 0, so $\mathrm{GL}(0) = \{\mathrm{id}_0\}$. $\chi_k = \mathbb{1}_G$ where k is the trivial representation.

Goal: If V, W are representations, build a representation $V \otimes W$ such that $\chi_{V \otimes W} = \chi_V \cdot \chi_W$.

5.1 Tensor Products

Suppose V and W are vector spaces over k with bases $v_1, \dots, v_m$ and $w_1, \dots, w_n$ respectively. We can view $V \oplus W$ as the set of pairs (v, w) with $v \in V$, $w \in W$ under pointwise operation or as the vector space with basis $v_1, \dots, v_m, w_1, \dots, w_n$.

Definition. The *tensor product* $V \otimes W$ of V and W is the k vector space with basis $w_i \otimes w_i$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ (so $\dim V \otimes W = (\dim V)(\dim W)$).

Example. If X and Y are finite then $kX \otimes kY$ has basis $\delta_x \otimes \delta_y$ for $x \in X$, $y \in Y$ and $\alpha_{X \times Y} : kX \otimes kY \to kX \times Y$, $\delta_x \otimes \delta_y \mapsto \delta_{(x,y)}$ extends to an isomorphism of vector spaces.

Notation. If $v = \sum \lambda_i v_i \in V$ and $w = \sum \mu_j w_j \in W$ then

$$v \otimes w := \sum_{i,j} \lambda_i \mu_j v_i \otimes w_j \in V \otimes W.$$

So under $\alpha_{X \times Y}$,

$$\alpha_{X \times Y}(f \otimes g)(x, y) = f(x)g(y)$$

Note that in general, not every element of $V \otimes W$ can be written in the form $V \otimes W$. For example, $v_1 \otimes w_1 + v_2 \otimes w_2$. The smallest number of summands needed is called the *rank* of the tensor.

Lemma. The map $V \times W \to V \otimes W$, $(v, w) \mapsto v \otimes w$ is bilinear.

Proof. We should prove that if $x, x_1, x_2 \in V$ and $y, y_1, y_2 \in W$ and $u_1, u_2 \in k$ then

$$(u_1 x_1 + u_2 x_2) \otimes y = u_1(x_1 \otimes y) + u_2(x_2 \otimes y)$$
$$x \otimes (u_1 y_1 + u_2 y_2) = u_1(x \otimes y_1) + u_2(x \otimes y_2)$$

We'll do the second and then appeal to symmetry. We write $x = \sum \lambda_i v_i$, $y_k = \sum \mu_j^k w_j$ for $k = 1, 2$. Then

$$x \otimes (u_1 y_1 + u_2 y_2) = \sum_{i,j} \lambda_i (u_1 \mu_j^1 + u_2 \mu_j^2)(v_i \otimes w_j)$$
$$u_1(x \otimes y_1) + u_2(x \otimes y_2) = \sum_{i,j} u_1 \lambda_i \mu_j^1 (v_i \otimes w_j) + \sum_{i,j} u_2 \lambda_I \mu_j^2 (v_i \otimes w_j)$$

These are equal. □

Exercise: Show that given U, V and W there is a one to one correspondence

$$\{\text{linear maps } V \otimes W \to U\} \to \{\text{bilinear maps } V \times W \to U\}$$

given by precomposition with the bilinear map $(v, w) \mapsto v \otimes w$.

Lemma. If $x_1, \ldots, x_m$ is any basis for V and $y_1, \ldots, y_n$ is any basis for W then $x_i \otimes y_j$ with $1 \leq i \leq m$ and $1 \leq j \leq n$ is a basis for $V \otimes W$. Thus the definition of $V \otimes W$ does not depend on the choice of basis.

Proof. It suffices to show that the given set spans $V \otimes W$ since it has size mn. But if $v_i = \sum_r A_{ri} x_r$ and $w_J = \sum_s V_{sj} y_s$ then

$$v_i \otimes w_j = \sum_{r,s} (A_{ri} B_{sj})(x_r \otimes y_s)$$

But $\{v_i \otimes w_j\}_{1 \le i \le m, 1 \le j \le n}$ span $V \otimes V \otimes W$ so we're done. $\square$

Remark (for enthusiast's)**.** In fact we could've defined $V \otimes W$ in a basis independent way. Let F be the (infinite dimensional) vector space basis $\langle v \otimes w \mid v \in V, w \in W \rangle$ and R the subspace spanned by elements

$$x \otimes (u_1 y_1 + u_2 y_2) - u_1(x \otimes y_1) - u_2(x \otimes y_2)$$
$$(u_1 v_1 + u_2 v_2) \otimes y - u_1(x_1 \otimes y) - u_2(x_2 \otimes y)$$

for all $x, x_1, x_2 \in V$, $y, y_1, y_2 \in W$, $u_1, u_2 \in k$. Then let $V \otimes W = F/R$.

Exercise: Show that for vector spaces U, V and W there is a natural (basis independent) isomorphism

$$(U \oplus V) \otimes W \to (U \otimes W) \oplus (V \otimes W).$$

Definition. Suppose V and W are vector spaces with bases $v_1, \dots, v_m$ and $w_1, \dots, w_n$ and $\varphi : V \to V$ and $\psi : W \to W$ are linear maps. We can define

$$\varphi \otimes \psi : V \otimes W \to V \otimes W (\varphi \otimes \psi) \qquad (v_i \otimes w_j) = \varphi(v_i \otimes \psi(w_j)$$

Example. If φ is represented by A and ψ is represented by B with respect to given bases, then if we order $V_i \otimes W_j$ lexicographically (i.e. $v_1 \otimes w_1$, $v_1 \otimes w_2$, $v_1 \otimes w_3$, ..., $v_1 \otimes w_n$, $v_2 \otimes w_1$, ..., $v_m \otimes w_n$), then $(\varphi \otimes \psi)$ is representede by the block matrix

$$\begin{pmatrix} A_{11}B & A_{12}B & \cdots & \\ A_{21}B & A_{22}B & \cdots & \\ \vdots & \vdots & \ddots & \\ & & & A_{mm}B \end{pmatrix}$$

Since

$$(\varphi \otimes \psi)(v_i \otimes w_j) = \left(\sum_k A_{ki} v_k \right) \otimes \left(\sum_l B_{lj} w_l \right) = \sum_{k,l} A_{ki} B_{lj} v_k \otimes w_l$$

Lemma. The linear map $\varphi \otimes \psi$ does not depend on the basis. Indeed,

$$(\varphi \otimes \psi)(v \otimes w) = \varphi(v) \otimes \psi(w) \qquad \forall v \in V, w \in W.$$

Proof. Writing $v = \sum \lambda_i v_i$ and $w = \sum \mu_j w_j$,

$$\begin{aligned}(\varphi \otimes \psi)(v \otimes w) &= (\varphi \otimes \psi)\left(\sum_{i,j} \lambda_i \mu_j v_i \otimes w_j\right) \\ &= \sum_{i,j} \lambda_i \mu_j \varphi(v_i) \otimes \psi(w_j) \\ &= \varphi(v) \otimes \psi(w)\end{aligned}$$

□

Remark. The proof really says that $V \times W \to V \otimes W$, $(v, w) \mapsto v \otimes w$ is bilinear and $\varphi \otimes \psi$ is the corresponding linear map $V \otimes W \to V \otimes W$ from an earlier exercise.

Lemma. Suppose $\varphi, \varphi_1, \varphi_2 \in \mathrm{Hom}_k(V, V)$ and $\varphi, \varphi_1, \varphi_2 \in \mathrm{Hom}_k(W, W)$. Then

(i) $(\varphi_1\varphi_2) \otimes (\psi_1\psi_2) = (\varphi_1 \otimes \psi_1)(\varphi_2 \otimes \psi_2) \in \mathrm{Hom}_k(V \otimes W, V \otimes W)$.

(ii) $\mathrm{id}_V \otimes \mathrm{id}_W = \mathrm{id}_{V \otimes W}$

(iii) $\mathrm{Tr}(\varphi \otimes \psi) = \mathrm{Tr}(\varphi)\,\mathrm{Tr}(\psi)$.

Proof.

(i) Given $v \in V$, $w \in W$ by the last lemma we can compute

$$\begin{aligned}(\varphi_1\varphi_2 \otimes \varphi_1\varphi_2)(v \otimes w) &= \varphi_1\varphi_2(v) \otimes \varphi_1\varphi_2(w) \\ &= (\varphi_1 \otimes \psi_1)(\varphi_2(v) \otimes \psi_2(w)) \\ &= (\varphi_1 \otimes \psi_1)(\varphi_2 \otimes \psi_2)(v \otimes w)\end{aligned}$$

We're done since all maps and linear and $\{v \otimes w\}$ spans $V \otimes W$.

(ii) Is clear.

(iii) By earlier example it suffices to see

$$\operatorname{Tr}\begin{pmatrix} A_{11}B & & & \\ & A_{22}B & & \\ & & \ddots & \\ & & & A_{nn}B \end{pmatrix} = \sum_{i,j} B_{ii}A_{jj} = (\operatorname{Tr} A)(\operatorname{Tr} B) \qquad \square$$

Definition. Given two representations (ρ, V) and $\sigma, W)$ of a group G we can define a representation $(\rho \otimes \sigma, V \otimes W)$ via

$$(\rho \otimes \sigma)(g) = \rho(g) \otimes \sigma(g)$$

This is a representation by parts (i) and (ii) of the last lemma, and $\chi_{\rho\otimes\sigma} = \chi_\rho \cdot \chi_\sigma$ by part (iii).

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Remark.

(1) Tensor product of representations defined last time is a generalisation of the tensor product of a representation and a 1-dimensional representation previously defined.

(2) If X and Y are finite sets with G-action

$$\begin{aligned} \alpha_{X\times Y} : kX \otimes kY &\to kX \times Y \\ \delta_x \otimes \delta_y &\mapsto \delta_{(x,y)} \end{aligned}$$

is an intertwining map.

Definition (Character ring)**.** The *character ring* $R(G)$ of a group G is defined by

$$R(G) := \{\chi_1 - \chi_2 \mid \chi_1, \chi_2 \text{are characters of } G\} \subset \mathcal{C}_G$$

Since $\chi_{V_1\oplus V_2} = \chi_{V_1} + \chi_{V_2}$, $R(G)$ is a subgroup of $\mathcal{C}_G$ under $+$. Since $\mathbb{1}_G$ is a character, $R(G)$ contains the multiplicative 1 in $\mathcal{C}_G$.

Since $\chi_{V_1\otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$,

$$(\chi_{V_1} - \chi_{V_2}) \cdot (\chi_{W_1} - \chi_{W_2}) = \chi_{(V_1\otimes W_1)\oplus(V_2\otimes W_2)} - \chi_{(V_2\otimes W_1)\oplus(V_1\otimes W_2)} \in R(G)$$

$R(G)$ is closed under $\cdot$ and so $R(G)$ is a subring of $\mathcal{C}_G$.

Observation: If (ρ, V) is a representation of G and (σ, W) is a representation of another group H, then

$$\begin{aligned} \rho \otimes \sigma : G \times H &\to \mathrm{GL}(V \otimes W) \\ (g,h) &\mapsto \rho(g) \otimes \sigma(h) \end{aligned}$$

is a representation of $G \times H$, by parts (i) and (ii) in the last lemma last time. Moreover,

$$(\chi_V \otimes \chi_W)(g,h) = \chi_{V \otimes W}(g,h) = \chi_V(g)\chi_W(h)$$

by part (iii) of the same lemma. Thus

$$\begin{aligned} R(G) \times R(H) &\to R(G \times H) \\ (\chi_V, \chi_W) &\mapsto \chi_V \otimes \chi_W \end{aligned}$$

defines a $\mathbb{Z}$-bilinear map.

The construction of $V \otimes W$ as a representation of G from last time can be viewed as the case $G = H$ in the construction followed by restriction along

$$\begin{aligned} G &\to G \times G \\ g &\mapsto (g,g) \end{aligned}$$

Proposition. Suppose G and H are finite groups, and $(\rho_1, V_1), \dots, (\rho_r, V_r)$ are the irreducible $\mathbb{C}$-representations of G and $(\sigma_1, W_1), \dots, (\sigma_s, W_s)$ are all the irreducible $\mathbb{C}$-representations of H.

For each $1 \le i \le r$, $1 \le j \le s$, $(\rho_i \otimes \sigma_j, V_i \otimes W_j)$ is a irreducible $\mathbb{C}$-representation of $G \times H$. Moreover, all irreducible $\mathbb{C}$-representations of $G \times h$ arise in this way.

We've seen this when G, H are abelian before since all these representations have degree 1 in this case.

Proof. Let $\chi_1, \dots, \chi_r$ be the characters of $\rho_1, \dots, \rho_r$, and $\psi_1, \dots, \psi_s$ the characters of $\sigma_1, \dots, \sigma_s$. The character of $\rho_i \otimes \sigma_j$ is $(\chi_i \otimes \psi_j)(g,h) = \chi_i(g)\psi_j(h)$. Then

$$\begin{aligned} \langle \chi_i \otimes \psi_j, \chi_k \otimes \psi_l \rangle_{G \times H} &= \frac{1}{|G \times H|} \sum_{(g,h) \in G \times H} \overline{\chi_i(g)\psi_j(h)} \chi_k(g)\psi_l(h) \\ &= \left(\frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \chi_k(g) \right) \left(\frac{1}{|H|} \sum_{h \in H} \overline{\psi_j(g)} \psi_l(g) \right) \\ &= \langle \chi_i, \chi_k \rangle_G \langle \psi_j, \psi_l \rangle_H \\ &= \delta_{ik}\delta_{jl} \end{aligned}$$

So the $\chi_i \otimes \psi_j$ are pairwise distinct and irreducible. Now

$$\sum_{i,j} (\dim V_i \otimes W_j)^2 = \left(\sum_i () \dim V_i)^2 \right) \left(\sum_j (\dim W_j)^2 \right) = |G||H| = |G \times H| \qquad \square$$

Question: If V, W are irreducible representations of G, can $V \otimes W$ be an irreducible representation of G? We've seen that if $\dim V = 1$ or $\dim W = 1$ then yes. Typically the answer is no.

Example. $G = S_3$

	1	(123)	(12)
$\mathbb{1}$	1	1	1
ε	1	1	-1
V	2	-1	0

clear that $\mathbb{1} \otimes W \simeq W$ always. $\varepsilon \otimes \varepsilon \cong \mathbb{1}$, $\varepsilon \otimes V = V$. Also, $V \otimes V$ has character χ_V^2.

$$\chi_V^2(e) = 2^2 = 4, \quad \chi_V^2((123)) = (-1)^2 = 1, \quad \chi_V^2((12)) = 0^2 = 0$$

$\chi_V^2 = \chi_V + \varepsilon + \mathbb{1}$.

In general if $\chi_1, \dots, \chi_r$ are all irreducible characters of G and $1 \leq i, j \leq r$ then

$$\chi_i \chi_j = \sum_{k=1}^{r} a_{i,j}^k \chi_k$$

for some $a_{i,j^k} \in \mathbb{N}_0$. These numbers $a_{i,j}^k$ determine the ring structure on $R(G)$ since $R(G) = \bigoplus_{i=1}^r \mathbb{Z}\chi_i$ as a group under $+$.

In fact, $V \otimes V, V \otimes V \otimes V, \dots$ are never irreducible if $\dim V > 1$.

5.2 Symmetric and exterior powers

For any vector space V we can define

$$\sigma_V := \sigma : V \otimes V \to V \otimes V \qquad \sigma(v \otimes w) = w \otimes v \ \forall v, w \in V$$

(Exercise: prove that $V \times V \to V \otimes V$, $(v, w) \mapsto w \otimes v$ is bilinear).

Notice $\sigma^2 = \mathrm{id}_{V\otimes V}$, so if $\operatorname{char} k \neq 2$, then σ decomposes $V \otimes V$ into eigenspaces.

$$S^2V := \{a \in V \otimes V \mid \sigma(a) = a\} \qquad \text{the } \textit{symmetric square} \text{ of } V$$
$$\Lambda^2V := \{a \in V \otimes V \mid \sigma(a) = a\} \qquad \text{the } \textit{exterior / alternating square} \text{ of } V$$

In fact $V \otimes V = S^2V \oplus \Lambda^2V$ is the isotypical decomposition of $V \otimes V$ as a representation of C_2. Suppose for now that $\operatorname{char} k \neq 2$.

