Linear Analysis

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Contents

Lectures

1 Normed Spaces and Linear Operators

Definition (Norm)**.** Let X be a real or complex vector space. A *norm* on X is a function $\|\bullet\| : X \to \mathbb{R}^+$ such that:

(i) $\|x\| = 0 \iff x = 0$

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all scalars λ and for all $x \in X$

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (triangle inequality)

A *normed space* is a pair $(X, \|\bullet\|)$ where X is a vector space and $\|\bullet\|$ is a norm on X.

The norm induces a metric space structure on X.

Definition (Banach space)**.** A normed space X is *Banach* if it is complete as a metric / topological space.

Definition (Unit ball)**.** It often helps to look at the *unit ball* of X, which is defined by

$$B = B(X) = B_X = \{x \in X \mid \|x\| < 1\}.$$

Remark. The unit ball is always convex.

Remark. Any set $B \subseteq \mathbb{R}^n$ which is closed, bounded, convex, symmetric about 0 and a neighbourhood of 0 defines a norm by:

$$\|x\| = \inf\{t > 0 \mid x \in tB\}.$$

B is the unit ball of that norm.

Theorem (Young's inequality)**.** Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and let $a, b \geq 0$. Then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Warning. Note that subspaces of normed vector spaces need not be closed topological subspaces!

Definition (Separable topological space)**.** A topological space is *separable* if it has a countable dense subset.

Theorem. Let X, Y be normed spaces and $T : X \to Y$ be linear. Then the following are equivalent:

(i) T is continuous

(ii) T is continuous at 0

(iii) $\exists k > 0$ such that $\|Tx\| \leq k\|x\|$ for all $x \in X$ (this condition is called *bounded*).

Notation. We will write:

$$L(X, Y) = \{T : X \to Y \mid T \text{ continuous and linear}\}$$

Note. The operator norm gives a norm on $L(X, Y)$, so $L(X, Y)$ is a normed vector space.

Theorem. Let X, Y be normed spaces with Y Banach. Then $L(X, Y)$ is Banach. In particular, X^* is always Banach.

Theorem (Dual of l_p)**.** If $1 \leq p < \infty$, then the dual of l_p is *isometrically isomorphic to* l_q, where q is the conjugate index.

We also have $c_0^* \cong l_1$, but l_∞^* does not have a particularly nice expression.

Lemma.

(i) Every finite dimensional normed space is Banach.

(ii) Every finite-dimensional subspace of a normed space is closed (useful to know!).

Proposition (Riesz's Lemma)**.** Let X be a normed space, and Y a closed subspace of X with $Y \neq X$. Then:

(i) $\forall \varepsilon > 0, \exists x \in X$ with $\|x\| = 1$ and $d(x, Y) \geq 1 - \varepsilon$

(ii) $\dim X < \infty$ gives $\exists x \in X$ with $\|x\| = 1$ and $d(x, Y) = 1$

Corollary. X an infinite dimensional normed space implies that there exists x_n in X with $\|x_n\| = 1$ and $\|x_n - x_m\| \geq 1$ for all $n \neq m$. In particular, $B(X)$ is not compact.

Lemma. This is a fact about metric spaces: for a subset A of a metric space X,

$$A \text{ is totally bounded } \iff \overline{A} \text{ is totally bounded.}$$

In particular if X is complete, then

$$A \text{ is totally bounded } \iff \overline{A} \text{ is compact.}$$

1.1 Compact operators

Definition (Compact operator)**.** Let X and Y be normed spaces. We say that $T : X \to Y$ linear is *compact* if $\overline{T(B_X)}$ is compact.

Remark. T compact implies $T(B_X)$ (totally) bounded, which implies that T is continuous.

Remark. $T : X \to Y$ is compact if and only if every bounded sequence (x_n) in X has a subsequence (x_{n_i}) such that (Tx_{n_i}) is convergent.

Remark. If Y is Banach, then $T : X \to Y$ is compact if and only if $T(B_X)$ is totally bounded.

Theorem. Let X, Y be normed vector spaces, with Y Banach. Then the compect operators from X to Y form a closed subspace of $L(X, Y)$.

Proposition. Let X, Y and Z be normed vector spaces and $T : X \to Y$, $S : Y \to Z$ linear maps.

(i) If S is compact, T continuous, then $S \circ T$ is compact.

(ii) If T is compact, S is continuous then $S \circ T$ is compact.

1.2 Open mapping lemma

Lemma (Open mapping lemma)**.** Let X, Y be normed, X Banach and let $T \in L(X, Y)$. Suppose that $B_Y \subseteq \overline{T(B_X)}$. Then:

(i) $B_Y \subseteq T(2B_X)$ (thus T is surjective)

(ii) Y is Banach

Note. $B_Y \subseteq \overline{T(B_X)}$ says that $T(B_X)$ is dense in B_Y.

