Graph Theory

May 30, 2024

Contents

Lectures

Start of lecture 1

Chapter I

Introduction

Contents

1. General Words of Advice

This course is meant to be quite elegant. Understanding the proofs in this course is essential. Exam questions will be about problems where you apply proof methods seen in lectured proofs.

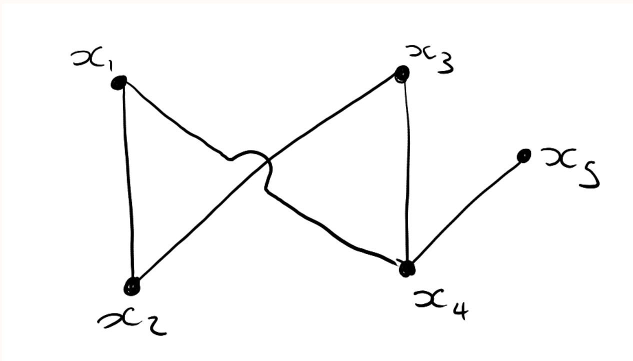
Proofs will be elegant, and will generally not involve calculations. We will use matrices sometimes (and in particular we may talk about eigenvalues) but you should never have to calculate a determinant of a matrix on any example sheet or exam question.

1.1. Basic Definitions

Definition (Graph)**.** A *graph* is a pair (V, E), where V is a set and E is a subset of $V^{(2)} = \{(x, y) : x, y \in V, x \neq y\}$, the unordered pairs from V.

V is the vertex set of G and E is the edge set of G.

Example. The graph below has $V = \{x_1, x_2, x_3, x_4, x_5\}$, $E = \{\{x_1, x_2\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x$



Definition (Order of a graph)**.** The *order* of G is $|G| = |V|$.

Definition (Size of a graph)**.** The size of G is $e(G) = |E|$.

Notes

(1) No self loops (edges starting and ending at the same vertex).

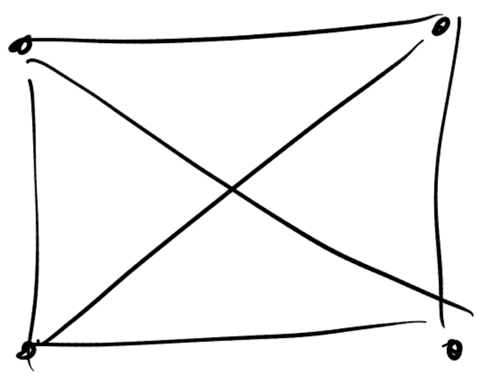
(2) No multiple edges (between every pair of vertices, there are exactly 0 or 1 edges).

(3) No directed edges.

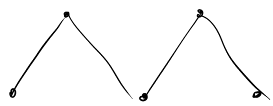
Examples

(1) The *empty graph* E_n: $V = \{x_1, \ldots, x_n\}$, $E = \emptyset$. So $|E_n| = n$, $e(E_n) = 0$.

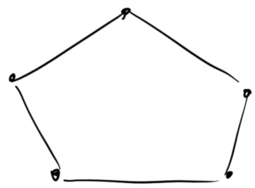
(2) The *complete graph* K_n: $V = \{x_1, \ldots, x_n\}$, $E = V^{(2)}$, so $|K_n| = n$, $e(K_n) = \binom{n}{2}$.



(3) The *path of length* n, P_n: $V = \{x_1, \ldots, x_{n+1}\}$, $E = \{x_1x_2, x_2x_3, \ldots, x_nx_{n+1}\}$, so $|P_n| = n + 1$, $e(P_n) = n$.

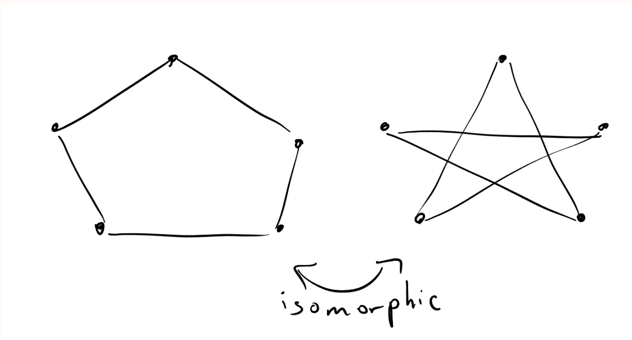


(4) The n-*cycle* C_n: $V = \{x_1, \dots, x_n\}$, $E = \{x_1x_2, x_2x_3, \dots, x_{n-1}x_n, x_nx_1\}$, so $|C_n| = n$, $e(C_n) = n$.



Definition (Graph Isomorphism)**.** Say graphs $G = (V, E)$ and $H = (V', E')$ are *isomorphic* if there exists bijection $f : V \to V'$ such that $xy \in E \iff f(x)f(y) \in E'$.

Example.



Definition (Subgraph)**.** Say H is a *subgraph* of G if $V' \subset V$, $E' \subset E$.

Notation. For $G = (V, E)$ and $xy \in E$, write $G - xy = (V, E \setminus \{x, y\})$.

Notation. For $G = (V, E)$ and $xy \in V^{(2)} \setminus E$, write $G + xy = (V, E \cup \{x, y\})$.

Definition (Neighbours)**.** If $xy \in E$, say x, y are *adjacent* or *neighbours*.

Definition (Neighbourhood)**.** For $x \in G$, the *neighbourhood* of x is $\Gamma(x) = \{y \in G : xy \in E\}$.

Definition (Vertex degree)**.** For $x \in G$, the *degree* of x is $d(x) = |\Gamma(x)|$.