Lemma. Suppose $v_1, \ldots, v_m$ is a basis for V.

(i) S^2V has basis $v_iv_j := \frac{1}{2}(v_i \otimes v_j + v_j \otimes v_i)$ for $1 \leq i \leq j \leq m$ ($v_jv_i = v_iv_j$ if $i > j$ allowed).

(ii) Λ^2V has basis $v_i \wedge v_j := \frac{1}{2}(v_i \otimes v_j - v_j \otimes v_i)$ for $1 \leq i < j \leq m$ ($v_j \wedge v_i = -v_i \wedge v_j$ if $i \geq j$ allowed)

Thus $\dim S^2V = \frac{1}{2}m(m+1)$ and $\dim \Lambda^2V = \frac{1}{2}m(m-1)$.

Proof. It is easy to check:

(i) $v_iv_j \in S^2V$ for all i, j.

(ii) $v_i \wedge v_j \in \Lambda^2V$ for all i, j.

(iii) The union of the claimed bases spans $V \otimes V$ and has size $m^2 = \dim V \otimes V$

So it follows from this that $V \otimes V = S^2V \oplus \Lambda^2V$. Everything else follows. □

You might want to ponder Example Sheet 2, Question 11 in this context.

Proposition. Let (ρ, V) be a representation of G.

(i) $V \otimes V = S^2V \oplus \Lambda^2V$ as a direct sum of representations of G.

(ii) For $g \in G$ such that $\rho(g)$ is diagonalisable (for convenience, but there exists a slightly more complicated proof not using this condition),

$$\chi_{S^2V}(g) = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2))$$
$$\chi_{\Lambda^2V} = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$$

Proof. For (i) we need that S^2V and Λ^2V are G-invariant, i.e. if $a \in V \otimes V$ such that

$\sigma(a) = \lambda a$ for $\lambda = \pm 1$, then $\sigma(ga) = \lambda ga\ \forall g \in G$. For this it suffices to show that $\rho(g)$ and σ commute $\forall g \in G$, i.e. $\sigma \in \mathrm{Hom}_G(V \otimes V, V \otimes V)$. BUt $\sigma \circ g(v \otimes w) = \sigma(gv \otimes gw)) = gw \otimes gv = g(w \otimes v) = g\sigma(v \otimes w)\ \forall g \in G$.

To prove (ii) it suffices to compute χ_{S^2V} since sum of RHS $= \chi_V^2 = \chi_{V \otimes V}$. Let $v_1, \dots, v_m$ form a basis for V such that $\rho(g)v_i = \lambda_i v_i$ for $1 \le i \le m$.

$$g(v_i \lambda_i) = \lambda_i \lambda_j v_i v_j$$

So

$$\chi_{S^2V}(g) = \sum_{1 \le i \le j \le m} \lambda_i \lambda_j$$

and

$$\chi_V(g)^2 + \chi_V(g^2) = \left(\sum_i \lambda_i\right)^2 + \sum_j (\lambda_j)^2 = 2\sum \lambda_i^2 + 2\sum_{i<j} \lambda_i \lambda_j = 2\chi_{S^2V}(g) \qquad \square$$

Exercise: Prove the formula for $\chi_{\Lambda^2 V}$ directly.

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Recall

$$\chi_{S^2V}(g) = \frac{1}{2}(\chi_V(g)^2 + \chi_v(g^2))$$
$$\chi_{\Lambda^2V}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$$

S_4	e	$(12)(34)$	(123)	(12)	(1234)
$\mathbb{1}$	1	1	1	1	1
ε	1	1	1	-1	-1
χ_3	3	-1	0	1	-1
$\varepsilon\chi_3$	3	-1	0	-1	1
χ_5	2	2	-1	0	0
χ_3^2	9	1	0	1	1
$\chi_3(g^2)$	3	3	0	3	-1
$S^2\chi_3$	6	2	0	2	0
$\Lambda^2\chi_3$	3	-1	0	-1	1

since $e^2 = e = ((12)(34))^2 = (12)^2$, $(123)^2 = (132)$, $(1234) = (13)(24)$.

Thus $S^2\chi_3 = \mathbb{1} + \chi_3 + \chi_5$, $\Lambda^2\chi_3 = \varepsilon\chi_3$. So given $1, \varepsilon, \chi_3$, we can construct the remaining characters from $S^2\chi_3$ and $\Lambda^2\chi_3$. More generally, for any vector sapce V we may consider

$$V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}} = V \otimes V^{\otimes(n-1)}$$

for $n \geq 1$ ($V^{\otimes 0} = k$, $V^{\otimes 1} = V$). Then for any $w \in S_n$, we can define an (invertible) linear map

$$\sigma(w) : V^{\otimes n} \to V^{\otimes n}$$
$$v_1 \otimes \cdots \otimes v_n \mapsto v_{w^{-1}(1)} \otimes \cdots \otimes v_{w^{-1}(n)}$$

for $v_1, \ldots, v_n \in V$.

Exercise: Show that this defines a representation of S_n on $V^{\otimes n}$ and that if V is a representation of G then the G-action and S_n-action on $V^{\otimes n}$ commute.

Thus we can decompose $V^{\otimes n}$ as a representation of S_n into isotypic components if $\operatorname{char} k = 0$ and each will be a G-invariant subspace of $V^{\otimes n}$.

Definition. If V is a vector space

(i) The n-th *symmetric power* of V is

$$S^n V := \{a \in V^{\otimes n} \mid \sigma(w)(a) = a \ \forall w \in S_n\}$$

(ii) The n-th *alternating / exterior power* of V is

$$\Lambda^n V := \{a \in V \mid \sigma(w)(a) = \varepsilon(w) a \ \forall w \in S_n\}$$

Note that for $n \geq 3$,

$$S^n V \oplus \Lambda^n V = \{a \in V^{\otimes n} \mid \sigma(w) a = a \ \forall w \in A_n\} \subsetneq V^{\otimes n}$$

We also define the following notation for $v_1, \ldots, v_n \in V$

$$v_1 v_2 \cdots v_n = \frac{1}{n!} \sum_{w \in S_n} v_{w(1)} \otimes \cdots \otimes v_{w(n)}$$
$$v_1 \wedge \cdots \wedge v_n = \frac{1}{n!} \sum_{w \in S_n} \varepsilon(w) v_{w(1)} \otimes \cdots \otimes v_{w(n)}$$

(for $\operatorname{char} k = 0$).

Exercise: Show that if $v_1, \ldots, v_d$ is a basis for V then

$$v_{i_1} \cdots v_{i_n} \mid 1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq d\}$$

is a basis for $S^n V$, and

$$v_{i_1} \wedge \cdots \wedge v_{i_n} \mid 1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq d\}$$

is a basis for $\Lambda^n V$. Hence given a basis for V with respect to which $\rho(g)$ is diagonal, compute $\chi_{S^n V}(g)$ and $\chi_{\Lambda^n V}(g)$ in terms of the eigenvlaues of $\rho(g)$.

For any vector space V, $\Lambda^{\dim V} V \simeq k$ and $\Lambda^n V = 0$ for $n > \dim V$.

Exercise: Show that if (ρ, V) is a representation of G then $\Lambda^{\dim V} V \simeq \det \rho$ as a representation of G.

Definition. Given a vector space V we can define the *tensor algebra* of V

$$TV := \bigoplus_{n \geq 0} V^{\otimes n}$$

as an infinite dimensional vector space. Then TV is a (non-commutative) graded ring with product

$$(v_1 \otimes \cdots \otimes v_r) \cdot (w_1 \otimes \cdots \otimes w_n) = v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_s \in V^{\otimes (r+s)}$$

For $v_1 \otimes \cdots v_r \in V^{\otimes r}$, $w_1 \otimes \cdots \otimes w_r \in V^{\otimes r}$ with graded quotient map. The *symmetric algebra* of V

$$SV := \frac{TV}{\{x \otimes y - y \otimes x \mid x, y \in V\}}$$

and *exterior algebra* of V

$$\Lambda V := \frac{TV}{\{x \otimes y + y \otimes x \mid x, y \in V\}}$$

One can show $SV \simeq \bigoplus_{n \geq 0} S^n V$ and $\Lambda V \simeq \bigoplus_{n \geq 0} \Lambda^n V$ in characteristic 0. This can be seen via

$$x_1 \cdots x_n \leftarrow\!\shortmid [x_1 \otimes \cdots \otimes x_n] \qquad x_1 \wedge \cdots \wedge x_n \leftarrow\!\shortmid [x_1 \otimes \cdots \otimes x_n]$$

SV is a commutative graded ring and ΛV is a graded commutative ($x \in \Lambda^r V$ and $y \in \Lambda^s V$ then $x \wedge y = (-1)^{rs} y \wedge x$) ring.

5.3 Duality

Recall that $\mathcal{C}_G$ has a $*$-operation given by $f^*(g) = f(g^{-1})$ for all $f \in \mathcal{C}_G$, $g \in G$. This also restricts to $R(G)$. Recall also if (ρ, V) is a representation of G then the *dual rep* (ρ^*, V^*) is defined by

$$\rho^*(g)(\theta)(v) = \theta(\rho(g^{-1})(v)) \qquad \forall v \in V, g \in G, \theta \in V^*$$

Lemma 2. $\chi_{V^*} = \chi_V^*$.

Proof. If $\rho(G)$ is represented by A with respect to a bais $v_1, \dots, v_d$ for V and $\varepsilon_1, \dots, \varepsilon_d$ is the dual basis for V^*, then $\rho(g^{-1})v_i = \sum_j (A^{-1})_{ji} v_j$ So

$$\rho^*(g)(\varepsilon_k)(v_i) = \varepsilon_k(\rho^{-1}(g)v_i) = \varepsilon_k \left(\sum_j (A^{-1})_{ji} v_j \right) = (A^{-1})_{ki}$$

and

$$\rho^*(g)(\varepsilon_k) = \sum_j (A^{-1})^\top_{jk} \varepsilon_j$$

i.e. $\rho^*(g)$ is represented by $(A^{-1})^\top$ with respect to this dual basis. Taking traces gives $\chi_{\rho^*}(g) = \chi_\rho(g^{-1}) = \chi_\rho^*(g)$. □

We say V is self-dual if $V \simeq V^*$ as representations of G. When G is finite and $k = \mathbb{C}$ then V is self-dual if and only if $\chi_V^* = \chi_V$ which happens if and only if $\chi_V(g) \in \mathbb{R}\ \forall g \in G$ since $\chi_V^* = \overline{\chi_V}$ in this case.

Example.

(1) $G = \langle x \rangle \simeq C_3$ and $V = \mathbb{C}$, $\rho : G \to \mathbb{C}^\times$, $x^j \mapsto e^{2\pi i j/3}$. Then $\rho^*(x^j) = e^{-2\pi i j/3}$ and V is not self-dual.

(2) $G = S_n$ since $[g]_{S_n} = [g^{-1}]_{S_n}\ \forall g \in S_n$, every representation of S_n is self-dual.

(3) Permutation representations are always self-dual.

We now have various ways of building representations of a group G.

- permutation representations.
- restrict representations of H to G along homomorphisms $\theta : G \to H$.
- tensor products.
- $S^n V$ and $\Lambda^n V$.
- decomposition of representations into irreducible components.
- character theoretically, e.g. row / column orthogonality in character table.

One more next time related to restriction from G to H for $H \leq G$ called induction.

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6 Induction

6.1 Construction

Suppose H is a subgroup of a (finite) group G. Then the restriction from G to H gives a way of building representations of H from representations of G. We want to go the other way and build representations of G from representations of H.

Recall that $[g]_G$ denotes the conjugacy class of g in G. So

$$\mathbb{1}_{[g]_G}(x) = \begin{cases} 1 & \text{if } x \text{ is conjugate to } g \text{ in } G \\ 0 & \text{otherwise} \end{cases}$$

We note that for $g \in G$,

$$[g^{-1}]_G = [g]_G^{-1} = \{y^{-1} \mid y \in [g]_G\}$$

since $(xgx^{-1})^{-1} = xg^{-1}x^{-1}$. So $\mathbb{1}^*_{[g]_G} = \mathbb{1}_{[g^{-1}]_G}$. If $H \leq G$ then $[g]_G \cap H$ is (possibly empty) union of H-conjugacy classes

$$[g]_G \cap H = \bigcup_{[h]_H \subseteq [g]_G} [h]_H$$

So $r : \mathcal{C}_G \to \mathcal{C}_H$; $r(f) = f|_H$ is a well-defined linear map with $r(\mathbb{1}_{[g]_G}) = \sum_{[h]_H \subseteq [g]_G} \mathbb{1}_{[h]_H}$. Since for every finite group G

$$\langle f_1, f_2 \rangle_G = \frac{1}{|G|} \sum_{g \in G} f_1^*(g) f_2(g)$$

defines a non-degenerate bilinear form on $\mathcal{C}_G$, the map r has an adjoint $r^* : \mathcal{C}_G \to \mathcal{C}_G$ given by

$$\langle r(f_1), f_2 \rangle_H = \langle f_1, r^*(f_2) \rangle_G$$

for $f_1 \in \mathcal{C}_G$, $f_2 \in \mathcal{C}_H$. In particular for $f \in \mathcal{C}_H$,

$$\langle \mathbb{1}_{[g^{-1}]_G}, r^*(f) \rangle_G = \frac{1}{|G|} \sum_{x \in [g]_G} r^*(f)(x) = \frac{1}{|C_G(g)|} r^*(f)(g)$$

On the other hand

$$\begin{aligned} \langle \mathbb{1}_{[g^{-1}]_G}, r^*(f) \rangle_G &= \langle r(\mathbb{1}_{[g^{-1}]_G}), f \rangle_H \\ &= \sum_{[h]_H \subseteq [g]_G} \frac{1}{|C_H(h)|} f(h) \end{aligned}$$

so combining these we see that

$$r^*(f)(g) = \sum_{[h]_H \subseteq [g]_G} \frac{|C_G(g)|}{|C_H(h)|} f(h) \tag{1}$$

Since $xgx^{-1} = ygy^{-1} \iff x^{-1}y \in C_G(g)$,

$$\begin{aligned} r^*(f)(g) &= \sum_{h \in [g] \cap H} \frac{|C_G(g)|}{|C_H(h)||[h]_H|} f(h) \\ &= \frac{1}{|H|} \sum_{x \in G} f^\circ(x^{-1}gx) \end{aligned}$$

where

$$f^\circ(g) = \begin{cases} f(g) & g \in H \\ 0 & \text{otherwise} \end{cases}$$

Question: Is $r^*(R(H)) \subseteq R(G)$? Suppose χ is a $\mathbb{C}$-character of H and ψ is an irreducible $\mathbb{C}$-character of G. Then

$$\langle r^*(\chi), \psi \rangle_G = \langle \chi, r(\psi) \rangle_H \in \mathbb{N}_0$$

by Orthogonality of characters since $r(\psi)$ is a character of H.

So writing $\mathrm{Irr}(G) = \{$irreducible $\mathbb{C}$-character of $G\}$,

$$r^*(\chi) = \sum_{\psi \in \mathrm{Irr}(G)} \langle \chi, \psi|_H \rangle_H \psi \tag{2}$$

is even a character of G.

Example. $G = S_3$ and $H = A_3 = \{e, (123), (132)\}$. If $f \in \mathcal{C}_H$ then by (1),

$$\begin{aligned} r^*(f)(e) &= \frac{|C_{S_3}(e)|}{|C_{A_3}(e)|} f(e) = \frac{6}{3} f(e) = 2f(e) \\ r^*(f)((12)) &= 0 \\ r^*(f)((123)) &= \frac{|C_{S_3}((123))|}{|C_{A_3}((123))|} f((123)) + \frac{|C_{S_3}((123))|}{|C_{A_3}((132))|} f((132)) \\ &= \frac{3}{3} f((123)) + \frac{3}{3} f((132)) = f((123)) + f((132)) \end{aligned}$$

Thus

A_3	1	(123)	(132)
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

S_3	1	(12)	(123)
$r^*(\chi_1)$	2	0	2
$r^*(\chi_2)$	2	0	-1
$r^*(\chi_3)$	2	0	-1

(where $\omega = e^{2\pi i/3}$ and we use the fact that $\omega + \omega^{-1} = 1$).