2 The Baire Category Theorem and Applications

Theorem (Principle of Uniform Boundedness)**.** Let X Banach and Y normed and let $\tau \subseteq L(X, Y)$. Suppose τ is pointwise bounded (i.e. $\forall x \in X$ there exists k such that $\|T(x)\| \le k$ for all $T \in \tau$). Then τ is uniformly bounded (i.e. there exists k such that $\|T\| \le k$ for all $T \in \tau$).

Corollary (Banach Steinhaus Theorem)**.** Let X Banach, Y normed. Let $T_1, T_2, \dots \in L(X, Y)$ such that $(T_n x)$ is convergent for all x: say $T_n x \to Tx$. Then T is a continuous linear map.

Theorem (Open Mapping Theorem)**.** Let X, Y be Banach, and $T \in L(X, Y)$ surjective. Then T is an open mapping, i.e. there exists k such that $\forall y \in Y$, $\exists x \in X$ with $Tx = y$ and $\|x\| \le k\|y\|$.

Corollary (Inversion Theorem)**.** Let X, Y be Banach, and $T \in L(X, Y)$. Then if T is bijective, then T^{-1} is continuous (hence T is an isomorphism).

Remark. If X, Y are Banach, $T \in L(X, Y)$ surjective, then Inversion Theorem gives that $\overline{T} : X/\ker T \to Y$ is an isomorphism.

Corollary (Comparable Banach norms are equivalent)**.** Let $\| \bullet \|_1$ and $\| \bullet \|_2$ be complete norms on a vector space V. Suppose that there exists $c > 0$ such that $\|x\|_2 \le c\|x\|_1$ for all $x \in V$. Then $\| \bullet \|_1$ and $\| \bullet \|_2$ are equivalent. "Comparable Banach norms are equivalent".

2.1 The Closed Graph Theorem

Theorem (Closed Graph Theorem)**.** Let X, Y be Banach and $T : X \to Y$ linear. Then T is continuous if and only if T has closed graph.

3 Spaces of Continuous Functions

4 Hilbert Spaces

Definition (Hilbert space)**.** A *Hilbert space* is a complete inner product space.

Proposition (Parallelogram law)**.** Let X be an inner product space and $x, y \in X$. Then:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Corollary. Let H be a Hilbert space. Then:

(i) F a closed subspace of H implies that $(F^\perp)^\perp = F$

(ii) $S \subseteq H \implies (S^\perp)^\perp = \overline{\langle S \rangle}$

(iii) $S \subseteq H$ has dense linear span if and only if $S^\perp = \{0\}$

Theorem (Riesz Representation Theorem)**.** Let H be a Hilbert space, $f \in H^*$. Then there exists $y \in H$ such that $f = \theta_y$ (i.e. $f = \langle \bullet, y \rangle$).

Corollary (Hilbert are self dual)**.** Let H be a Hilbert space. Then the map $\theta : y \mapsto \theta_y$ is an isometric conjugate-linear isomorphism from H to H^*. "H is self-dual."

Corollary. Let X be a separable inner product space. Then X has an orthonormal basis.

Aim: We will show that for an orthonormal basis, we have

$$x = \sum_n (x, e_n) e_n$$

for all $x \in X$.

Remark. This is false for a general Banach space. For example, $1, t, t^2, \dots$ have dense linear space in $C_{\mathbb{R}}([-1,1])$ but $|t| \neq \sum_k c_k t^k$.

5 Spectral Theory

A useful corollary of the Inversion Theorem for Banach spaces:

Corollary 1. For a Banach space X, $T \in L(X)$ invertible $\iff$ T injective and T surjective.

Theorem. Let X be a Banach space, $T \in L(X)$. Then $\|T\| < 1$ implies $I - T$ is invertible with

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

Notation. For a Banach space X, we write

$$G(X) = \{T \in L(X) \mid T \text{ is invertible}\}.$$

Theorem. Let X be a Banach space. Then:

(i) G is open in $L(X)$

(ii) The function $T \mapsto T^{-1}$ from G to G is continuous

(iii) Let (T_n) in G, $T \in L(X)$ with $T_n \to T$ but $T \notin G$. Then $\|T_n^{-1}\| \to \infty$

Proposition. Let X be a complex Banach space, and $T \in L(X)$. Then $\sigma(T)$ is a closed subset of

$$\{z \in \mathbb{C} \mid |z| \leq \|T\|\}.$$

In particular, $\sigma(T)$ is compact.