Definition (Degree Sequence)**.** If $V = \{x_1, \ldots, x_n\}$ then the *degree sequence* of G is $d(x_1), \ldots, d(x_n)$.

Definition (Minimum and Maximum Degree)**.** G has *maximum degree* $\Delta(G) = \max\{d(x_1), \ldots, d(x_n)\}$ and *minimum degree* $\delta(G) = \min\{d(x_1), \ldots, d(x_n)\}$.

Definition (Regular Graph)**.** If the degree of each vertex in G is k, then we say that G is *k-regular* (or just *regular*).

For example, C_n is 2-regular, and K_n is $(n-1)$-regular. In this course, unless otherwise stated, V is finite (we will study infinite graphs briefly).

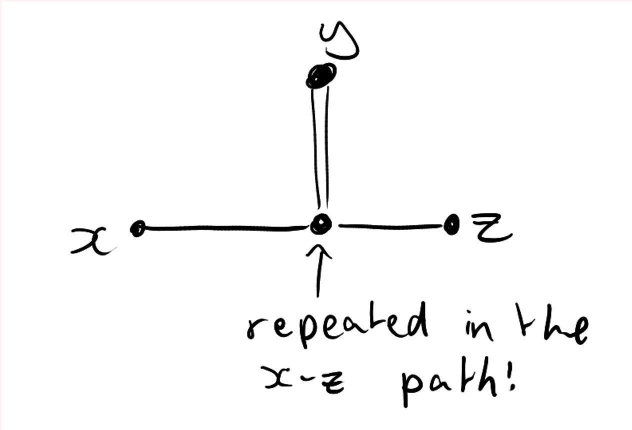
1.2. Connectedness

Definition (Path)**.** For $x, y \in G$, an $x - y$ path is a sequence $x_1, \ldots, x_k$ $(k \geq 1)$ of distinct vertices of G such that $x_1 = x$, $x_k = y$ and $x_i x_{i+1} \in E \forall i$. Its *length* is $k - 1$.

Notation. Write $x \sim y$ if there exists a path from x to y (an $x - y$ path).

Note that this is an equivalence relation.

Remark. Given an $x-y$ path and a $y-z$ path, concatenating them does *not* necessarily give an $x-z$ path.



However, it always *contains* an $x-z$ path – indeed, if we set i to be the least $1 \leq i \leq k$ such that there exists $k \leq j \leq l$ with $x_j = x_k$, then $x_1, \dots, x_i, x_{j+1}, \dots, x_l$ is an $x-z$ path.

Definition (Component). The equivalence classes of $\sim$ are the *components* of G. The component $[x]$ of a vertex is $[x] = \{y : \exists x - y \text{ path in } G\}$.

Definition (Connected Graph). Say G is connected if $\forall x, y \in G$, there exists an $x-y$ path in G.

In other words, has one component or has no vertices.

Definition (Walk). An $x-y$ walk is a sequence $x_1, \dots, x_k$ $(k \geq 1)$ such that $x_1 = x$, $x_k = y$ and $x_i x_{i+1} \in E \forall i$.

So a walk is a path where we allow repeats.

Remark. There exists a walk from x to y if and only if there exists a path from x to y.

Start of

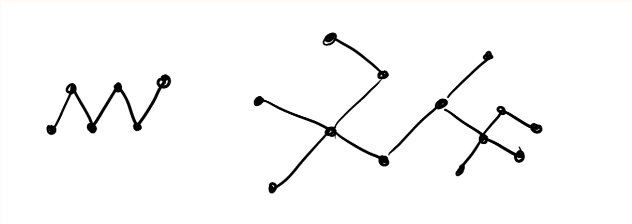
lecture 2

1.3. Trees

Definition (Acyclic graph)**.** Say graph G is *acyclic* if it has no cycle.

Definition (tree)**.** A *tree* is a connected acyclic graph.

Example. The following are both trees:



Definition (leaf)**.** In a tree, a vertex of degree 1 is a *leaf* or *enovertex*.

Proposition 1. Let G be a graph. Then the following are equivalent:

(i) G is a tree.

(ii) G is minimal connected (G is connected, and $G - xy$ is disconnected for all $xy \in E$)

(iii) G is maximal acyclic (G is acyclic, $G + xy$ has a cycle for all $xy \in V^{(2)} \setminus E$)

Proof.

(i) $\implies$ (ii) Suppose $G - xy$ connected, so $G - xy$ has an $x - y$ path P say. But then Pyx is a cycle in G.

(ii) $\implies$ (i) Suppose G has a cycle C, and pick $xy \in E(C)$. Then $G - xy$ connected. Indeed, for any $a, b \in V$ have an ab path in G. If this path uses xy, replace xy with $C - xy$ to obtain an $a - b$ walk in $G - xy$.

(i) $\implies$ (iii) For any $xy \notin E$: G has an $x - y$ path P, so $G + xy$ has the cycle Pyx.

(iii) $\implies$ (i) If G is disconnected, choose x, y in different components. Then $G + xy$ is acyclic. □

Proposition 2. Every tree T ($|T| \geq 2$) has a leaf.

Proof. Choose a longest path $P = x_1, \dots, x_k$ in T (since T finite). Then $\Gamma(x_k) \subset P$ (by maximality of P), but also $\Gamma(x_K) \cap P = \{x_{k-1}\}$ (since T acyclic). So x_k is a leaf. □

Note. Actually always have at least 2 leaves, since same argument shows x_1 is a leaf.