Thus $r^*(\chi_1) = \mathbb{1} + \varepsilon$ and $r^*(\chi_2) = r^*(\chi_3) = \chi_V$ where V is the 2-dimensional irreducible representation of S_3 consisted with formula (2) since $r(\mathbb{1}) = r(\varepsilon) = \chi_1$ and $r(\chi_V) = \chi_2 + \chi_3$.

Note that χ is an irreducible character of G, $r^*(\chi)$ can be an irreducible character of G but need not be in general. Also note that $r^*(\chi)(e) = \frac{|G|}{|H|}\chi(e)$.

We'd like to build a representation of G with character $r^*(\chi)$ given a representation W of H with character χ.

Suppose that X is a finite set and W is a k-vector space we may define

$$\mathcal{F}(X, W) = \{f : X \to W\}$$

the k-vector space of functions X to W.

So $\mathcal{F}(X, k) = kX$. Then $\dim \mathcal{F}(X, W) = |X| \dim W$ since if $w_1, \ldots, w_d$ is a basis for W then

$$(\delta_x w_i \mid x \in X, 1 \le i \le d)$$

is a basis for $\mathcal{F}(X, W)$.

If K is a group and X has a K-action and W is a representation of k then $\mathcal{F}(X, W)$ is a representation of K via $(k \cdot f)(x) = k \cdot (f(k^{-1}x))$ for all $f \in \mathcal{F}(X, W), k \in K, x \in X$. For example if $W = k$ is the trivial representation, then $\mathcal{F}(X, W) = kX$ as representations of K.

Now suppose $H \leq G$ are finite groups then G can be viewed as a set with $G \times H$-action via $(g \cdot h) \cdot x = gxh^{-1}$, $\forall (g,h) \in G \times H, x \in X$.

If W is a representation of G then we can view W sa a representation of $G \times H$ via

$$(g,h) \cdot w = h \cdot w$$

Now $\mathcal{F}(G,W)$ is a representation of $G \times H$ via

$$((g,h) \cdot f)(x) = hf(g^{-1}xh) \qquad \forall (g,h) \in G \times H, f \in \mathcal{F}(G,W), x \in G$$

So it can be viewed as a representation of G and as a representation of H via $g \mapsto (g, e_H)$ and $h \mapsto (e_G, h)$ respectively and the actions commute.

Now

$$\begin{aligned} \mathcal{F}(G,W)^H &= \{f \in \mathcal{F}(G,W) \mid (e,h)f = f \ \forall h \in H\} \\ &= \{f \in \mathcal{F}(G,W) \mid f(xh) = h^{-1}f(x) \ \forall h \in H, x \in G\} \end{aligned}$$

is a G-invariant subspace of $\mathcal{F}(G,W)$ since if $(e,h) \cdot f = f$ then for $g \in G$,

$$(e,h)(g,e)f = (g,e)(e,h)f = (g,e)f$$

Example. $\mathcal{F}(G,k)^H \simeq kG/H$ if k is the trivial representation of G.

Definition (Induced representation)**.** Suppose H is a subgroup of a finite group G and W is a representation of H. The *induced representation*

$$\operatorname{Ind}_H^G W = \mathcal{F}(G,W)^H$$

is a representation of G.

Lemma. $\dim \operatorname{Ind}_H^G W = \frac{|G|}{|H|} \dim W$.

Proof. Let $X = G/H$ be the left cosets of H in G and let $x_1, \dots, x_{|G/H|}$ be coset representatives. Then

$$\begin{aligned} \theta : \mathcal{F}(G,W)^H &\to \mathcal{F}(X,W) \\ \theta(f)(x_i H) &= f(x_i) \end{aligned}$$

is a k-linear map with inverse $\varphi(l)(x_i h) = h^{-1}l(x_i)$ for all $l \in \mathcal{F}(X,W), h \in H$, $i = 1, \dots, |G/H|$. $\square$

Theorem (Frobenius reciprocity)**.** If V is a representation of G and W is representation of G then

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) \simeq \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$$

where $\mathrm{Res}_H^G V$ is the restriction of V to H.

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Corollary. If $k = \mathbb{C}$ then

$$\langle \chi_V, \chi_{\mathrm{Ind}_H^G W} \rangle_G = \langle \chi_V|_H, \chi_W \rangle_G$$

In particular, $\chi_{\mathrm{Ind}_H^G W} = \Gamma^*(\chi_W)$.

Proof of Frobenius reciprocity. We'll prove $\mathrm{Hom}_G(V, \mathcal{F}(G, W)) \simeq \mathrm{Hom}_k(V, W)$ as representations of H and then deduce the result by taking H-invariants. Here the action of H on the left hand side is given by

$$(h \cdot \theta)(v) = h \cdot \theta(v) \qquad \theta \in \mathrm{Hom}_G(V, \mathcal{F}(G, W)), v \in V, h \in H$$

so $\mathrm{Hom}_G(V, \mathcal{F}(G, W))^H = \mathrm{Hom}_G(V, \mathcal{F}(G, W)^H) = \mathrm{Hom}_G(V, \mathrm{Ind}_H^G W)$. Note that this means

$$\begin{aligned} (h \cdot \theta)(v)(x) &= h(\theta(v)(xh)) \\ h(\theta(h^{-1}x^{-1}v)(w)) \end{aligned}$$

$\forall x \in G$ since θ is G-invariant. We can define a linear map

$$\begin{aligned} \psi : \mathrm{Hom}_G(V, \mathcal{F}(G, W)) &\to \mathrm{Hom}_k(V, W) \\ \psi(\theta)(v) &= \theta(v)(e) \end{aligned} \tag{$*$}$$

We claim ψ is an H-intertwining map. First we prove for $h \in H$, $\theta \in \mathrm{Hom}_G(V, \mathcal{F}(G, W))$, $v \in V$.

$$\begin{aligned} h \cdot (\psi(\theta))(v) &= h(\psi(\theta)(h^{-1}v)) \\ &= h(\theta(h^{-1}v)(e)) \\ &= (h \cdot \theta)(v)(e) && \text{by } (*) \text{ for } x = e \\ &= \psi((h \cdot \theta))(v) \end{aligned}$$

and ψ is H-equivariant.

Given $\varphi \in \mathrm{Hom}_k(V, W)$ we can define

$$\begin{aligned} \varphi_G &\in \mathrm{Hom}_k(V, \mathcal{F}(G, W)) \\ \varphi_G(v)(x) &= \varphi(x^{-1}v) \qquad \forall x \in G, v \in V \end{aligned}$$

Then for all $g, x \in G$, $v \in V$,

$$\varphi_G(gv)(x) = \varphi(x^{-1}gv) = \varphi_G(v)(g^{-1}x) = (g \cdot \varphi_G(v))(x)$$

i.e. $\varphi_G \in \operatorname{Hom}_G(V, \mathcal{F}(G, W))$. We can compute $\psi(\varphi_G)(v) = \varphi_G(v)(e) = \varphi(v)$, $\varphi \in \operatorname{Hom}_k(V, W), v \in V$ and

$$\psi(\theta)_G(v)(x) = \psi(\theta)(x^{-1}v) = \theta(x^{-1}v)(e) = x^{-1}\theta(v)(e) = \theta(v)(x)$$

for $\theta \in \operatorname{Hom}_G(V, \mathcal{F}(G, W)), x \in G, v \in V$. Thus $\varphi \mapsto \varphi_G$ is an inverse to ψ. $\square$

Remark. We could instead have computed $\chi_{\operatorname{Ind}_H^G V}$ directly and shown that it is equal to $r^*(\chi_W)$ and then deduced Frobenius reciprocity from this when $k = \mathbb{C}$.

6.2 Mackey Theory

This is the study of representations like $\operatorname{Res}_K^G \operatorname{Ind}_H^G W$ for H, k subgroups of G and W a representation of H. We can (and will) use it to characterise when $\operatorname{Ind}_H^G W$ is irreducible as a representation of G (when $k = \mathbb{C}$). If H, K are subgroups of G then $H \times K$ acts on G via

$$(h, k) \cdot g = kgh^{-1}$$

An orbit of this action is called a *double coset*. We write

$$KgH = \{kgh \mid k \in K, h \in H\}$$

for the orbit containing g.

Definition. $K \backslash G / H = \{KgH \mid g \in G\}$ is the set of double cosets.

For any representation (ρ, W) of H and $g \in G$ we can define a representation $({}^g\rho, {}^gW)$ bw

$$\rho^g : {}^gH \to \operatorname{GL}(W)$$
$$ghg^{-1} \mapsto \rho(h)$$

where ${}^gH := gHg^{-1} \leq G$.

Theorem (Mackey's restriction formula)**.** If G is a finite group with subgroups H and K and W is a representation of H then

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W \simeq \bigoplus_{KgH \in K\backslash H/H} \operatorname{Ind}_{K \cap {}^gH}^K \operatorname{Res}_{{}^gH \cap K}^H {}^gW$$

Proof. Note that

$$\begin{aligned}\operatorname{Ind}_H^G W &= \mathcal{F}(G,W)^H \\ &= \mathcal{F}\left(\coprod_{KgH\in K\backslash G/H} KgH, W\right)^H \\ &\cong \bigoplus_{KgH\in K\backslash G/H} \mathcal{F}(KgH,W)^H\end{aligned}$$

as representations of K.

So it suffices to show

$$\mathcal{F}(KgH,W)^H \simeq \mathcal{F}(K,{}^gW)^{K\cap {}^gH}$$

as representations of K. We'll defer this to next time. □

Corollary (Character version of Mackey restriction)**.** If χ is a character or a representation of a H then

$$(\operatorname{Ind}_H^G \chi)|_K = \sum_{KgH\in K\backslash G/H} \operatorname{Ind}_{{}^gH\cap K}^K ({}^g\chi|_{{}^gH\cap K})$$

Exercise: Prove this corollary directly using characters.

Corollary (Mackey's irreducibility criterion)**.** If $H \le G$ and W is a $\mathbb{C}$-representation of H then $\operatorname{Ind}_H^G W$ is irreducible if and only if

(i) W is irreducible as a representation of H

(ii) For each $g \in G \setminus H$, the two representations $\operatorname{Res}_{{}^gH\cap H}^{{}^gH} {}^gW$ and $\operatorname{Res}_{{}^gH\cap H}^{H} W$ of $H \cap {}^gH$ have no irreducible subrepresentations in common.

Proof.

$$\begin{aligned}\langle \chi_{\operatorname{Ind}_H^G W}, \chi_{\operatorname{Ind}_H^G W}\rangle_G &= \langle \chi_W, \chi_{\operatorname{Res}_H^G \operatorname{Ind}_H^G W}\rangle_G && \text{(Frobenius reciprocity)} \\ &= \sum_{HgH\in H\backslash G/H} \langle \chi_W, \chi_{\operatorname{Ind}_{H\cap {}^gH}^H \operatorname{Res}_{{}^gH\cap H}^{{}^gH} {}^gW}\rangle_H && \text{(Mackey's restriction formula)} \\ &= \sum_{HgH\in H\backslash G/H} \langle \operatorname{Res}_{{}^gH\cap H}^H \chi_W, \operatorname{Res}_{{}^gH\cap H}^{{}^gH} \chi_{{}^gW}\rangle && \text{(Frobenius reciprocity)}\end{aligned}$$

So $\operatorname{Ind}_H^G W$ is irreducible if and only if RHS is 1. The term for the double coset HeH is $\langle \chi_W, \chi_W\rangle_H \ge 1$ and all the other terms are ≥ 0 so irreducibility is equivalent to $\langle \chi_W, \chi_W\rangle_H = 1$ and all otehr temrs are 0.

$\langle \chi_W, \chi_W \rangle_H = 1$ if and only if condition (1).

$\langle \operatorname{Res}^H_{H\cap {}^gH} \chi_W, \operatorname{Res}^{{}^gH}_{{}^gH\cap H} \chi_{{}^gW} \rangle = 0$ if and only if (ii) for g. □

Note for condition (ii) we only need to check for a family of double cosets excluding $HeH = H$.

Corollary. If $H \trianglelefteq G$ and W is an irreducible representation of H then

$$\operatorname{Ind}^G_H W \text{ is irreducible} \iff {}^g\chi_W \neq \chi_W \qquad \forall g \in G \setminus H$$

$({}^g\chi_W(ghg^{-1}) = \chi_W(h))$.

Proof. Since $H \trianglelefteq G$, ${}^gH = H$ for all $g \in G$ and ${}^gW$ is irreducible since W is. So by Mackey's irreducibility criterion,

$$\begin{aligned} \operatorname{Ind}^G_H W \text{ is irreducible} &\iff W \not\simeq {}^gW \qquad \forall g \in G \setminus H \\ &\iff \chi_W \neq {}^g\chi_w \qquad \forall g \in G \setminus H \end{aligned}$$

□

Example.

(1) $H = \langle r \rangle \simeq C_n$ the subgroup of rotations in $G = D_{2n}$. The irreducible characters of H are all of the form

$$\chi_k(r^j) = e^{2\pi ijk/n}$$

We see that $\operatorname{Ind}^G_H \chi_K$ is irreducible if and only if

$$\chi_K(rj) \neq \chi(r-j) \text{ for some } j \iff \chi_K \text{ is not real valued}$$

(2) $G = S_n$, $H = A_n$. If $g \in S_n$ is a cycle type that splits in A_n and χ is an irreducible character of A_n taking different values on the two classes, then

$$\operatorname{Ind}^{S_n}_{A_n} \chi$$

is irreducible.

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Recall that for $g \in G$,

$$\mathcal{F}(KgH, W)^H = \{f : KgH \to W \mid f(xh) = h^{-1}f(x) \ \forall x \in KgH, h \in H\}$$

is a representation of K via

$$(kf)(x) = f(k^{-1}x) \qquad \forall x \in KgH \in k \in K.$$

Last time we reduced the proof of Mackey's restriction formula to the following lemma:

Lemma. There is an isomorphism of representations of K

$$\mathcal{F}(KgH, W)^H \xrightarrow{\sim} \mathcal{F}(K, {}^gW)^{K \cap {}^gH}.$$

Proof. Let $\Theta : \mathcal{F}(KgH, W)^H \to \mathcal{F}(K, {}^gW)$, $\Theta(f)(k) = f(kg)$. If $k' \in K$,

$$(k' \cdot \Theta(f))(k) = \Theta(f)(k'^{-1}k) = f(k'^{-1}g) = (k' \cdot f)(kg) = \Theta(k'f)(k)$$

i.e. Θ is k-linear. If $ghg^{-1} \in K$ for some $h \in H$, then

$$\begin{aligned} \Theta(f)(kghg^{-1}) &= f(kgh) \\ &= \rho(h)^{-1}f(kg) \\ &= ({}^g\rho)(ghg^{-1})\Theta(f)(k) \end{aligned}$$

i.e. $\operatorname{Im}\Theta \le \mathcal{F}(K, {}^gW)^{K \cap {}^gH}$. We try to define an inverse to Θ via

$$\begin{aligned} \psi : \mathcal{F}(K, {}^gW)^{K \cap {}^gH} &\to \mathcal{F}(KgH, W)^H \\ \psi(f)(kgh) &= \rho(h^{-1})f(k) \end{aligned}$$

If $k_1gh_1 = k_2gh_2$ then $k_2^{-1}k_1 = g(h_2h_1^{-1})g^{-1} \in K \cap {}^gH$.

$$\begin{aligned} f(k_2) &= f(k_1(k_2^{-1}k_1)^{-1}) \\ &= ({}^g\rho)(gh_2h_1^{-1}g^{-1})f(k_1) \\ \rho(h_2h_1^{-1})f(k_1) \end{aligned}$$

So $\rho(h_2)^{-1}f(k_2) = \rho(h_1^{-1})f(k_1)$ i.e. $\psi(f)$ is well-defined. Moreover if $f \in \mathcal{F}(KgH, W)^H$, then

$$\psi\Theta(f)(kgh) = \rho(h)^{-1}\Theta(f)(k) = \rho(h^{-1})f(kg) = f(kgh).$$

and if $f \in \mathcal{F}(K, {}^gW)^{{}^gH \cap K}$. Also

$$\Theta\psi(f)(k) = \psi(f)(kg) = f(k)$$

so ψ is inverse to Θ. $\square$

6.3 Frobenius Groups

Theorem (Frobenius 1901)**.** Let G be a finite group acting transitively on a set X. If each $g \in G \setminus \{e\}$ fixes at most one element of X then

$$K = \{e\} \cup \{g \in G \mid gx \neq x \ \forall x \in X\}$$

is a normal subgroup of G of order $|X|$.