Proposition. Let X be a complex Banach space, and $T \in L(X)$, $\lambda \in \mathbb{C}$. Then:

(i) λ is an eigenvalue of T implies λ is an approximate eigenvalue of T

(ii) λ an approximate eigenvalue of T implies $\lambda \in \sigma(T)$

Theorem. Let X be a complex Banach sace, $T \in L(X)$. Then $\partial\sigma(T) \subseteq \sigma_{ap}(T)$.

Theorem (Spectral Mapping Theorem)**.** Let X be a complex Banach space, $T \in L(X)$. Let P be a non-constant polynomial. Then $\sigma(P(T)) = P(\sigma(T))$.

Warning. We did not use the "fact" that A not invertible $\implies$ AB not invertible (or BA not invertible).

This is false (for example, left and right shift).

Definition (Spectral value)**.** Let X be a complex Banach space, $T \in L(X)$. The *spectral value* of T is:

$$r(T) = \sup\{|\lambda| \mid \lambda \in \sigma(T)\}.$$

Certainly have that $r(T) \leq \|T\|$.

Corollary. Let X be a complex Banach space and $T \in L(X)$. Then $r(T) \leq \inf_{n\geq 1} \|T^n\|^{1/n}$.

Theorem (Non-emptiness of the spectrum)**.** Let H be a non-zero Hilbert space, and $T \in L(H)$. Then $\sigma(T) \neq \emptyset$.

5.1 Spectral Theory of Hermitian Operators (all spaces are complex here)

Proposition. Let H be a Hilbert space, $T \in L(H)$. Then $\sigma(T^*) = \{\overline{\lambda} \mid \lambda \in \sigma(T)\}$.

Theorem. Let H be a Hilbert space, $T \in L(H)$ Hermitian. Then $\sigma(T) \subseteq \mathbb{R}$.

Remark. We know that any operator T on a (non-zero) Hilbert space H has an approximate eigenvalue ($\sigma(T) \neq \emptyset$ so $\partial\sigma(T) \neq \emptyset$). But if T is Hermitian, then in fact every $\lambda \in \sigma(T)$ is an approximate eigenvalue ($\sigma(T) \subseteq \mathbb{R}$, so $\sigma(T) = \partial\sigma(T)$).

Proposition. Let H be a Hilbert space, $T \in L(H)$ Hermitian. Then $r(T) = \|T\|$.

Proposition. Let H be a Hilbert space, $T \in L(H)$, and Y a subspace of H. Then if T acts on Y, then T^* acts on $Y^\perp$. In particular for T Hermitian, if T acts on Y then T acts on $Y^\perp$.

5.2 Spectral Theory of Compact Operators

Proposition. Let X be an infinite dimensional Banach space, and $T \in L(X)$ compact. Then 0 is an approximate eigenvalue of T.

Note. 0 need not be an eigenvalue of T, for example $T : l_2 \to l_2$ defined by

$$T\left(\sum_n x_n e_n\right) = \sum_n \frac{1}{2^n} x_n e_n.$$

Proposition. Let X be a Banach space, $T \in L(X)$ compact, and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$. Then if $\lambda \in \sigma_{ap}(T)$, then $\lambda \in \sigma_p(T)$.

Proposition. Let X be a Banach space, $T \in L(X)$ compact. Then every eigenspace $E(\lambda)$ for $\lambda \neq 0$ is finite dimensional.

Lemma. Let X be a Banach space, $T \in L(X)$ compact. Then $\forall \delta > 0$, T has only finitely many eigenvalues with $|\lambda| > \delta$.

Theorem (Spectral Theorem for Compact Operators)**.** Let X be a Banach space, $T \in L(X)$ compact. Then:

(i) Either $\sigma(T)$ is finite or $\sigma(T) = \{0, \lambda_1, \lambda_2, \ldots\}$ where $\lambda_n \to 0$

(ii) $\lambda \in \sigma(T)$, $\lambda \neq 0 \implies \lambda \in \sigma_{ap}(T)$

(iii) $\lambda \in \sigma(T)$, $\lambda \neq 0 \implies \dim E(\lambda) < \infty$

Theorem (Spectral Theorem for Compact Hermitian Operators)**.** Let H be a separable Hilbert space, $T \in L(H)$ compact Hermitian. Then:

(i) H has an orthonormal basis (e_n) consisting of eigenvectors of T

(ii) The corresponding eigenvalues $\lambda_n \to 0$ (if $\dim H = \infty$)

Theorem. Let H be a Hilbert space, $T \in L(H)$ compact Hermitian. Then there exists a closed subspace Y of H and an orthonormal basis (e_n) for Y and (λ_n) in $\mathbb{R}$ such that $\forall x \in H$:

$$x = \sum_n x_n e_n + z; z \in Y^\perp \implies Tx = \sum_n \lambda_n x_n e_n.$$

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