Proof (alternative). Suppose $d(x) \geq 2$ for all $x \in T$. Choose x_1, x_2 such that $x_1 x_2 \in E$. Then choose $x_3, x_4, \dots$ as follows: given x_{k-1}, let x_k be a neighbour of x_{k-1} not equal to x_{k-2}. Since G is finite, we must repeat, which gives a cycle. □

We can describe this proof as "go for a walk".

Notation. For a graph G, $W \subset V$, write $G[W]$ for the graph $(W, E \cap W^{(1)})$ (the subgraph spanned by W).

Notation. For a graph G, $x \in V$, write $G - x$ for $G[V \setminus \{x\}]$.

Proposition 3. Every tree T on n vertices ($n \geq 1$) has $e(T) = n - 1$.

Proof. Induction on n. $n = 1$ is trivial.

Given a tree T on n vertices, $n \geq 2$: let x be a leaf of T. Define $T - x$ is a tree on $n - 1$ vertices, so $e(T - x) = n - 2$ (induction). Hence $e(T) = n - 1$. □

Definition (Spanning tree)**.** In a connected graph $G = (V, E)$, a *spanning tree* is a subgraph T that is a tree with $V(T) = V$.

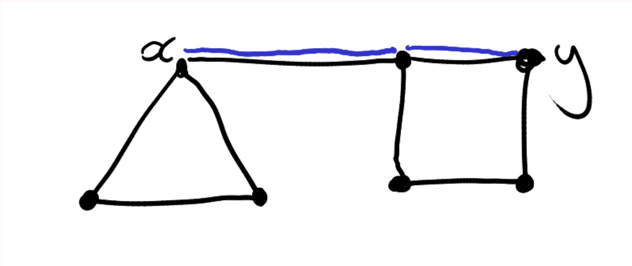
Lemma. Every connected graph G has a spanning tree.

Proof. Keep removing edges until we are minimally connected (which is equivalent to being a tree). □

Note. For a tree T, T is the *unique* spanning tree of T (by minimal connectedness, for example).

Definition (distance)**.** For $x, y \in G$, the *distance* $d(x, y)$ is the length of a shortest $x - y$ path in G.

Example. In the following graph, x and y have distance 2:



Most proofs of Proposition 3 work by induction. Some books even say that every proof must use induction. This isn't true. Just to prove this point, we show the below proof.

Proof (non-inductive proof of Proposition 3). We will show that any tree T has a spanning tree of $n - 1$ edges (and since the only spanning tree of a tree is itself, this will mean we are done).

Fix $x_0 \in T$. For each $x \in V \setminus \{x_0\}$, let $xx', \dots, x_0$ be a shortest $x - x_0$ path (so $d(x', x_0) = d(x, x_0) - 1$). Let $T' = \{xx' : x \in V \setminus \{x_0\}\}$. So $e(T') = n - 1$ (we can't count an edge twice because we can only go in the direction that brings us closer to x_0).

Claim: T' is a tree. **Proof of claim:**

connected For any x, $xx'(x')' \dots$ must reach x_0.

connected Suppose C is a cycle in T'. Choose $x \in C$ at greatest distance from x_0. Then *both* neighbours of x on C are at distance $\le d(x, x_0)$ from x_0. Contradiction construction of T'. □

Start of lecture 3

Definition (Forest)**.** A *forest* is an acyclic graph.

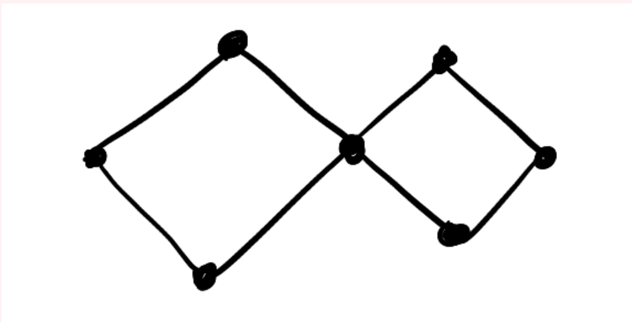
A *cutvertex* is a vertex x such that $G - x$ is connected.

Definition (Bridge)**.** For a connected graph G, an edge $xy \in G$ is a *bridge* if $G - xy$ is disconnected.

Remark. Every edge in a tree is a bridge.

Lemma. If G has a bridge, then it has a curvertex (as long as $|G| > 2$).

Remark. If G has a cutvertex, it doesn't necessarily have a bridge. For example:

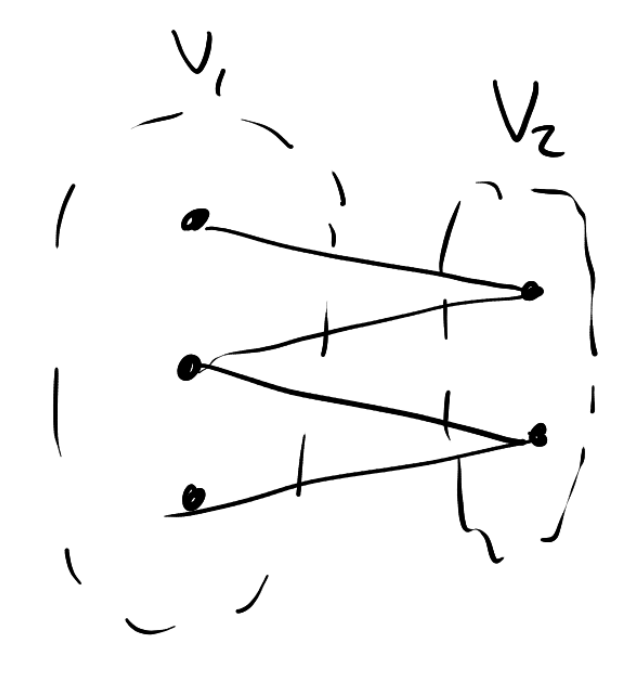


1.4. Bipartite Graphs

Definition (Bipartite Graph)**.** A graph $G = (V, E)$ is *bipartite* if there exists a partitition $V = V_1, V_2$ of V such that $E \subset \{xy : x \in V_1, y \in V_2\}$.