Definition (Frobenius group)**.** A *Frobenius group* is a finite group G that has a transitive action on a set X with $1 < |X| < |G|$ such that each $g \in G \setminus \{e\}$ fixes at most one element of X. It follows from Frobenius 1901 that Frobenius groups can't be simple. The subgroup K is called the *Frobenius kernel* and any of the subgroups

$$\mathrm{Stab}_G(x)$$

for $x \in X$ are called *Frobenius complements.*

Example.

(1) $G = D_{2n}$. For n odd acting on vertices of an n-gon in the usual way. The reflections fix precisely one vertex and the non-trivial solutions fix no vertices.

(2)
$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a, b \in \mathbb{F}_p, a \neq 0 \right\}$$

acting on

$$X = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} \middle| x \in \mathbb{F}_p \right\}$$

by matrix multiplication.

Note. No proof of Frobenius 1901 is known that does not use representation theory!

Proof of Frobenius 1901. Fix $x \in X$ and let $H = \mathrm{Stab}_G(x)$ so $|G| = |H||X|$ by Orbit Stabiliser theorem. By hypothesis if $g \in G \setminus H$ then

$$\{e\} = \mathrm{Stab}_G(gx) \cap \mathrm{Stab}_G(x).$$

Thus

(i) $\left|\bigcup_{g\in G} gHg^{-1}\right| = \left|\bigcup_{x\in X} \mathrm{Stab}_G(x)\right| = (|H|-1)|X|+1$

(ii) If $h_1, h_2 \in H$ then $[h_1]_H = [h_2]_H \iff [h_1]_G = [h_2]_G$.

(iii) $C_G(h) = C_H(h)$ if $h \in H \setminus \{e\}$.

By (i),

$$|K| = \left|\{e\} \cup \left(G \setminus \bigcup_{x\in X} \mathrm{Stab}_G(x)\right)\right| = |H||X| - ((|H|-1)|X|+1)+1 = |X|$$

as claimed. We must show $K \trianglelefteq G$.

If χ is any character of H, we can compute $\mathrm{Ind}_H^G\chi$:

$$\begin{aligned}\mathrm{Ind}_H^G\chi(g) &= \sum_{[h]_H\subset[g]_G} \frac{|C_G(x)|}{|C_H(h)|}\chi(h) \\ &= \begin{cases} \frac{|G|}{|H|}\chi(e) & \text{if } g = e \\ \chi(h) & \text{if } [g]_G = [h]_G \neq \{e\} \text{ by (i) and (ii)} \\ 0 & \text{if } g \in K \setminus \{e\} \end{cases}\end{aligned}$$

Suppose $\mathrm{Irr}(H) = \{\chi_1, \dots, \chi_r\}$ and let

$$\boxed{\theta_i = \mathrm{Ind}_H^G\chi_i - \chi_i(e)\mathbb{1}_G - \chi_i(e)\,\mathrm{Ind}_H^G\mathbb{1}_H \in R(G)}$$

(this is sort of the magic bit). So

$$\theta_i(g) = \begin{cases} \chi_i(e) & g = e \\ \chi_i(h) & [g]_G = [h]_G \text{ for some } h \in H \\ \chi_i(e) & g \in K \end{cases}$$

If θ_i were a character of a representation of G then the kernel of the representation would contain K. Since $\theta_i \in R(G)$, $\theta_i = \sum n_i\psi_i$ for $n_i \in \mathbb{Z}$ and $\psi_i \in \mathrm{Irr}(G)$. Now we calculate:

$$\begin{aligned}\langle\theta_i, \theta_i\rangle_G &= \frac{1}{|G|}\sum_{g\in G} |\theta_i(g)|^2 \\ &= \frac{1}{|G|}\left(\sum_{h\in H\setminus\{e\}} \frac{|G|}{|H|}|\chi_i(h)|^2 + \sum_{k\in K} |\chi_i(e)|^2\right) \\ &= \frac{|X|}{|G|}\sum_{h\in H} |\chi_i(h)|^2 \\ &= \langle\chi_i, \chi_i\rangle_H \\ &= 1\end{aligned}$$

So $\sum n_j^2 = 1$ and $\theta_i = \pm\psi_j$ for some j. But $g_i(e) = \chi_i(e) > 0$ so $\theta_i \in \mathrm{Irr}(G)$.

To finish we write

$$\theta = \sum_{i=1}^{r} \chi_i(e)\theta_i$$

and so $\theta(h) = \sum_{i=1}^{r} \chi_i(e)\chi_i(h) = 0$ for $h \in H \setminus \{e\}$ by column orthogonality. Also $\theta(k) = \sum_{i=1}^{r} \chi_i(e)^2 = |H|$ by column orthogonality. Thus K is the kernel of the representation corresponding to θ. □

In his thesis, John Thompson proved (among other things), that K must be nipotent or equivalently the product of its Sylow-p-subgroups.

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7 Arithmetic Properties of Characters

We'll assume G is finite and $k = \mathbb{C}$.

7.1 Arithmetic results

The following facts will be proved in Number Fields next term.

Definition. $x \in \mathbb{C}$ is an *algebraic integer* if it is a root of a monic polynomial with integer coefficients.

Facts

(1) The algebraic integers form a subring $\mathcal{O}$ of $\mathbb{C}$.

(2) Any subring of $\mathbb{C}$ that is finitely generated as an (additive) abelian group is contained in $\mathcal{O}$.

(3) If $x \in \mathcal{O} \cap \mathbb{Q}$ then $x \in \mathbb{Z}$ (see)

For (1) and (2), see GRM Example Sheet 4, Q13 from 2023. For (3), see Numbers & Sets Example Sheet 3, Q3 from 2021.

Lemma. If χ is a character of G then $\chi(g) \in \mathcal{O}$ for all $g \in G$.

Proof. We know that $\chi(g)$ is a sum of n-th roots of unity ($n = |G|$, say). Each such n-th root of unity satisfies $X^n - 1$ and so lies in $\mathcal{O}$. So $\chi(g) \in \mathcal{O}$ by Fact (1). □

The group algebra

We now want to make the k-vector space kG into a ring. There are two sensible ways to do this. One is by pointwise multiplication making kG a commutative ring. More usefully for us right now is the *comvolution product*

$$(f_1 f_2)(g) = \sum_{x \in X} f_1(gx) f_2(x^{-1}) = \sum_{\substack{x,y \in G \\ xy = g}} f_1(x) f_2(y).$$

that makes kG into a (usually) non-commutative ring.

We can verify that $\delta_g\delta_h = \delta_{gh}$ for all $g, h \in G$. So we can rephrase the product as

$$\left(\sum g \in G\lambda_g\delta_g\right)\left(\sum_{h\in G}\mu_H\delta_h\right) = \sum_{k\in G}\left(\sum{}_{gh=k}^{g,h\in G}\lambda_j\mu_h\right)\delta_k.$$

From now on we'll have this product in mind when we view kG as a ring. A (finitely generated) kG-module is "the same" as a representation of G. Given a representation (ρ, V) of G we make V into a (finitely generated) kG-module via

$$f \cdot v = \sum_{g\in G} f(g)\rho(g)v \qquad \forall v \in V, f \in kG.$$

Conversely, given a finitely generated kG-module M, the underlying k-vector space is a representation of G via

$$\rho(g)(m) = \delta_{g\cdot m} \qquad \forall m \in M, g \in G.$$

Moreover, under this correspondence G-linear maps correspond to kG-module homomorphisms.

Exercise: Suppose kX is a permutation representation of G. Calculate the action of $f \in kG$ on kX under the correspondence.

It will prove useful to study $Z(kG)$, the *centre* of kG; that is the subring of kG consisting of elements $f \in kG$ such that $fh = hf$ for all $h \in kG$. This is because for $f \in Z(kG)$,

$$\sum_{g\in G} f(g)\rho(g) \in \operatorname{Hom}_G(V, V)$$

for every representation (ρ, V) of G.

Lemma. Suppose $f \in kG$. Then $f \in Z(kG)$ if and only if $f \in \mathcal{C}_G$ the space of class functions. In particular, $\dim_k Z(kG) = \#\text{conjugacy classes in } G$.

Proof.

$$\begin{aligned} f \in kG &\iff fh = hf \quad \forall h \in kG \\ &\iff f\delta_g = \delta_g f \quad \forall g \in G \\ &\iff \delta_{g^{-1}} f\delta_g = f \qquad\qquad \text{(since } \delta_e = 1 \text{ and } \delta_{g^1}\delta_g = \delta_e) \end{aligned}$$

But

$$\begin{aligned} (\delta_{g^{-1}} f\delta_g)(x) &= \sum_{g\in G}(\delta_{g^{-1}} f)(xy^{-1})\delta_g(y) \\ &= (\delta_{g^{-1}} f)(xg^{-1}) \\ &= f(gxg^{-1}) \qquad \forall g \in G \end{aligned}$$

So $f \in Z(kG)$ if and only if $f \in \mathcal{C}_G$ as required. □

Remark. The multiplication on $Z(kG)$ and $\mathcal{C}_G$ will not be the same even though their k-vector space structures are the same even though both are commutative.

Notation. Given $g \in G$ define the *class sum*

$$C_{[g]_G}(x) = \begin{cases} 1 & x \in [g]_G \\ 0 & x \notin [g]_G \end{cases}$$

Then if $[g_1]_G, \dots, [g_r]_G$ is a list of conjugacy classes in G we write

$$C_i := C_{[g_i]_G}.$$

We used to write $\mathbb{1}_{[g_i]_G}$ for C_i. We have switched to draw attention to the different multiplication.

Proposition.

$$C_i C_j = \sum_{l=1}^{r} a_{ij}^l C_l$$

where

$$a_{ij}^l = |\{(x,y) \in [g_1]_G \times [g_j]_G \mid xy = g_l\} \in \mathbb{Z}.$$

The a_{ij}^l are called the *structure constants* of $Z(kG)$.

Proof. Since $Z(kG)$ is a ring,

$$C_i C_j = \sum_{l=1}^{r} a_{ij}^l C_l$$

for some $a_{ij}^l \in k$. But we explicitly compute

$$\begin{aligned} a_{ij}^l &= (C_i C_j)(g_l) \\ &= \sum_{x,y \in G} C_i(x) C_j(y) \\ &= |\{(x,y) \in [g_i]_G \times [g_j]_G \mid xy = g_l\}| \end{aligned}$$

as required. □

SUppose now that (ρ, V) is an irreducible representation of G. Then we've seen that if $z \in Z(kG)$, then

$$z : V \to V, \quad zv = \sum_{g \in G} z(g)\rho(g) \in \mathrm{Hom}_G(V,V) = k\mathrm{id}_V$$

(k algebraically closed). So we get a k-algebra homomorphism

$$\omega_\rho : Z(kG) \to k$$

where $z \in Z(kG)$ acts by $\omega_{\rho(z)}\mathrm{id}_V$ on V. Taking traces we see

$$\begin{aligned}(\dim V_i)\omega_{\rho(z)} &= \sum_{g\in G} z(g)\chi_\rho(g) \\ \implies \omega_\rho(z) &= \sum_{g\in G} \frac{z(g)\chi_\rho(g)}{\chi_\rho(e)} \\ \implies \omega_\rho(C_i) &= \frac{\chi_\rho(g_i)}{\chi_\rho(e)}|[g_i]_G| \qquad (\dagger)\end{aligned}$$

We now see that ω_ρ only depends on χ_ρ so we can write $\omega_{\chi_\rho} = \omega_\rho$.

Lemma. The values $\omega_\chi(C_i) \in \mathcal{O}$ for all irreducible characters χ.

Note that htis is not immediately clear as $\frac{1}{\chi(e)} \notin \mathcal{O}$ for $\chi(e) \neq 1$.

Proof. Since ω_χ is an algebra homomorphism,

$$\omega_\chi(C_i)\omega_\chi(C_j) = \sum_{l=1}^{r} a_{ij}^l \omega_\chi(C_l) \qquad (*)$$

so the subring of $\mathbb{C}$ generated by the $\omega_\chi(C_i)$ is a finitely generated abelian group under $+$. By Fact (2) in Section 7.1, it follows that ecah $\omega_\chi(C_i) \in \mathcal{O}$. □

Lemma.

$$a_{ij}^l = \frac{|G|}{|C_G(g_i)||C_G(g_j)|} \sum_{\chi\in\mathrm{Irr}(G)} \frac{\chi(g_i)\chi(g_j)\chi(g_l^{-1})}{\chi(e)}.$$

In particular, a_{ij}^l is deterined by the character table.

Proof. By $(*)$ and $(\dagger)$

$$\frac{\chi(g_i)}{\chi(e)}|[g_i]_G|\frac{\chi(g_j)}{\cancel{\chi(e)}}|[g_j]_G| = \sum_{l=1}^{r} \frac{a_{ij}^l \chi(g_l)|[g_l]_G|}{\cancel{\chi(e)}}$$

TODO □

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7.2 Degree of irreducible representations

Theorem. If V is a simple representation of G then $\dim V \mid |G|$.

Proof. Let $\chi = \chi_V$ and we'll show $\frac{|G|}{\dim V} \in \mathcal{O} \cap \mathbb{Q} = \mathbb{Z}$.

$$\begin{aligned} \frac{|G|}{\dim V} &= \frac{1}{\chi(e)} \sum_{g \in G} \chi(g)\chi(g^{-1}) \\ &= \frac{1}{\chi(e)} \sum_{i=1}^{r} |[g_i]_G| \chi(g_i)\chi(g_i^{-1}) \\ &= \sum_{i=1}^{r} \omega_\chi(g_i)\chi(g_i^{-1}) \\ &\in \mathcal{O} \end{aligned}$$

since $\mathcal{O}$ is a ring and $\chi(g_i^{-1})$ and $\omega_\chi(g_i)$ are all in $\mathcal{O}$. But also $\frac{|G|}{\dim V} \in \mathbb{Q}$, so $\frac{|G|}{\dim V} \in \mathbb{Z} = \mathcal{O} \cap \mathbb{Q}$ as required. □

Example.

(1) If G is a p-group and χ is an irreducible character then $\chi(e)$ is always a power of p. In particular, if $|G| = p^2$ then as $\sum_{\chi \in \mathrm{Irr}(G)} \chi(e)^2 = |G|$ we see that $\chi(e) = 1$ for all $\chi \in \mathrm{Irr}(G)$, i.e. G is abelian.

(2) If G is A_n or S_n and $p > n$ is prime, then p can't divide the degree of an irreducible representation.

In fact a strange result is true.

Theorem (Burnside (1904))**.** If (ρ, V) is a simple representation of G then we have $\dim V \bigm| |G/Z(G)|$.

Compare to $|[g]_G| = \frac{|G|}{|C_G(g)|} \Bigm| \frac{|G|}{|Z(G)|}$ for all $g \in G$.

Proof. If $Z = Z(G)$ then by Schur's Lemma $\rho|_Z : Z \to \mathrm{GL}(V)$ has image contained in $k^* \mathrm{id}_V$. $\rho(z) = \lambda_z \mathrm{id}_V$ say for each $z \in Z$. For each $m \geq 2$ consider the irreducible representation of $G^m = \underbrace{G \times G \times \cdots \times G}_{m \text{ times}}$ given by $\rho^{\otimes m} : G^m \to \mathrm{GL}(V^{\otimes m})$. If $z =$

$(z_1, \dots, z_m) \in Z^m$ then

$$\begin{aligned} \rho^{\otimes m}(z) &= \sum_{i=1}^{m} \lambda_{z_i} \mathrm{id}_{V^{\otimes m}} \\ &= \lambda_{(\prod_{i=1}^m z_i)} \mathrm{id}_{V^{\otimes m}} \end{aligned}$$

So if $\prod_{i=1}^m z_i = 1$ then $z \in \ker \rho^{\otimes m}$. So $V^{\otimes m}$ can be viewed as an irreducible degree $(\dim V)^m$ representation of $\frac{G^m}{Z'}$ where

$$Z' = \left\{ (z_1, \dots, z_m) \in Z^m \;\middle|\; \prod_{i=1}^m z_i = 1 \right\} \le Z^m.$$

Moreover $|Z'| = |Z|^{m-1}$. So by previous theorem $(\dim V)^m \mid \frac{|G|^m}{|Z|^{m-1}}$. Now if p is a prime and $p^a \mid \dim V$ then $p^{am} \mid \frac{|G|^m}{|Z|^{m-1}} = \left|\frac{G}{Z}\right|^m |Z|$. By taking m large enough that $p^m \nmid |Z|$, we see that $p^a \mid \left|\frac{G}{|}\right|$. Thus $\dim V \mid \left|\frac{G}{Z}\right|$ as claimed. □

Proposition. If G is a simple group then G has no irreducible representations of degree 2.

Proof. If G is abelian then all irreducible representations have degree 1. So we may assume that G is non-abelian. If $|G|$ is even then $\exists x \in G$ of order 2. By Example Sheet 2, Question 2, if χ is an irreducible character of G then $\chi(x) \equiv \chi(e) \pmod 4$. So if $\chi(e) = 2$ then $\chi(x) = \pm 2$ so $\rho(x) = \pm I$. Thus $\rho(x) \in Z(\rho(G))$, ※. (G is non-abelian and simple, and ρ is non-trivial).

Now if $|G|$ is odd, we're done by (either of) today's theorems so far. □

7.3 Burnside's $p^a q^b$ theorem

Lemma. Suppose $0 \neq \alpha \in \mathcal{O}$ is of the form $\frac{1}{m} \sum_{i=1}^m \lambda_i$ for some $\lambda_i \in \mathbb{C}$ such that $\lambda_i^n = 1$ for some $n \in \mathbb{N}$. Then $|\alpha| = 1$ (and so all λ_i are equal).

Sketch-proof (non-examinable). See Galois Theory for the details.

By assumption, $\alpha \in \mathbb{Q}(\varepsilon)$, $\varepsilon = e^{2\pi i/n}$. Let $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}(\varepsilon)/\mathbb{Q})$. It is known that

$$\{\beta \in \mathbb{Q}(\varepsilon) \mid \sigma(\beta) = \beta \ \forall \sigma \in \mathcal{G}\} = \mathbb{Q}.$$

Consider $N(\alpha) = \prod_{\sigma \in \mathcal{G}} \sigma(\alpha)$. It is easy to verify that $\sigma(N(\alpha)) = N(\alpha)$ for all $\sigma \in \mathcal{G}$, i.e. $N(\alpha) \in \mathbb{Q}$. Moreover, $N(\alpha) \in \mathcal{O}$ since if α satisfies a monic integer polynomial then every $\sigma(\alpha)$ $(\sigma \in \mathcal{G})$ satisfies the same polynomial. Thus $N(\alpha) \in \mathbb{Z}$. But for each $\sigma \in \mathcal{G}$,

$$|\sigma(\alpha)| = \left| \frac{1}{m} \sum_{i=1}^{m} \sigma(\lambda_i) \right| \leq 1.$$

So $N(\alpha) = \pm 1$ and $|\alpha| = 1$. $\square$

Lemma. Suppose χ is an irreducible character of G and $g \in G$ such that $\chi(e)$ and $|[g]_G|$ are coprime. Then $|\chi(g)| = \chi(e)$ or $|\chi(g)| = 0$.

Note that if $|\chi(g)| = \chi(e)$ then g acts as a scalar on the corresponding representation V and so $\rho(g) \in Z(\rho(G))$.

Proof. By Bezout's lemma, we can find $a, b \in \mathbb{Z}$ such that $a\chi(e) + b|[g]_G| = 1$. Then

$$a\chi(g) + b\left(\frac{|[g]_G|\chi(g)}{\chi(e)}\right) = \frac{\chi(g)}{\chi(e)} =: \alpha \in \mathcal{O}$$

Since $\chi(g)$ is a sum of $\chi(e)$ $|G|$-th roots of unity, it follows from the last lemma that $\alpha = 0$ or $|\alpha| = 1$. $\square$

Proposition. If G is a finite non-abelian group with $g \neq e$ such that $|[g]_G|$ has prime power order, then G is not simple.

Proof. Suppose for contradiction that G is simple and $g \in G \setminus \{e\}$ such that $|[g]_G| = p^r$ for some prime p. If $\chi \in \mathrm{Irr}(G) \setminus \{\mathbb{1}_G\}$ then $|\chi(g)| < \chi(e)$ since otherwise $\rho(g)$ is a scalar and lies in $Z(\rho(g)) = 1$. Thus by the last lemma, for every non-trivial character χ, either $p \mid \chi(e)$ or $\chi(g) = 0$. By column orthogonality,

$$0 = \sum_{\chi \in \mathrm{Irr}(G)} \chi(e)\chi(g)$$

Thus

$$-\frac{1}{p} = \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi \neq \mathbb{1}}} \frac{\chi(e)}{p}\chi(g) \in \mathcal{O} \cap \mathbb{Q} = \mathbb{Z} \quad ※$$

$\square$

Theorem (Burnside 1904)**.** Let p, q be primes and G a group of order $p^a q^b$ with $a, b \geq 0$ and $a + b \geq 2$. Then G is not simple.

Proof. Without loss of generality $b > 0$. Let $\mathbb{Q}$ be a Sylow-p-subgroup of G and pick $g \in Z(\mathbb{Q}) \setminus \{e\}$ (possible since $\mathbb{Q}$ is a q-group). Now $q^b \mid |C_G(g)|$ so $|[g]_G| = p^r$ for some $0 \leq r \leq a$. The Theorem follows from the last proposition. □

Remark.

(1) It follows that every group of order $p^a q^b$ is soluble, i.e. there exists a chain of subgroups $G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n = \{e\}$ such that for all i, $G_{i+1} \trianglelefteq G_i$ and G_i / G_{i+1} is abelian.

(2) Note that $|A_5| = 2^2 \cdot 3 \cdot 5$ so a finite simple group can have precisely 3 prime factors. Conjugacy classes are $1, 15, 20, 12, 12$ not prime power order.

(3) The first purely group theoretic proof of the $p^a q^b$-theorem first appeared in 1972.

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8 Topological Groups

In this chapter, $k = \mathbb{C}$. This is important, because we will be using topological properties of $\mathbb{C}$ (contrary to previously, where we normally are just using the fact that it is algebraically closed).

8.1 Definitions and Examples

Definition (Topological group)**.** A *topological group* G is a group G which also has the structure of a topological space such that the multiplication map $G \times G \to G$, $(x, y) \mapsto xy$ and the inversion map $G \to G$, $x \mapsto x^{-1}$ are both continuous.

Examples

(1) $\mathrm{GL}_n(\mathbb{C})$ with the subspace topology from $\mathrm{Mat}_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$, since

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \qquad \text{and} \qquad A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

are both continuous. More generally, if V is any $\mathbb{C}$-vector space we can give $\mathrm{GL}(V)$ the topology that makes the isomorphism $\mathrm{GL}(V) \to \mathrm{GL}_n(\mathbb{C})$ (given by choosing a basis) a homeomorphism. Since conjugation on $\mathrm{GL}_n(\mathbb{C})$, $X \mapsto P^{-1}XP$ is continuous for all $P \in \mathrm{GL}_n(\mathbb{C})$, this does not depend on the choice of basis.

(2) G finite with discrete topology since all amps $G \times G \to G$ and $G \to G$ are continuous.

(3) $\mathrm{O}(n) = \{A \to \mathrm{GL}_n(\mathbb{R}) \mid A^\top A = I\}$, $\mathrm{SO}(n) = \{A \in \mathrm{O}(n) \mid \det A = 1\}$.

(4) $\mathrm{U}(n) = \{A \in \mathrm{GL}_n(\mathbb{C}) \mid \overline{A}^\top A = I\}$, $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) \mid \det(A) = 1\}$. In particular, $\mathrm{U}(1) = S^1 = (\{x \in \mathbb{C}^\times \mid |z| = 1\}, \cdot)$.

(5) $*$ (non-examinable) G a profinite group such as $\mathbb{Z}_p$ the completion of $\mathbb{Z}$ with respect to p-adic metric.

Definition (Representation of a topological group)**.** A *representation of a topological group* G is a continuous homomorphism

$$\rho : G \to \mathrm{GL}(V)$$

(V a vector space over $\mathbb{C}$).

Remark.

(1) If G is a (finite) group with the discrete topology then every function $G \to \mathrm{GL}(V)$ is continuous and we recover the old definition.

(2) If X is any topological space then $\alpha : X \to \mathrm{GL}_n(\mathbb{C})$ is continuous if and only if $\alpha_{ij} : X \to \mathbb{C}$, $\alpha_{ij}(x) := \alpha(x)_{ij}$ is continuous for all i, j.

8.2 Compact groups

Our most powerful when studying finite groups was the operator $\frac{1}{|G|}\sum_{g\in G}$. We want to replace $\sum$ by $\int$.

Definition (Haar integral)**.** For G a topological group and $C(G,\mathbb{R}) = \{f : G \to \mathbb{R} \mid f \text{ continuous}\}$, a linear map $\int_G : C(G,\mathbb{R}) \to \mathbb{R}$ is called a *Haar integral* if

(i) $\int_G \mathbb{1}_G = 1$ (So $\int_G$ is normalised so that total volume is 1).

(ii) $\int_G f(xg)\mathrm{d}g = \int_G f(g)\mathrm{d}g = \int_G f(gx)\mathrm{d}g$ for all $x \in G$ (so $\int_G$ is translation invariant). (we write $\int_G f(g)\mathrm{d}g = \int_G f$ and $\int_G f(xg)\mathrm{d}g$ means apply $\int_G$ to $g \mapsto f(xg) \in C(G,\mathbb{R})$).

(iii) $\int_G f \geq 0$ if $f(g) \geq 0$ for all $g \in G$ (positivity).

Example.

(1) If G is finite then $\int_G f = \frac{1}{|G|}\sum_{g\in G} f(g)$ is a Haar integral.

(2) If $G = S^1$, $\int_G f = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\mathrm{d}\theta$ is a Haar integral.

Note that for any $\mathbb{R}$-vector space V, $\int_G$ induces a linear map (also called $\int_G$)

$$\int_G : C(G,V) \to V$$

Under the identification $V \simeq V^{**}$ for $\theta \in V^*$, $f \in C(G,V)$,

$$\theta\left(\int_G f\right) = \int_G \theta(f(g))\mathrm{d}g$$

More concretely, if $v_1, \dots, v_n$ is a basis for V and $f \in C(G,V)$ then

$$f = \sum_{i=1}^{n} f_i v_i$$

with $f_i \in C(G,\mathbb{R})$ and

$$\int_G f = \sum_{i=1}^{n} \left(\int_G f_i \right) v_i.$$

This map is also translation invariant and sends a constant function to its only value. Moreover if $\alpha : V \to W$ is linear and $f \in C(G,V)$, then

$$\alpha \left(\int_G f \right) = \int_G \alpha(f).$$

In particular if V is a $\mathbb{C}$-vector space $v \mapsto iv$ is $\mathbb{R}$-linear so $\int_G : C(G,V) \to V$ is $\mathbb{C}$-linear.

Theorem. If G is a compact Hausdorff group then there is a unique Haar integral on G.

Proof. Omitted. □

All the examples in Section 8.1 are compact Hausdorff except $\mathrm{GL}_n(\mathbb{C})$ which is not compact. We'll follow standard practice in this field and write "compact" to mean "compact and Hausdorff".

Corollary (Weyl's unitary trick)**.** If G is a compact topological group then every representation (ρ, V) of G is unitary.

Proof. As for finite groups, let $\langle \bullet, \bullet \rangle$ be an inner product on V. Then

$$(v,w) := \int_G \langle \rho(g)v, \rho(g)w \rangle \mathrm{d}g$$

is the required G-invariant inner product. Since, for $x \in G$ and $v, w \in V$,

$$\begin{aligned} (\rho(x)v, \rho(x)w) &= \int_G \langle \rho(gx)v, \rho(gx)w \rangle \mathrm{d}g \\ &= \int_G \langle \rho(g)v, \rho(g)w \rangle \mathrm{d}g && \left(\text{by } G\text{-invariance of } \int_G\right) \\ &= (v,w) \end{aligned}$$

Clearly $(\bullet, \bullet)$ is an inner product by using $\mathbb{C}$-linearity of $\int_G$ and positivity of $\int_G$. □

Remark. It follows that every compact subgroup of $\mathrm{GL}_n(\mathbb{C})$ is conjugate to a subgroup of $\mathrm{U}(n)$.

Corollary. All representations of a compact group are completely reducible.

If $G \to \mathrm{GL}(V)$ is a representation then $\chi_\rho := \operatorname{Tr} \rho$ is a continuous class function on G.

Lemma. If U is a representation of G compact then

$$\dim U^G = \int_G \chi.$$

Proof. Let $\pi \in \mathrm{Hom}_k(U, U)$ be defined by $\pi = \int_G \rho \in \mathrm{Hom}_k(U, U)$. If $x \in G$ then

$$\rho(x) \cdot \pi = \rho(x) \int_G \rho(g) \mathrm{d}g = \int_G \rho(xg) \mathrm{d}g = \pi$$

since $\int_G$ is translation invariant. So $\operatorname{Im} \pi \le U^G$. If $u \in U^G$ then

$$\pi(u) = \left(\int_G \rho(g) \mathrm{d}g \right)(u) = \int_G \rho(g) u \mathrm{d}g = \int_G u = u$$

Thus π is a projection onto U^G. So

$$\dim U^G = \operatorname{Tr} \pi = \operatorname{Tr} \left(\int_G \rho \right) = \int_G \operatorname{Tr} \rho = \int_G \chi.$$

□

Corollary (Orthogonality of characters)**.** If G is a compact group and V, W are irreducible representations of G then

$$\langle \chi_V, \chi_W \rangle_G = \begin{cases} 1 & \text{if } V \simeq W \\ 0 & \text{if } V \not\simeq W \end{cases}$$

where

$$\langle f_1, f_2 \rangle_G = \int_G \overline{f_1}(g) f_2(g) \mathrm{d}g$$

To prove this as in the finite case we use $\chi_v(g^{-1}) = \overline{\chi_V}(g)$. This holds because V is unitary.

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8.3 Worked example: S^1

Goal: Understand the representations of S^1. Since

$$f \mapsto \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\mathrm{d}\theta$$

is a Haar integral, these representations are all unitary and hence completely reducible. So it is enough to understand the irreducible (unitary) representations of S^1.

By Schur's Lemma all such have degree 1, i.e. we have a correspondence

$$\{\text{irreducible representations of } S^1\} \leftrightarrow \{\text{continuous group homomorphisms } S^1 \to S^1\}.$$

Since $\mathbb{R} \to S^1$, $x \mapsto e^{2\pi i x}$ induces an isomorphism of topological groups

$$\mathbb{R}/\mathbb{Z} \xrightarrow{\sim} S_1$$

$$\left\{ \begin{matrix} \text{continuous group} \\ \text{homomorphisms } S^1 \to S^1 \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{continuous group} \\ \text{homomorphisms } \theta : R \to S^1 \end{matrix} \;\middle|\; \ker\theta \geq \mathbb{Z} \right\}$$

Fact: If $f : \mathbb{R} \to S^1$ is a continuous function with $f(0) = 1$ there is a unique continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(0) = 0$ and $f(x) = e^{2\pi i \alpha(x)}$ for all $x \in \mathbb{R}$.