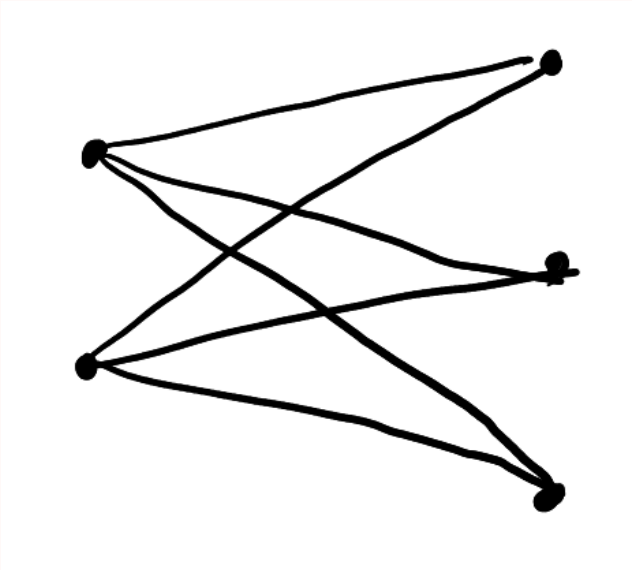
"No edges inside the two parts."

Example. A path is a bipartite graph:



Definition. $K_{m,n}$ denotes the *complete bipartite graph*, which has vertex set $V_1 \cup V_2$ (V_1, V_2 disjoint), with $|V_1| = m$, $|V_2| = n$ and edge set $E = \{xy : x \in V_1, y \in V_2\}$. So $e(K_{m,n}) = mn$.

Example. $K_{2,3}$ looks like:



We will show that a graph is bipartite if and only if it has no odd length cycles, but first we need to introduce the concept of a circuit.

Definition (Circuit)**.** A *circuit* in a graph is a closed walk, i.e. a walk $x_1, \dots, x_k$ with $x_k = x_1$.

Lemma. If G has an odd circuit, then it has an odd cycle.

Proof. Suppose $x_1, \dots, x_k = x_1$ is an odd circuit that is not a cycle. Say $x_i = x_j$ for $i < j$. Then one of $x_i x_{i+1} \dots x_j$ or $x_j x_{j+1} \dots x_{k-1} x_1 x_2 \dots x_i$ is a shorter circuit. Done by induction on length. □

Remark. We have crucially used oddness in this proof. The lemma is false if "odd" is replaced by "even".

Proposition 4. A graph G is bipartite if and only if G has no odd cycle.

Proof. $\Rightarrow$ A cycle's vertices must alternate between V_1 and V_2.

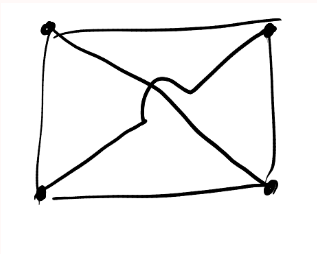
$\Leftarrow$ Without loss of generality, assume G is connected (since if each component of G is bipartite then so is G). Fix $x_0 \in V$. Let $V_1 = \{x : d(x, x_0)\text{even}\}$ and $V_2 = \{x : d(x, x_0)\text{odd}\}$. If we had an edge xy with $x, y \in V_1$ or $x, y \in V_2$, then xy together with shortest paths from x and y to x_0 would form an odd circuit. Hence G would contain an odd cycle, contradiction. □

1.5. Planar graphs

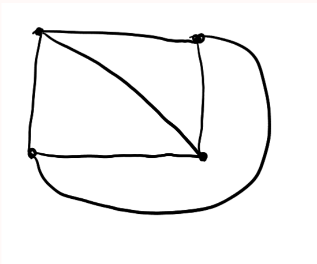
Definition (Planar Graph)**.** A *planar graph* is a graph that can be drawn in the plane without crossing edges.

Definition (Plane Graph)**.** A *plane graph* is a drawing of a planar graph in which none of the edges cross.

Example. K_4 is a planar graph, but the following is not a plane graph:



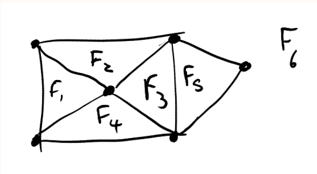
On the other hand, this drawing of K_4 *is* a plane graph:



Definition (Face)**.** For a plane graph G, $\mathbb{R}^2 - G$ splits up into connected regions called *faces*.

The *boundary* of a face consists of the vertices and edges of G that touch it.

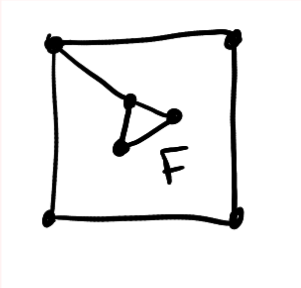
Example. The following has 6 faces:



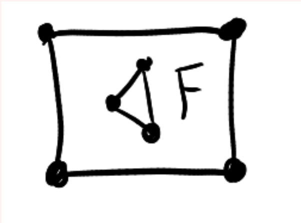
F_1 has boundary a 3-cycle, and F_6 has boundary a 5-cycle.

Warning.