$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{\alpha}{\nearrow} & \big\downarrow e^{2\pi i x} \\ \mathbb{R} & \xrightarrow{f} & S^1 \end{array}$$

Sketch proof. On small intervals we can define $\alpha(x) = \frac{1}{2\pi i \log x}$ and we choose the branch of log so that $\alpha(0) = 0$ and α is continuous. □

Lemma. If $\theta : \mathbb{R} \to S^1$ is a continuous group homomorphism there is $\psi : \mathbb{R} \to \mathbb{R}$ a continuous group homomorphism such that $\theta(x) = e^{2\pi i \psi(x)}$ for all $x \in \mathbb{R}$.

Proof. Our fact uniquely determines $\psi : \mathbb{R} \to \mathbb{R}$ continuous function such that $\psi(0) = 0$ and $\theta(x) = e^{2\pi i \psi(x)}$. We must show ψ is a group homomorphism. To this end we consider

$$\Delta : \mathbb{R}^2 \to \mathbb{R}$$
$$\Delta(a, b) = \psi(a + b) - \psi(a) - \psi(b)$$

We must show $\Delta \equiv 0$. It is easy to see that Δ is continuous. Also,

$$e^{2\pi i \Delta(a,b)} = \theta(a + b)\theta(a)^{-1}\theta(b)^{-1} = 1$$

so Δ takes values in $\mathbb{Z}$. So as $\mathbb{R}^2$ is connected, Δ is constant ($\mathbb{Z}$ is discrete). But $\Delta(0, 0) = 0$, so $\Delta \equiv 0$ as required. □

Lemma. If $\psi : (\mathbb{R}, +) \to (\mathbb{R}, +)$ is a continuous group homomorphism then $\exists \lambda \in \mathbb{R}$ such that $\psi(x) = \lambda x$ for all $x \in \mathbb{R}$.

Proof. Let $\lambda = \psi(1)$. Then $\psi(n) = \lambda n$ for all $n \in \mathbb{Z}$ (ψ is a homomorphism). So

$$m\psi\left(\frac{n}{m}\right) = \psi(n) = \lambda n$$

for all $\frac{n}{m} \in \mathbb{Q}$ (ψ is a homomorphism), i.e. $\psi(x) = \lambda x$ for all $x \in \mathbb{Q}$. But $\mathbb{Q}$ is dense in $\mathbb{R}$, so $\psi(x) = \lambda x$ for all $x \in \mathbb{R}$. □

Theorem (Representations of S^1)**.** Every irreducible representation of S^1 is 1-dimensional and is of the form $z \mapsto z^n$ for some $n \in \mathbb{Z}$.

Proof. We've already seen that if $\rho : S^1 \to \mathrm{GL}_d(\mathbb{C})$ is an irreducible representation then $d = 1$ and $\rho(S^1) \le S^1$. Moreover ρ induces a continuous homomorphism $\theta : \mathbb{R} \to S^1$ given by $\theta(x) = \rho(e^{2\pi i x})$. By the last two lemmas, there exists $\lambda \in \mathbb{R}$ such that $\theta(x) = e^{2\pi i \lambda x}$ for all $x \in \mathbb{R}$. Since $\theta(1) = \rho(e^{2\pi i}) = \rho(1) = 1$, we deduce $e^{2\pi i \lambda} = 1$, i.e. $\lambda \in \mathbb{Z}$. So $\rho(e^{2\pi i x}) = (e^{2\pi i x})^\lambda$ for $\lambda \in \mathbb{Z}$. □

The theorem says that the "character table" of S^1 has rows given by χ_n for $n \in \mathbb{Z}$, $\chi_n(z) = z^n$.

(unitary 1-dimensional characters of $\mathbb{Z}$ are all of the form $n \mapsto e^{in\theta}$ for some $e^{i\theta} \in S^1$).

Notation. $\mathbb{Z}[z, z^{-1}) = \left\{ \sum_{n \in \mathbb{Z}} a_n z^n \;\middle|\; a_n \in \mathbb{Z}, \sum_{n \in \mathbb{Z}} |a_n| < \infty \right\} j$ a ring under natual operations.

If V is any representation of S^1 then it decomposes as a direct sum of 1-dimensional subrepresentations and its character $\chi_V = \sum_{n \in \mathbb{Z}} a_n z^n$ with all $a_n \ge 0$ and $\sum a_n = \dim V$, where as usual a_n is the number of copies of $(z \mapsto z^n)$ in V.

So

$$R(S^1) = \{\chi - \chi' \mid \chi, \chi' \text{ are characters of } S^1\} = \mathbb{Z}[z, z^{-1})$$

By orthogonality of characters,

$$\langle \chi_n, \chi_m \rangle_{S^1} = \frac{1}{2\pi} \int_0^{2\pi} e^{2\pi i (m-n)\theta} \mathrm{d}\theta = d_{m,n}$$

$$a_n = \langle \chi_n, \chi_V \rangle_{S^1} = \frac{1}{2\pi} \int_0^{2\pi} \chi_V(e^{i\phi}) e^{-in\phi} \mathrm{d}\phi$$

and so

$$\chi_V(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \left(\frac{1}{2\pi} \int_0^{2\pi} \chi_V(e^{i\phi}) e^{-in\phi} \mathrm{d}\phi \right) e^{in\phi}$$

So Fourier decomposition of χ_V decomposes χ_V into irreducible characters and the FOurier mode is the multiplicity.

Remark. In fact by the theory of Fourier series any continuous function on S^1 can be approximated uniformly by a finite $\mathbb{C}$-linear combination of χ_n. Moreover the χ_n form a complete orthonormal set in the Hilbert space

$$L^2(S^1) = \left\{ f : S^1 \to \mathbb{C} \,\middle|\, \int_0^{2\pi} |f(e^{i\theta})|^2 \mathrm{d}\theta < \infty \right\} / \sim$$

of square integrable functions on S^1, i.e. every function in $L^2(S^1)$ has a unique expression as

$$f(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \left(\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\phi}) e^{-in\phi} \mathrm{d}\phi \right) e^{in\phi}$$

converging with respect to the norm $\|f\|^2 = \int_0^{2\pi} |f(e^{i\theta})|^2 \mathrm{d}\theta$. We can phrase this as

$$L^2(S^1) = \hat{\bigoplus}_{n\in\mathbb{Z}} \mathbb{C}\chi_n$$

($\hat{\bigoplus}$ means complete direct sum), which is an analogue of

$$\mathbb{C}G = \bigoplus_{V\in\mathrm{Irr}(G)} (\dim V)V$$

for finite groups (cf Peter Weyl Theorem).

8.4 Second worked example SU(2)

Recall $\mathrm{SU}(2) = \{A \in \mathrm{GL}_2(\mathbb{C}) \mid \overline{A}^\top A = I, \det A = 1\}$. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(2)$$

then as $\det A = 1$,

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$$

Thus $d = \overline{a}$ and $c = -\overline{b}$. Moreover, $|a|^2 + |b|^2 = 1$. In this way,

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \;\middle|\; a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$$

which is homeomorphic to $S^3 \subset \mathbb{R}^4 \simeq \mathbb{C}^2$. More precisely if

$$\mathbb{H} = \mathbb{R} \cdot \mathrm{SU}(2) = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \right\} \subset M_2(\mathbb{C})$$

Then $\|A\|^2 = \det A$ defines a norm on $\mathbb{H} \simeq \mathbb{R}^4$ and $\mathrm{SU}(2)$ is the unit sphere in $\mathbb{H}$ with respect to this norm. If $A \in \mathrm{SU}(2)$,

$$\|AX\| = \|X\| = \|XA\| \qquad \forall X \in \mathbb{H}$$

(since $\det A = 1$). So $\mathrm{SU}(2)$ acts on $\mathbb{H}$ on both left and right by isometries. So after normalisation, usual integration on S^3 defines a Haar integral on $\mathrm{SU}(2)$, i.e.

$$\int_{\mathrm{SU}(2)} f = \frac{1}{2\pi^2} \int_{S^3} f$$

Here $\frac{1}{2\pi^2}$ is the volume of S^3 in $\mathbb{R}^4$ with respect to usual measure. We now try to understand conjugacy classes in $\mathrm{SU}(2)$. Let

$$T = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \;\middle|\; z \in S^1 \right\} \le \mathrm{SU}(2)$$

Proposition.

(i) Every conjugacy clsas in $\mathrm{SU}(2)$ contains an element of T

(ii) More precisely, if O is a conjugacy class in $\mathrm{SU}(2)$, $O \cap T = \{t, t^{-1}\}$ for some $t \in T$. If $t = t^{-1}$, $t = \pm I$ and $O = \{t\}$.

(iii) There is a continuous bijection

$$\begin{aligned} \{\text{conjugacy classes in } \mathrm{SU}(2)\} &\to [-1, 1) \\ A &\mapsto \frac{1}{2} \operatorname{Tr} A \end{aligned}$$

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Proof.

(i) Every unitary matrix has an orthonormal basis of eigenvectors. That is, for $A \in \mathrm{SU}(2)$, there exists $P \in \mathrm{U}(2)$ such that $P^{-1}AP \in T$. Then if $Q = \frac{1}{\sqrt{\det P}} P \in \mathrm{SU}(2)$, $Q^{-1}AQ = P^{-1}AP \in T$, i.e. $[A]_{\mathrm{SU}(2)} \cap T \neq \emptyset$.

(ii) If $A = \pm I$ the claim is clear. Otherwise

$$\begin{aligned} [A]_{\mathrm{SU}(2)} &= [t]_{\mathrm{SU}(2)} && \text{for some } t \in T \\ &= \{gtg^{-1} \mid g \in \mathrm{SU}(2)\} \end{aligned}$$

All elements of gtg^{-1} have the same eigenvalues as t. So if $t' = gtg^{-1} \in T$ then $t' \in n\{t^{\pm 1}\}$, i.e. $[A]_{\mathrm{SU}(2)} \cap T \subseteq \{t^{\pm 1}\}$. But if

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SU}(2)$$

then $sts^{-1} = t^{-1}$.

(iii) $[A]_{\mathrm{SU}(2)} \mapsto \frac{1}{2} \operatorname{Tr} A$ is well-defined and injective since conjugate matrices have the same trace and if $\frac{1}{2} \operatorname{Tr} A = \frac{1}{2} \operatorname{Tr} B$ for $A, B \in \mathrm{SU}(2)$, since $\det A = \det B = 1$, then A and B have the same characteristic polynomial and hence the same eigenvalues, so by (ii) they are conjugate. Moreover

$$\frac{1}{2} \operatorname{Tr} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \cos\theta$$

so the image of our map is $[-1, 1)$. □

Corollary. A (continuous) class function $f : \mathrm{SU}(2) \to \mathbb{C}$ is determined by its restriction to T and $f|_T$ is even, i.e.

$$f\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\right) = f\left(\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}\right) \qquad \forall z \in S^1.$$

We'll write

$$f(z) = f\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\right)$$

for $z \in S^1$ i.e. identify T and S^1.

Notation. We'll write

$$\begin{aligned} \mathbb{Z}[z, z^{-1})^{\mathrm{ev}} &= \{f \in \mathbb{Z}[z, z^{-1}) \mid f(z) = f(z^{-1})\} \\ &= \left\{\sum a_n z^n \mid a_n \in \mathbb{Z}, a_n = a_{-n} \forall n \in \mathbb{Z}\right\} \end{aligned}$$

Lemma. If χ is a character of a representation of SU(2) then $\chi|_T \in \mathbb{Z}[z, z^{-1}]^{\mathrm{ev}}$.

It follows that $R(\mathrm{SU}(2)) \lesssim \mathbb{Z}[z, z^{-1}]^{\mathrm{ev}}$ and we're going to see that it is an equality.

Proof. If V is a representation of SU(2) and χ its character, then

$$\chi|_T = \chi_{\mathrm{Res}_T^{\mathrm{SU}(2)} V}$$

since every character of $T \cong S^1$ lies in $\mathbb{Z}[z, z^{-1}]$ and $\chi|_T$ is even, we're done. □

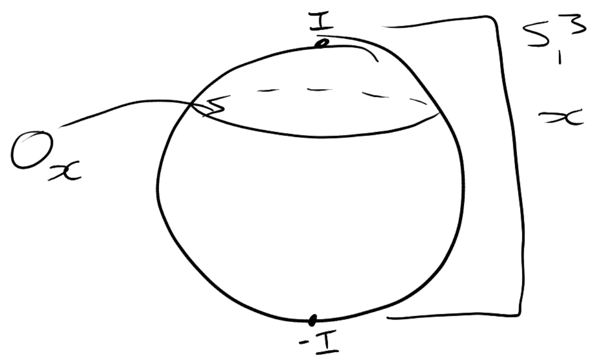
Let's write

$$O_x = \left\{ A \in \mathrm{SU}(2) \mid \frac{1}{2} \operatorname{Tr} A = x \right\}$$

for $x \in [-1, 1)$. These are the conjugacy classes of SU(2), $O_1 = \{I\}$, $O_{-1} = \{-I\}$ and for $-1 < x < 1$. There is some (unique) $\theta \in (0, \pi)$ such that $\cos\theta = x$ and

$$O_x = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \;\middle|\; (\operatorname{Im} a)^2 + |b|^2 = 1 - x^2 = \sin^2\theta \right\}$$

(since $\operatorname{Re} a = x$). That is O_x is a 2-sphere of radius $|\sin\theta|$.



Thus if f is a class function on SU(2), since f is constant on $O_{\cos\theta}$,

$$\begin{aligned}
\int_{\mathrm{SU}(2)} f(g)\mathrm{d}g &= \frac{1}{2\pi^2} \int_0^\pi \left[\int_{O_{\cos\theta}} f(e^{i\theta}) \right] \mathrm{d}\theta \\
&= \frac{1}{2\pi^2} \int_0^\pi 4\pi \sin^2\theta f(e^{i\theta}) \mathrm{d}\theta \\
&= \frac{1}{\pi} \int_0^{2\pi} f(e^{i\theta}) \sin^2\theta \mathrm{d}\theta
\end{aligned}$$

since f is even. Note this is normalised correctly since

$$\frac{1}{\pi}\int_0^{2\pi} \sin^2\theta \mathrm{d}\theta = \frac{\pi}{\pi} = 1.$$

So we can compute $\langle f, g\rangle_{\mathrm{SU}(2)}$ on class functiosn (and so characters) as

$$\langle f, g\rangle_{\mathrm{SU}(2)} = \frac{1}{\pi}\int_0^{2\pi} \overline{f}(e^{i\theta})g(e^{i\theta})\sin^2\theta \mathrm{d}\theta$$

8.5 Representations of SU(2)

Let V_n be the $\mathbb{C}$-vector space of homogeneous polynomials in x and y of degree n. So

$$V_n = \bigoplus_{i=1}^{n} \mathbb{C}x^i y^{n-i}$$

has dimension $n+1$. $\mathrm{GL}_2(\mathbb{C})$ acts on V_n via

$$\rho_n : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}(V_n)$$
$$\rho_n\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(f(x,y)) = f(ax+cy, bx+dy)$$

i.e.

$$\rho_n\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) x^i y^i = (ax+cy)^i(bx+dy)^j.$$

Example. $V_0 = \mathbb{C}$ is the trivial representation.

$V_1 = \mathbb{C}x \oplus \mathbb{C}y$ is the natural representation of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}^2$ with respect to basis x, y.

$V_2 = \mathbb{C}x^2 \oplus \mathbb{C}xy \oplus \mathbb{C}y^2$ and with respect to this basis,

$$\rho_2\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left(\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}\right)$$

In general $V_n \simeq S^n V_1$ as representations of $\mathrm{GL}_2(\mathbb{C})$.