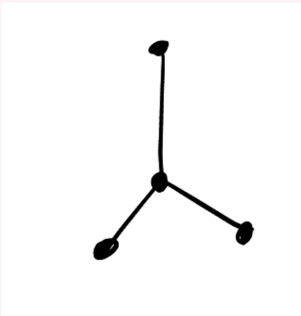
(1) Boundary of a face need not be a cycle, for example



(2) Boundary of a face need not even be connected, for example:



(3) The two faces on either side of an edge may be the same, for example:



Example.

(1) Every tree is planar, with exactly 1 face. Proof is by induction via remove a leaf (remember that this is the usual way to attack a question about trees).

(2) The empty graph E_n is planar.

(3) Every cycle is planar (via the obvious drawing).

Question: Which graphs are planar? How do we tell if a graph is / isn't planar?

Formal Bit

Definition. For $x, y \in RR^2$, a *polygonal arc* from x to y is a finite union of (closed) straight-line segments $\overline{x_1x_2} \cup \overline{x_2x_3} \cup \cdots \cup \overline{x_{k-1}x_k}$ such that $x_1 = x$, $x_k = y$ and the $\overline{x_ix_{i+1}}$ are disjoint except for $\overline{x_ix_{i+1}} \cap \overline{x_{i+1}x_{i+2}} = \{x_{i+1}\}$.

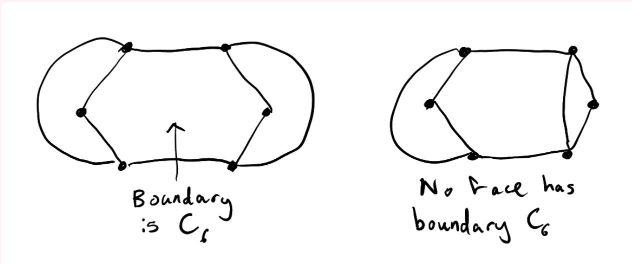
For a graph G, with $V = \{v_1, \dots, v_n\}$, a *drawing* of G consists of distinct points $x_1, \dots, x_n \in \mathbb{R}^2$, together with, for each $v_iv_j \in E$, a polygonal arc p_{ij} from x_i to x_j such that $p_{ij} \cap p_{kl} = \emptyset$ if i, j, k, l distinct and $p_{ij} \cap p_{jk} = \{x_j\}$ for i, k distinct.

On $\mathbb{R}^2 - G$, define $x \sim y$ if there exists a polygonal arc in $\mathbb{R}^2 - G$ from x to y. Clearly (ish), this is an equivalence relation. The components are the *faces* of G. The *boundary* of a face is the intersection of G with the closure of the face.

We'll use simple faces about $\mathbb{R}^2$, like "boundary of a face consists of vertices and (whole) edges" or "a cycle has 2 faces" - all can be proved by induction on the number of straight-line segments.

Start of lecture 4

Remark. A planar graph G can have inequivalent drawings:



Despite this, perhaps is it the case that the number of faces is always the same? It turns out the answer is yes.

Theorem 5 (Euler's Formula)**.** Let G be a connected plane graph with n vertices, m edges and f faces. Then $n - m + f = 2$.

Note. We do need G connected. For example, E_n has n vertices, 0 edges and 1 face.

Proof. If G acyclic, then G is a tree, so $m = n - 1$, $f = 1$.

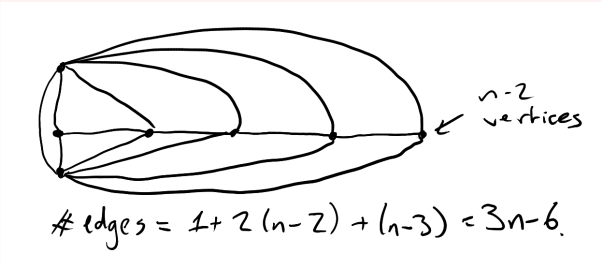
If G has a cycle, pick edge e on a cycle and let $G' = G - xy$. Then G' has n vertices, $m - 1$ edges, $f - 1$ faces (as e on a cycle). So $n - (m-1) + (f-1) = 2$ (induction), i.e. $n - m + f = 2$. □

Theorem 6. Let G be a planer graph with n vertices ($n \geq 3$) and m edges. Then $m \leq 3n - 6$.

Note.

(1) This is a *linear* bound – whereas in general a graph could have up to $\binom{n}{2} = \frac{n^2-n}{2}$ edges.

(2) Bound is best-possible, for example



Proof. Without loss of generality, assume G is connected (else add edges to make G connected).

If we sum, for each face, the number of edges on its boundary, we obtain $\geq 3f$, since each face has ≥ 3 edges (for $n > 3$, and for $n = 3$ the theorem is trivial anyway).

Also, each edge is counted at most twice. Hence $3f \leq 2m$, so by Euler's Formula, $n - m + \frac{2}{3}m \geq 2$, i.e. $n - \frac{m}{3} \geq 2$, so $m \leq 3n - 6$ as desired. □

Corollary 7. K_5 is not planar.

Proof. $n = 5$, $m = 10$, violating $m \leq 3n - 6$ from Theorem 6. □

Hence any G containing K_5 as a subgraph is not planar, for example K_n for $n \geq 5$.

Definition (Subdivision)**.** A *subdivision* of a graph G is obtained by replacing some edges of G by (distinct) paths.

Hence any G containing any subdivided K_5 is non-planar.

Definition (Girth)**.** The *girth* of a graph is the length of a shortest cycle in G (and if G has no cycles we say girth ∞).

Theorem. Let G be a planar graph with girth $\geq g$. Then

$$m \leq \max\left(\frac{g}{g-2}(n-2), n-1\right)$$

Using this, we may check that $K_{3,3}$ is not planar.

Hence any G that contains a subdivided $K_{3,3}$ or subdivided K_5 must not be planar.

Theorem (Kuratowski's Theorem)**.** G is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

Proof. Not proved in this course, because the proof is very long and not terribly interesting. □

So, to show a graph G is planar, just draw it. To show a graph G isn't planar, find a subdivided K_5 or $K_{3,3}$. Kuratowski's Theorem says that this method will always work. But in any proof of this form, we don't actually have to quote this theorem!

Chapter II

Connectivity and Matchings

Contents

2. Matchings and Hall's Marriage Theorem

Say that a set of edges is *independent* if none of the edges have any vertices in common.

Let G be bipartite, classes X, Y. A matching from X to Y consists of $|X|$ independent edges (no vertices in common). When does G have a matching. 'Mathmaker' terminology: X is boys, Y is girls. An edge from x to y if x knows y. Want to marry off each boy with a girl he knows.

Fails if there exists $x \in X$ with $d(x) = 0$, or if there exists distinct $x, x' \in X$ with $d(x) = d(x') = 1$, $\Gamma(x) = \Gamma(x')$.

Write $\Gamma(A)$ for $\bigcup_{x \in A} \Gamma(x)$ for any $A \subset X$. Then certainly *must* have $|\Gamma(A)| \ge |A|$ for all $A \subset X$. Do there exist any other possible obstructions?

Start of lecture 5

Theorem 1 (Hall's Marriage Theorem)**.** Let G be a bipartite graph, vertex classes X, Y. Then G has a matching from X to Y if and only if $|\Gamma(A)| \ge |A|$ $\forall A \subset X$ (this condition is sometimes called "Hall's condition").

Proof 1. Induction on $|X|$, noting $|X| = 1$ is trivial. Given G with $|X| > 1$:

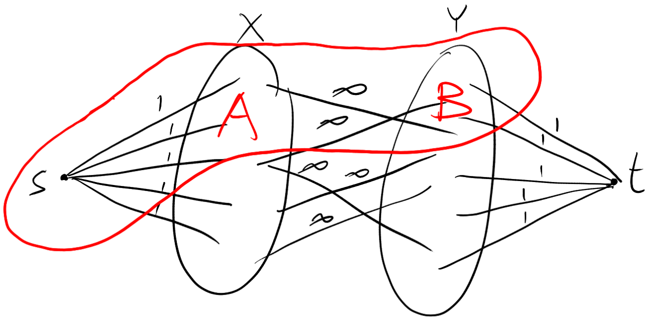
Question: Do we have $|\Gamma(A)| > |A|$ for all $A \subset X$ $(A \neq \emptyset, X)$?

If yes, pick $x \in X$ and $y \in \Gamma(x)$ and set $G' = G - x - y$. Then $\forall A \subset X - \{x\}$, $|\Gamma_{G'}(A)| \ge |A|$, because $|\Gamma_G(A)| \ge |A| + 1$ $(A \neq \emptyset)$. So G' has a matching from $X - \{x\}$ to $Y - \{y\}$ by induction hypothesis. Together with xy, this gives a matching from X to Y.

If no, then we have some $A \subset X$ $(A \neq \emptyset, X)$ with $|\Gamma(A)| = |A|$. Let $B = \Gamma(A)$. Let $G' = G[A \cup B]$, $G'' = G[X \setminus A \cup Y \setminus B]$. For $C \subset A$, we have $|\Gamma_G(C)| \ge |C|$, so $|\Gamma_{G'}(C)| \ge |C|$ (as $\Gamma(A) = B$), so by induction, G' has a matching from A to B. For $C \subset X \setminus A$, we have $|\Gamma_G(C \cup A)| \ge |C| + |A|$, so $|\Gamma_G(C \cup A)$ contains at least $|C|$ points of $Y \setminus B$ (as $\Gamma(A) = B$ and $|B| = |A|$). But $\Gamma(A)$ is disjoint from $Y \setminus B$, so $|\Gamma_{G''}(C)| \ge C$, so by induction, G'' has a matching. □

Proof 2. Form a directed network by adding source s, joined to each $x \in X$ by an edge of capacity 1. Add a sink t joined to each $y \in Y$ by an edge of capacity 1, and direct each $xy \in G$ from X to Y with capacity ∞ (some large integer).

Then a matching is precisely an integer-valued flow of value $|X|$, so done by integrality form of max flow min cut if every cut has capacity $\ge |X|$.



Suppose there is a cut (S, S^c), where $S = \{s\} \cup A \cup B$ with capacity $< |X|$ ($A \subset X$, $B \subset Y$). We must have $\Gamma(A) \subset B$ (else S has infinite capacity). Hence S has capacity $|X| - |A| + |B| \geq |X|$ since $|B| \geq |A|$ (as $B \supset \Gamma(A)$). So a cut of capacity less than $|X|$ is impossible. □

Definition (deficient matching)**.** A *matching of deficiency* d in bipartite G on X, Y consists of $|X| - d$ independent edges.

Corollary 2 (Defect Hall)**.** Let G be a bipartite graph with classes X, Y. Then G has a matching from X to Y of deficiency d if and only if $|\Gamma(A)| \geq |A| - d \; \forall A \subset X$.

Proof.

$\Rightarrow$ Trivial.