Since SU(2) is a subgroup of $\mathrm{GL}_2(\mathbb{C})$ we can view these V_n as representations of SU(2) by restriction. In fact we'll see the V_n are precisely the irreducible representations of SU(2).

Let's compute $\chi_{V_n}|_T$ of (ρ_n, V_n).

$$\rho_n\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}\right)(z^i y^j) = (zx)^i (z^{-1}y)^j = z^{i-j}x^i y^j$$

So for each $0 \le j \le n$, $\mathbb{C}x^j y^{n-j}$ is a 1-dimensional representation of T with character z^{2j-n} and

$$\chi_{V_n}(z) = z^n + z^{n-2} + z^{n-4} + \cdots + z^{2-n} + z^{-n} = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}} \in \mathbb{Z}[z, z^{-1}]^{\text{ev}}$$

Theorem. Each V_n is irreducible as a representation of SU(2).

Proof. Let $0 \neq W \le V_n$ be SU(2)-invariant. We must show $W = V_n$. W is also T-invariant as $\text{Res}_T^{\text{SU}(2)} V_n = \bigoplus_{j=0}^n \mathbb{C}x^j y^{n-j}$ is as direct sum of non-isomorphic 1-dimensional subrepresentations. $(*)$ W has a basis that is a subset of $\{x^j y^{n-j} \mid 0 \le j \le n\}$ (uniqueness of isotypical decomposition). Thus $x^j y^{n-j} \in W$ for some $0 \le j \le n$.

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1\end{pmatrix} x^j yn - j = \frac{1}{\sqrt{2}}(x-y)^j (x+y)^{n-j} \in W$$

so by $(*)$, $x^n \in W$. Repeat the same calculation for $j = n$, we get $(x-y)^n \in W$. So $x^i y^{n-i} \in W$ for all $0 \le i \le n$ and $W = V_n$. □

Exercise (Alternative proof)**:** Show

$$\langle \chi_{V_n}, \chi_{V_n}\rangle_{\text{SU}(2)} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{e^{(n+1)i\theta} - e^{-(n+1)i\theta}}{e^{i\theta} - e^{-i\theta}}\right)^2 \sin^2\theta \mathrm{d}\theta = 1$$

Theorem. Every irreducible representation of SU(2) is isomorphic to V_n for some $n \ge 0$.

Proof. Let V be an irreducible representation of SU(2), $\chi_V|_T \in \mathbb{Z}[z, z^{-1}]^{\text{ev}}$. Let $\chi_n = \chi_{V_n}|_T$ for $n \ge 0$, so $\chi_0 = 1$, $\chi_1 = z + z^{-1}$, $\chi_2 = z^2 + 1 + z^{-2}$ etc. It is easy to see that there exists $\lambda_0, \lambda_1, \ldots, \lambda_m \in \mathbb{Z}$ such that $\chi_V|_T = \sum_{i=0}^n \lambda_i \chi_i$. By Orthogonality of characters,

$$\lambda_i = \langle \chi_V, \chi_{V_i}\rangle_{\text{SU}(2)} = \begin{cases} 1 & \text{if } V \simeq V_i \\ 0 & \text{if } V \not\simeq V_i\end{cases}$$

Since $\chi_V \neq 0$ there is some i such that $\lambda_i = 1$ and $\chi_V = \chi_{V_i}$. □

We want to understand $\otimes$ for representations of SU(2). We know

$$\chi_{V \otimes W} = \chi_V \chi_W$$

for representations V and W of any group G. Let's compute some examples:

$$\chi_{V_1 \otimes V_2}(z) = (z + z^{-1})^2 = z^2 + 2 + z^{-2} = \chi_{V_2} + \chi_{V_0}$$

so $\chi_{V_1} \otimes \chi_{V_1} \simeq V_2 \oplus V_0$.

$\chi_{V_1 \otimes V_2}(z) = (z + z^{-1})(z^2 + 1 + z^{-2}) = z^3 + z + z^{-1} + z + z^{-1} + z^{-3} = \chi_{V_3}(z) + \chi_{V_1}(z)$

so $V_1 \otimes V_2 \simeq V_3 \oplus V_1$.

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Proposition (Cletsch-Gordon rule)**.** For $n, m \in \mathbb{N}$,

$$V_n \otimes V_m \simeq V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{|n-m|}$$

Proof. Without loss of generality $n \ge m$. Then

$$\begin{aligned}(\chi_n \chi_m)(z) &= \left(\frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}} \right) (z^m + z^{m-1} + \cdots + z^{-m}) \\ &= \sum_{j=0}^{m} \left(\frac{z^{n+m+1-2j} - z^{-n+m+1-2j}}{z - z^{-1}} \right) \\ &= \sum_{j=0}^{m} \chi_{n+m-2j}(z)\end{aligned}$$

□

8.6 Representations of SO(3)

Proposition. The action of SU(2) on the 3D $\mathbb{R}$-normed vector space

$$\left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \;\middle|\; a + \bar{a} = 0 \right\} \subseteq M_2(\mathbb{C})$$

with norm $\|A\|^2 = \det A$ by conjugation induces an isomorphism of topological groups

$$\frac{\mathrm{SU}(2)}{\{\pm I\}} \to \mathrm{SO}(3).$$

Proof. See Example Sheet 4, Question 4 (for more hints see lecturers notes from 2012). □

Corollary. Every irreducible representation of SO(3) is of the form V_{2n} for some $n \geq 0$.

Proof. It follows from the previous proposition that the irreducible representation of SO(3) correspond to irreducible representations of SU(2) whose kernel contains $\pm I$. But it is easy to see that $-I$ acts on V_n by $(-1)^n$. □

9 Character table of $\mathrm{GL}_2(\mathbb{F}_q)$

9.1 $\mathbb{F}_q$

Let $p > 2$ be a prime, $q = p^a$ a power of p for some $a > 0$ and let $\mathbb{F}_q$ be the field with q elements. We know that $\mathbb{F}_q^\times \simeq C_{q-1}$ and $\mathbb{F}_q^\times \to \mathbb{F}_q^\times$, $x \mapsto x^2$ is a homomorphism with kernel $\{\pm 1\}$. Thus half the elements are squares and half are not. Moreover, $x \mapsto x^{\frac{q-1}{2}}$ sends squares to 1 and non-squares to -1. Let $\varepsilon \in \mathbb{F}_q^\times$ be a fixed non-square. So $\varepsilon^{\frac{q-1}{2}} = -1$ and let

$$\mathbb{F}_{q^2} = \{a + b\sqrt{\varepsilon} \mid a, b \in \mathbb{F}_q\}$$

the field extension of $\mathbb{F}_q$ (with respect to the obvious operations) of order q^2.

Every element of $\mathbb{F}_q$ has a square root in $\mathbb{F}_q^2$, since if $\lambda \in \mathbb{F}_q$ is a non-square then $\frac{\lambda}{\varepsilon}$ is a square, μ^2 say, and $(\sqrt{\varepsilon}\mu)^2 = \varepsilon\mu^2 = \lambda$. Thus every quadratic polynomial with coefficients in $\mathbb{F}_q$ factorises over $\mathbb{F}_q^2$. Notice $(a + b\sqrt{\varepsilon})^q = a^q + b^q\varepsilon^{\frac{q-1}{2}} = a - b\sqrt{\varepsilon}$ since $p \mid \binom{q}{i}$ for $0 < i < q$.

Thus the roots of an irreducible quadratic over $\mathbb{F}_q$ are of the form λ, λ^q ($\lambda \mapsto \lambda^q$ is like complex conjugation).

9.2 $\mathrm{GL}_2(\mathbb{F}_q)$ and its conjugacy classes

We want to compute the character table of the group

$$\mathrm{GL}_2(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{F}_q, ad - bc \neq 0 \right\}.$$

The order of $\mathrm{GL}_2(\mathbb{F}_q)$ is equal to the number of bases for $\mathbb{F}_q^2$ over $\mathbb{F}_q$. This is $(q^2-1)(q^2-q) = q(q-1)^2(q+1)$. First we compute the conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q) =: G$. We know from linear algebra (rational canonical form) that for $A \in G$, $[A]_G$ is determined by $m_A(x)$, the minimal polynomial and $\deg m_A(x) \leq 2$ (Cayley-Hamilton). Moreover $m_A(0) \neq 0$.

There are 4 cases:

C1se 1: $m_A(x) = (x - \lambda)$ for some $\lambda \in \mathbb{F}_q^\times$. Then $A = \lambda I$ so $C_G(A) = G$ and $|[A]_G| = |\{\lambda I\}| = 1$. There are $q - 1$ such classes – one for each λ.

C2se 2: $m_A(x) = (x - \lambda)^2$ for some $\lambda \in \mathbb{F}_q^\times$. Then

$$[A]_G = \left[\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right]_G$$

Now

$$C_G\left(\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\right) = \left\{\begin{pmatrix}a & b\\ 0 & a\end{pmatrix} \;\middle|\; a, b \in \mathbb{F}_q, a \neq 0\right\}$$

(compute!). So

$$|[A]_G| = \frac{q(q-1)^2(q+1)}{(q-1)q} = (q-1)(q+1).$$

There are $q - 1$ such classes – one for each λ.

C3se 3: $m_A(x) = (x - \lambda)(x - \mu)$ for $\lambda, \mu \in \mathbb{F}_q^\times$ distinct. So

$$[A]_G = \left[\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\right] = \left[\begin{pmatrix}\mu & 0\\ 0 & \lambda\end{pmatrix}\right]_G$$

Moreover

$$C_G\left(\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\right) = \left\{\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix} \;\middle|\; a, d \in \mathbb{F}_q^\times\right\} =: T$$

so

$$|[A]_G| = \frac{q(q-1)^2(q+1)}{(q-1)^2} = q(q+1)$$

There are $\binom{q-1}{2}$ such classes – one for each pair λ, μ.

C4se 4: $m_A(x)$ is irreducible over $\mathbb{F}_q$ of degree 2. So for some $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, $\alpha = \lambda + \mu\sqrt{\varepsilon}$ for some $\lambda, \mu \in \mathbb{F}_q$, $\mu \neq 0$,

$$\begin{aligned} m_A(x) &= (x - \alpha)(x - \alpha^q) \\ &= (x^2 - (\alpha + \alpha^q)x + \alpha\alpha^q) \\ &= (x^2 - (\operatorname{Tr} A)x + \det A) \end{aligned}$$

Then

$$[A]_G = \left[\begin{pmatrix}\lambda & \varepsilon\mu\\ \mu & \lambda\end{pmatrix}\right]_G = \left[\begin{pmatrix}\lambda & -\varepsilon\mu\\ -\mu & \lambda\end{pmatrix}\right]_G$$

Since these matrices have trace $2\lambda = \alpha + \alpha^q$ and det $\lambda^2 - \varepsilon\mu^2 = \alpha\alpha^q$. Now

$$C_g\left(\begin{pmatrix}\lambda & \varepsilon\mu\\ \mu & \lambda\end{pmatrix}\right) = \left\{\begin{pmatrix}a & \varepsilon b\\ b & a\end{pmatrix} \;\middle|\; a, b \in \mathbb{F}_q, a^2 - \varepsilon b^2 \neq 0\right\} =: K$$

If $a^2 - \varepsilon^2 b = 0$, then $a^2 = \varepsilon b^2$ so ε is a square or $a = b = 0$. So $|K| = q^2 - 1$ and

$$|[A]_G| = \frac{q(q-1)^2(q+1)}{(q-1)(q+1)} = q(q-1).$$

There are $\frac{q(q-1)}{2} = \binom{q}{2}$ such classes – one for each pair $\{\alpha, \alpha^q\} \subset \mathbb{F}_{q^2} \setminus \mathbb{F}_q$.

In summary:

Rep A	C_G	$\lvert[A]_G\rvert$	# classes
λI	G	1	$q-1$
$\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$	$\left\{\begin{pmatrix}a & b\\ 0 & a\end{pmatrix}\right\}$	$(q-1)(q+1)$	$q-1$
$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$	T	$q(q+1)$	$\binom{q-1}{2}$
$\begin{pmatrix}\lambda & \varepsilon\mu\\ \mu & \lambda\end{pmatrix}$	K	$q(q-1)$	$\binom{q}{2}$

The groups T and K are both called *maximal tori*, i.e. they are maximal subgroups such that they are conjugate to a diagonal subgroup in $\mathrm{GL}_2(\mathbb{F})$ for some $\mathbb{F}/\mathbb{F}_q$. T is called *split* and K is called *non-split*.

Some other important subgroups of G are

- The subgroup of scalar matrices (the centre of G):

$$Z = \{\lambda I \mid \lambda \in \mathbb{F}_q^\times\}.$$

- A Sylow-p-subgroup of G

$$N = \left\{\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} \;\middle|\; b \in \mathbb{F}_q\right\}$$

  so

$$ZN = \left\{\begin{pmatrix}a & b\\ 0 & a\end{pmatrix} \;\middle|\; a, b \in \mathbb{F}_q, a \neq 0\right\}$$

- A *Borel subgroup* of G

$$B = \left\{\begin{pmatrix}a & b\\ 0 & d\end{pmatrix} \;\middle|\; a, d \in \mathbb{F}_q^\times, b \in \mathbb{F}_q\right\}$$

Then $N \trianglelefteq B$ and $B/N \simeq T \simeq \mathbb{F}_q^\times \times \mathbb{F}_q^\times \simeq C_{q-1} \times C_{q-1}$.

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9.3 The character table of B

As a warm-up we compute the character table of

$$B = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \;\middle|\; a, d \in \mathbb{F}_q^\times, b \in \mathbb{F}_q\right\} \leq G,$$

a group of order $(q-1)^2 q$.

Recall

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & b \end{pmatrix} \;\middle|\; b \in \mathbb{F}_q \right\} \trianglelefteq B$$

and

$$\begin{gathered} B/N \simeq T \simeq \mathbb{F}_q^\times \mathbb{F}_q^\times \\ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} N \to \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \end{gathered}$$

The conjugacy classes in B are

Rep	C_B	size of class	#classes
$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$	B	1	$q-1$
$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	ZN	$q-1$	$q-1$
$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$	T	q	$(q-1)(q-2)$

Moreover if Θ_q : {representations $\mathbb{F}_q^\times \to \mathbb{C}^\times$} then Θ_q is a cyclic group of order $q-1$ under pointwise operations since $\mathbb{F}_q^\times \simeq C_{q-1}$ and for each pair $\theta, \phi \in \Theta_q$ we can define a 1-dimensional representation of B (factoring through B/N) given by

$$\chi_{\theta,\phi}\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \theta(a)\phi(d)$$

giving $(q-1)^2$ 1-dimensional representations. We will build the remaining irreducible **??** of B by induction from ZN.

$$\begin{gathered} ZN \simeq \mathbb{F}_q^\times \times (\mathbb{F}_q, +) \\ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto (a, a^{-1}b) \end{gathered}$$

so given a 1-dimensional representation $\gamma : (\mathbb{F}_q, +) \to \mathbb{C}^\times$ and $\theta \in \Theta_q$, we can define a 1-dimensional representation of ZN

$$\begin{gathered} \rho_{\theta,\gamma} : ZN \to \mathbb{C}^\times \\ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto \theta(a)\gamma(a^{-1}b) \end{gathered}$$

Now $ZN \trianglelefteq B$, so by Mackey's irreducibility criterion,

$$\operatorname{Ind}_{ZN}^{B} \rho_{\theta,\gamma} \text{ is irreducible} \iff {}^g\rho_{\theta,\gamma} \neq \rho_{\theta,\gamma}$$

for all $gZN \in B/ZN \setminus ZN/ZN$. Since

$$\left\{ t_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \;\middle|\; \lambda \in \mathbb{F}_q^\times \right\}$$

is a coset representative for B/ZN and

$$\begin{aligned} {}^{t_\lambda}\rho_{\theta,\gamma}\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) &= \rho_{\theta,\gamma}\left(\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}\right) \\ &= \rho_{\theta,\gamma}\left(\begin{pmatrix} a & \lambda b \\ 0 & a \end{pmatrix}\right) \\ &= \theta(a)\gamma(a^{-1}\lambda b) \end{aligned}$$

We see that

$$\begin{aligned} {}^{t_\lambda}\rho_{\theta,\gamma} = \rho_{\theta,\gamma} &\iff \gamma(\lambda b) = \gamma(b) && \forall b \in (\mathbb{F}_q, +) \\ &\iff \gamma((\lambda - 1)b) = 1 && \forall b \in \mathbb{F}_q \\ &\iff \gamma = \mathbb{1}_{\mathbb{F}_q} \text{ or } \lambda = 1 \end{aligned}$$

So $\operatorname{Ind}_{ZN}^B \rho_{\theta,\gamma}$ is irreducible if and only if $\gamma \neq \mathbb{1}_{\mathbb{F}_q}$.