$\Leftarrow$ Form G' from G by adding d new points to Y, joined to all points of X. Then $\forall A \subset X$, $\Gamma_{G'}(A)| \geq |A|$, so G' has a matching (Hall's Marriage Theorem). In this matching, at least $|X| - d$ are paired into Y. □

Definition (transversal)**.** A *transversal* for sets $S_1, \dots, S_n$ consists of some distinct $x_1, \dots, x_n$ with $x_i \in S_i \; \forall i$.

Example. $S_1 = \{a, b, c\}$, $S_2 = \{a, b\}$, $S_3 = \{c, d\}$, $S_4 = \{b, d\}$. Then $x_1 = b$, $x_2 = a$, $x_3 = c$, $x_4 = d$ is a transversal.

When is there a transversal? Clearly need

$$\left|\bigcup_{i\in A} S_i\right| \geq |A| \qquad \forall A \subset \{1, \dots, n\}$$

This should remind you of Hall's Marriage Theorem.

Corollary 3. Sets $S_1, \dots, S_n$ have a transversal if and only if

$$\left|\bigcup_{i\in A} S_i\right| \geq |A| \qquad \forall A \subset \{1, \dots, n\}$$

Proof. $\Rightarrow$ Trivial.

$\Leftarrow$ WLOG each S_i finite. Form a bipartite graph G with vertex classes $X = \{1, \dots, n\}$ and $Y = \bigcup_i S_i$ by joining $i \in X$ to $j \in Y$ if $j \in S_i$. Then a transversal of $S_1, \dots, S_n$ is precisely a matching from X to Y. But

$$|\Gamma_G(A)| = \left|\bigcup_{i\in A} S_i\right| \geq |A| \qquad \forall A \subset X,$$

so done by Hall's Marriage Theorem. □

Remark. Corollary 3 is actually *equivalent* to Hall, since if G is bipartite, classes X and Y with $X = \{x_1, \dots, x_n\}$, then a matching from X to Y is exactly a transversal of the sets $\Gamma(x_1), \dots, \Gamma(x_n)$.

Start of lecture 6

To see a typical application of Hall's Marriage Theorem, consider G a finite group and H a subgroup of G. We have the left cosets $L_1, \dots, L_k$ say $g_1H, \dots, g_kH$ and right cosets $R_1, \dots, R_k$, say $Hg'_1, \dots, Hg'_k$, where k is the index of H in G.

Can we pick the same representatives? In other words, can we pick some $g_1, \dots, g_k$ such that the left cosets are $g_1H, \dots, g_kH$ and the right cosets are $Hg_1, \dots, Hg_k$, up to reordering? Equivalently, can we reorder the R_i such that $L_i \cap R_i \neq \emptyset$ $\forall i$? In other words, we seek a matching in the bipartite graph with vertex sets $X = \{L_1, \dots, L_k\}$, $Y = \{R_1, \dots, R_k\}$, for which L_i is joined to R_i if their intersection is non-empty.

By Hall's Marriage Theorem, it is enough to show $|\Gamma(A)| \geq |A|$ $\forall A \subset X$. In other words, we need $\bigcup_{i\in A} L_i$ meets at least $|A|$ right cosets. But $\bigcup_{i\in A} L_i = |A||H|$, $\bigcup_{i\in A} L_i$ must meet at least $|A|$ right cosets as each right coset has size $|H|$.

3. Connectivity

How "connected" is a connected graph? A path P is connected, $P-x$ is disconnected for some $x \in P$. A cycle C is connected, and also $C-x$ is connected $\forall x \in C$, but $C-x-y$ is disconnected for some $x, y \in C$.

If G is a "cube", then $G-x-y$ is connected $\forall x, y \in G$, but can be disconnected by removing three vertices.

Definition (Connectivity)**.** For a graph G with $|G| \geq 2$, the *connectivity* of G, written $\kappa(G)$, is the least size of a set $S \subset V$ such that $G-S$ is disconnected or a single point. If $\kappa(G) \geq k$, we say G is k-connected. Equivalently, G is k-connected if and only if $|G| > k$ and for all $S \subset V$ with $|S| < k$, $G-S$ is connected.

Example.

(1) No tree is 2-connected.

(2) C_n is 2-connected but not 3-connected.

(3) The "cube" is 3-connected.

Warning. We can have $\kappa(G-x) > \kappa(G)$. For example, consider a cycle joined to a vertex x. Then $\kappa(G) = 1$, but $G-x = C_n$, so $\kappa(G-x) = 2$.

Remark. We must have $\kappa(G) \leq \delta(G)$, since for any $x \in G$, removing $\Gamma(x)$ from G disconnects it.

We have that if G is 1-connected, then there is an ab-path in G for all distinct $a, b \in G$.

It would be nice if G being k-connected implies there exists a family of k independent paths from a to b.

Definition (independent paths)**.** We say that two ab-paths P_1 and P_2 are *independent* if $P_1 \cap P_2 = \{a, b\}$.

Definition (ab-separator)**.** For distinct $a, b \in G$, say $S \subset V \setminus \{a, b\}$ separates a and b, or is an ab-separator, if a and b are in different components of $G - S$.

Equivalently, every ab-path meets S.

Theorem 4 (Menger's Theorem)**.** Let a and b be distinct non-adjacent vertices in a graph G such that every ab-separator has size at least k. Then G contains a family of k independent ab-paths.

Remark.

(1) The converse is trivial. It we have k independent paths from a to b, then any separator S must meet each path.

(2) An equivalent formulation says that the minimum size of an ab-separator is the maximum size of an independent family of ab-paths.

(3) We need a and b to be non-adjacent, else there would be no separators!

(4) We cannot just "pick a point on each of a maximum-sized independent family of ab-path" to prove this.