Now since

$$(\operatorname{Ind}_{ZN}^B) = \sum_{[g]_{ZN} \subseteq [b]_B} \frac{|C_B(b)|}{|C_{ZN}(g)|}\chi(g).$$

We can compute

$$\begin{aligned} (\operatorname{Ind}_{ZN}^B \rho_{\theta,\gamma})\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) &= \frac{|B|}{|ZN|}\rho_{\theta,\gamma}\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = (q-1)\theta(\lambda) \\ (\operatorname{Ind}_{ZN}^B \rho_{\theta,\gamma})\left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right) &= \sum_{b \in \mathbb{F}_q^\times} \frac{|BN|}{|BN|}\rho_{\theta,\gamma}\left(\begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}\right) \\ &= \theta(\lambda)\left(\left(\sum_{b \in \mathbb{F}_q} \gamma(\lambda^{-1}b)\right) - 1\right) \\ &= \theta(\lambda)(q\langle \gamma, \mathbb{1}_{\mathbb{F}_q}\rangle_{(\mathbb{F}_q,+)} - 1) \\ &= \begin{cases} -\theta(\lambda) & \gamma \neq \mathbb{F}_q \\ (q-1)\theta(\lambda) & \gamma = \mathbb{1}_{\mathbb{F}_q} \end{cases} \\ \operatorname{Ind}_{ZN}^B = \left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right) &= 0 \qquad (\lambda \neq \mu) \end{aligned}$$

Let $\mu_\theta := \operatorname{Ind}_{ZN}^B \rho_{\theta,\gamma}$ for $\gamma \neq \mathbb{1}_{\mathbb{F}_q}$, noting this does not depend on the choice of γ. Then each μ_θ is irreducible by earlier calculation and we have $q-1$ irreducible representations of B all of degree $q-1$.

Thus the character table of B is

	$\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$	$\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$	$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$	
$\chi_{\theta,\phi}$	$\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\mu)$	$\theta, \phi \in \Theta_q$
μ_θ	$(q-1)\theta(\lambda)$	$-\theta(\lambda)$	0	$\theta \in \Theta_q$

Remark.

(1) The 0 in the bottom right corner appears in $(q-1)$ rows and $(q-1)(q-2)$ columns. They are all forced to be 0 by a Lemma from Section 7.3, since order of the conjugacy class is q and the dimension of an irreducible representation are coprime and these elements don't act by scalars since the representations are faithful and the elements are not in the centre.

(2)
$$B = Z \times \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q \right\}$$
and the second factor is a Frobenius group, so Example Sheet 3, Question 10 tells us that **??** of the second factor arise essentially as we have constructed them.

9.4 The character table of G

As $\det : G \to \mathbb{F}_q^\times$ is a surjective group homomorphism, for each $\theta \in \Theta_q$, $\theta \circ \det : G \to \mathbb{C}^\times$ is a distinct 1-dimensional representation of G. We get $q-1$ in all. Next we'll do induction from B. Define
$$s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in G,$$
and note that
$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & a + b\beta \\ d & \beta d \end{pmatrix}$$
These elements are all disinct. Hence
$$|BsN| = q|B| = |G \setminus B|$$
(G/B has order $q+1$). Thus $BsN = BsB$ and $G = B \amalg BsB$ (Bruhat decomposition) and $B \backslash G / B = \{B, BsB\}$. By the proof of Mackey's irreducibility criterion,
$$\langle \mathrm{Ind}_B^G \chi, \mathrm{Ind}_B^G \chi \rangle_G = \langle \chi, \chi \rangle_B + \langle \mathrm{Res}^B_{B \cap {}^sB} \chi, \mathrm{Res}^{{}^sB}_{B \cap {}^sB} {}^s\chi \rangle_{B \cap {}^sB}$$
Now
$$s \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} s^{-1} = \begin{pmatrix} d & 0 \\ b & a \end{pmatrix}$$

So $B \cap {}^sB = T$ and

$$\langle \operatorname{Ind}_B^G \chi, \operatorname{Ind}_B^G \chi \rangle_G = \langle \chi, \chi \rangle_B + \langle \chi|_T, {}^s\chi|_T \rangle_T$$

where

$$({}^s\chi)\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) = \chi\left(\begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}\right)$$

Thus $W_{\theta,\gamma} := \operatorname{Ind}_B^G \chi_{\theta,\phi}$ is irreducible if $\theta \neq \phi \in \Theta_q$. These are called *principal series representations*. We can also compute $W_{\theta,\theta}$ has 2 distinct irreducible summands:

$$\langle \operatorname{Ind}_B^G \mu_\theta, \operatorname{Ind}_B^G \mu_\theta \rangle = 1 + \frac{1}{|T|} \sum_{\lambda \in \mathbb{F}_q^\times} |(q-1)\theta(\lambda)|^2 = 1 + \frac{(q-1)^3}{(q-1)^2} = q.$$

For any character χ of B,

$$\begin{aligned}
(\operatorname{Ind}_B^G \chi)(g) &= \sum_{[b]_B \subseteq [g]_G} \frac{|C_G(g)|}{|C_B(b)|} \chi(b) \\
(\operatorname{Ind}_B^G \chi)\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) &= (q+1)\chi\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) \\
(\operatorname{Ind}_B^G \chi)\left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right) &= \chi\left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right) \\
(\operatorname{Ind}_B^G \chi)\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right) &= \chi\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right) + \chi\left(\begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}\right) \\
(\operatorname{Ind}_B^G \chi)\left(\begin{pmatrix} \lambda & \varepsilon\mu \\ \mu & \lambda \end{pmatrix}\right) &= 0
\end{aligned}$$

Notice $W_{\theta,\phi} \simeq W_{\phi,\theta}$ so so get $\binom{q-1}{2}$ principal series representations. Also, $W_{\theta,\theta} = \chi_\theta \otimes W_{\mathbb{1},\mathbb{1}}$ and

$$W_{\mathbb{1},\mathbb{1}} = \operatorname{Ind}_B^G \mathbb{1} = \mathbb{C}(G/B)$$

is a permutation representation. Thus $W_{\mathbb{1},\mathbb{1}} = \mathbb{C} \oplus V_1$ with $V_{\mathbb{1}}$ an (explicit) irreducible representation of degree q (the *Steinberg representation*) and $W_{\theta,\theta} = \chi_\theta \oplus V_\theta$ where $V_\theta = \theta \otimes V_{\mathbb{1}}$ (a twisted Steinbery representation). We've explicitly constructed $(q-1) + \binom{q-1}{2} + (q-1)$ irreducible representations. We have $\binom{q}{2}$ irreducible representations to go. It will turn out that they are indexed by irreducible representations of K such that $\varphi \neq \varphi^q$ up to $\varphi \leftrightarrow \varphi^q$. We won't explicitly construct these representations, just their characters.

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We have found:

$$\operatorname{Ind}_B^G \mu_\theta(g) = \begin{cases} (q^2-1)\theta(\lambda) & [g]_G = \left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right]_G \\ -\theta(\lambda) & [g]_G = \left[\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right]_G \\ 0 & \text{otherwise} \end{cases}$$

and

$$\langle \operatorname{Ind}_B^G \mu_\theta, \operatorname{Ind}_B^G \mu_\theta \rangle_G = q.$$

Our next strategy is to induce characters from K:

$$\begin{aligned} \mathbb{F}_{q^2} &\to M_2(\mathbb{F}_q) \\ \lambda + \mu\sqrt{\varepsilon} &\mapsto \begin{pmatrix} \lambda & \varepsilon\mu \\ \mu & \lambda \end{pmatrix} \end{aligned}$$

induces an isomorphism of rings $\mathbb{F}_{q^2}$ to $K \cup \{0\}$. We will identify these rings. Under this identification,

$$\begin{aligned} \mathbb{F}_q^\times &\leftrightarrow Z \\ \lambda &\leftrightarrow \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \end{aligned}$$

Moreover

$$\begin{pmatrix} \lambda & \varepsilon\mu \\ \mu & \lambda \end{pmatrix}^q = \begin{pmatrix} \lambda & -\varepsilon\mu \\ -\mu & \lambda \end{pmatrix}$$

since $(\lambda + \mu\sqrt{\varepsilon})^q = (\lambda - \mu\sqrt{\varepsilon})$.

We want to understand $\operatorname{Ind}_K^G \varphi$ for an (irreducible) character φ of K. First we consider the double cosets $K \backslash G / K$ and then use Mackey to compute $\langle \operatorname{Ind}_K^G \varphi, \operatorname{Ind}_K^G \varphi \rangle_G$. For $k \in K$, $g \in G$,

$$\begin{aligned} kgK = gK &\iff g^{-1}kg \in K \\ &\iff g^{-1}kg \in \{k, k^q\} \end{aligned}$$

($[k]_G \cap K = \{k, k^q\}$). Writing

$$t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we get

$$t^{-1} \begin{pmatrix} \lambda & \varepsilon\mu \\ \mu & \lambda \end{pmatrix} t = \begin{pmatrix} \lambda & -\varepsilon\mu \\ -\mu & \lambda \end{pmatrix}$$

so $kgK = gK \iff g^{-1}kg = k$ or $(tg)^{-1}k(tg) = k$. Furthermore since

$$C_G\left(\begin{pmatrix} \lambda & \varepsilon\mu \\ \mu & \lambda \end{pmatrix}\right) = \begin{cases} K & \text{if } \mu \neq 0 \\ G & \text{if } \mu = 0 \end{cases}$$

so $kgK = gK \iff gK \in \{K, tK\}$ or $k \in Z$. It follows that

$$|KgK| = \begin{cases} |K| & g \in K \cup tK \\ \underbrace{\left|\frac{K}{2}\right| |K|}_{=(q^2-1)(q+1)} & \text{otherwise} \end{cases}$$

so there are

$$\frac{|G| - 2|K|}{\left|\frac{K}{2}\right| |K|} = \frac{\frac{|G|}{|K|} - 2}{\left|\frac{K}{2}\right|} = \frac{q(q-1) - 2}{q+1} = q - 2$$

double cosets of size $\left|\frac{K}{2}\right| |K|$.

Now $K \cap {}^tK = K$, $K \cap {}^gK = Z$ if $g \notin K \cup tK$. Thus by Mackey,

$$\langle \operatorname{Ind}_K^G \varphi, \operatorname{Ind}_K^G \rangle_G = \langle \varphi, \varphi \rangle_K + \langle \varphi, {}^t\varphi \rangle_K + \sum_{g \in K \backslash G / K - \{K, tK\}} \langle \varphi|_Z, {}^g\varphi|_Z \rangle_Z.$$

Since ${}^g\varphi|_Z = \varphi|_Z$ for all $g \in G$, ${}^t\varphi = \varphi^q$. So if φ has degree 1.

$$\langle \operatorname{Ind}_K^G \varphi, \operatorname{Ind}_K^G \varphi \rangle_G = \begin{cases} q - 1 & \varphi \neq \varphi^q \\ q & \varphi = \varphi^q \end{cases}$$

Next we compute

$$\operatorname{Ind}_K^G \varphi(g) = \begin{cases} q(q-1)\varphi(\lambda) & g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\ \varphi(\alpha) + \varphi^q(\alpha) & g = \alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q \\ 0 & \text{otherwise} \end{cases}$$

We can compute

$$\begin{aligned} \langle \operatorname{Ind}_B^G \mu_\theta \operatorname{Ind}_K^G u \rangle_G &= \frac{1}{|G|} \left(\sum_{\lambda \in Z} (q^2 - 1)\overline{\theta(\lambda)} q(q-1)\varphi(\lambda) + 0 \right) \\ &= |Z| \langle \theta, \varphi|_Z \rangle_Z \\ &= q - 1 & \text{if } \theta = \varphi|_Z \end{aligned}$$

If $\beta_\varphi = \operatorname{Ind}_B^G \mu_\theta = \operatorname{Ind}_B^G \mu_\theta - \operatorname{Ind}_K^G \varphi$ for $\theta = \varphi|_Z$.

$$\begin{aligned}
\langle \beta_\varphi, \beta_\varphi \rangle_G &= \langle \operatorname{Ind}_B^G \mu_\theta, \operatorname{Ind}_B^G \mu_\theta \rangle_G - 2\langle \operatorname{Ind}_B^G \mu_\theta, \operatorname{Ind}_K^G \varphi \rangle_G + \langle \operatorname{Ind}_K^G \varphi, \operatorname{Ind}_K^G \varphi \rangle \\
&= q - 2(q-1) + \begin{cases} q-1 & \varphi \neq \varphi^q \\ q & \varphi = \varphi^q \end{cases} \\
&= \begin{cases} 1 & \text{if } \varphi \neq \varphi^q \\ 2 & \text{if } \varphi = \varphi^q \end{cases}
\end{aligned}$$

Also

$$\beta_\varphi \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = (q^2 - 1) - (q)(q-1) = q - 1 > 0$$

It follows that β_φ is an irreducible character whenever $\varphi \neq \varphi^q$. Since $\beta_\varphi = \beta_{\varphi^q}$, $\varphi^{q^2} = \varphi$ and

$$|\{\varphi : \varphi^q \neq \varphi\}| = q - 1$$

We et

$$\frac{(q^2-1)-(q-1)}{2} = \binom{q}{2}$$

irreducible characters in this way.

	$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$	α, α^q	#reps
χ_θ	$\theta(\lambda)^2$	$\theta(\lambda)^2$	$\theta(\lambda)\theta(\mu)$	$\theta(\alpha^{q+1})$	$q-1$
V_θ	$q\theta(\lambda)^2$	0	$\theta(\lambda)\theta(\mu)$	$-\theta(\alpha^{q+1})$	$q-1$
$W_{\theta,\phi}$	$(q+1)\theta(\lambda\phi(\lambda)$	$\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\mu) + \phi(\lambda)\theta(\mu)$	0	$\binom{q-1}{2}$
β_φ	$(q-1)\varphi(\lambda)$	$-\varphi(\lambda)$	0	$-(\varphi + \varphi^q)(\alpha)$	$\binom{q}{2}$

We have not compute the representations corresponding to β_φ explicitly. These are known as *discrete series representations.*

Drufield found these in l-adic étale cohomology groups of an explicit algebraic curve $X/\mathbb{F}_q$. They can also be found as p-adic de Rham cohomology groups over a similar space. These can be viewed as generalisations of 'functions on X'. This work was generalised by Deligne-Lusztig for all "finite groups of lie type". Our computation also allows us to compute the character table of $\mathrm{PGL}_2(\mathbb{F}_q) = \frac{\mathrm{GL}_2(\mathbb{F}_q)}{Z}$ as its representations are just irreducible representations of G where Z acts trivially, i.e. χ_θ, V_θ for $\theta^2 = \mathbb{1}$, $W_{\theta,\theta^{-1}}$ for $\theta \neq \theta^{-1}$ and β_φ such that $\varphi|_Z = \mathbb{1}_Z$ (i.e. $\varphi^{q+1} = \mathbb{1}$ as well as $\varphi^q \neq \varphi$). We can then also compute the character table of $\mathrm{PSL}_2(\mathbb{F}_q) = \frac{\mathrm{SL}_2(\mathbb{F}_q)}{(\pm I)}$ which has index 2 in $\mathrm{PGL}_2(\mathbb{F}_q)$. These groups $\mathrm{PSL}_2(\mathbb{F}_q)$ are simple if $q \geq 5$ and this can be seen from the character table.

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