(5) Menger's Theorem generalises Hall's Marriage Theorem. Indeed, given bipartite G on X, Y, form G' by adding a, b where $\Gamma(a) = X$, $\Gamma(b) = Y$. Then, G has a matching if and only if G' has a family of $|X|$ independent ab-paths. So, by Menger's Theorem, it is enough to show that each ab-separator S has size of at least $|X|$. Let $S = A \cup B$ be a separator, with $A \subset X$ and $B \subset Y$. Since S is a separator, we must have $\Gamma_G(X \setminus A) \subset B$. So $|S| = |A| + |B| \ge |A| + \Gamma_G(X \setminus A)| \ge |A| + |X \setminus A| = |X|$.

Start of lecture 7

Proof of Menger's Theorem. Let k be the minimum size of an $a-b$ separator. We want k independent $a-b$ paths. If not possible, pick a minimal counterexample (say minimum k, then minimum $e(G)$). Note $k \ge 2$ (the theorem is trivial for $k = 1$). Let S be a separator $|S| = k$.

Case 1: $S \ne \Gamma(a)$, $S \ne \Gamma(b)$. Form a graph G' from G, by replacing the component of b in $G - S$ with one point b', joined to each point of S. Then $e(G') < e(G)$, because otherwise $\Gamma(b) \subset S$ whence $\Gamma(b) = S$ (by minimality). Also, no $a - b'$ separator in G' has size $< k$ (as it would also be an $a-b$ separator in G). Hence there exist k independent $a-b'$ paths in G', i.e. have $a-S$ paths $P_1, \dots, P_k$ in G, disjoint except at a. Similarly, have $b-S$ paths $Q_1, \dots, Q_k$ in G, disjoint except at b. Note that no P_i meets a Q_j (else S is not a separator). So can pair up the P_i and Q_j to obtain k independent $a-b$ paths.

Case 2: S a separator, $|S| = k \implies S = \Gamma(a)$ or $S = \Gamma(b)$. Must have $\Gamma(a)$, $\Gamma(b)$ disjoint. Indeed suppose $x \in \Gamma(a) \cap \Gamma(b)$. Then $G - x$ has no $a-b$ separator of size $< k-1$ (as, with x, it forms an $a-b$ separator in G). So by minimality, $G-x$ has $k-1$ independent $a-b$ paths. But now add the path axb to obtain k independent $a-b$ paths in G. Let $ax_1x_2\dots x_rb$ ($r \ge 2$) be a shortest path from a to b. In $G - x_1x_2$, must have a separator of size $k-1$, S say (minimality). Have $S \neq \emptyset$ as $k \ge 2$. So $S \cup \{x_1\}, S \cup \{x_2\}$ are separators in G of size k. Now, $S \cup \{x_2\} \neq \Gamma(a)$, since $x_2 \notin \Gamma(a)$ – so $S \cup \{x_2\} = \Gamma(b)$. Also, $S \cup \{x_1\} \neq \Gamma(b)$, since $x_1 \notin \Gamma(b)$ – so $S \cup \{x_1\} = \Gamma(a)$. Hence $\Gamma(a)$ and $\Gamma(b)$ are not disjoint, contradiction. □

Remark. Can also prove Menger's Theorem with vertex-capacity from max-flow min-cut. Indeed, form a network by directing each edge $xy \in G$ as $\vec{xy}$ and $\vec{yx}$, and give each vertex capacity 1. Then a family of k independent $a-b$ paths is exactly an integer-valued flow of size k from a to b. So done by integrality form of max-flow min-cut (as each vertex cut has capacity $\ge k$).

Theorem 5. Let G be a graph, $|G| > 1$. Then G is k-connected if and only if for all distinct $a, b \in G$, G has k independent $a-b$ paths.

This is "the right way to think about k-connectedness".

Theorem 5 is also often called "Menger's Theorem".

Proof.

$\Leftarrow$ Certainly G connected and $|G| > k$. Also, for $|S| \le k-1$, $G - S$ is connected (else pick a, b in different components of $G - S$, contradiction).

$\Rightarrow$ If a, b non-adjacent then finish by Menger's Theorem.

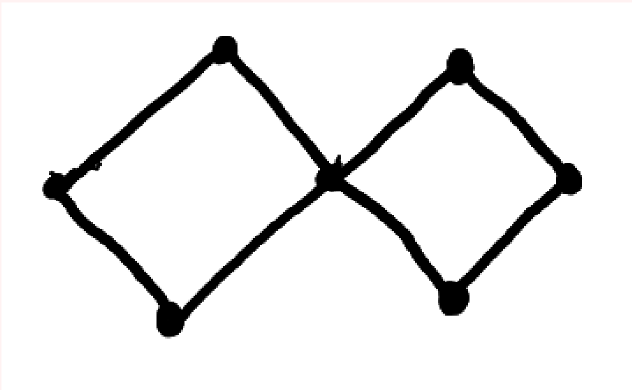
Otherwise, if a, b are adjacent, then have $G - ab$, for which Menger's Theorem gives $k-1$ independent $a-b$ paths (as no $a-b$ separator in $G - ab$ has size $< k-1$). Add in ab to obtain k independent $a-b$ paths in G. □

Definition (edge-connectivity)**.** For G connected, $|G| > 1$, the *edge-connectivity* of G, written $\lambda(G)$, is the least $|W|$, where $W \subset E(G)$ has $G - W$ disconnected.

Say G is *l-edge-connected* if $\lambda(G) \geq l$.

Example. G being 1-edge-connected is equivalent to being connected ($|G| > 1$).

G being 2-edge connected is equivalent to G being connected with no bridge ($|G| > 1$).

Note. 

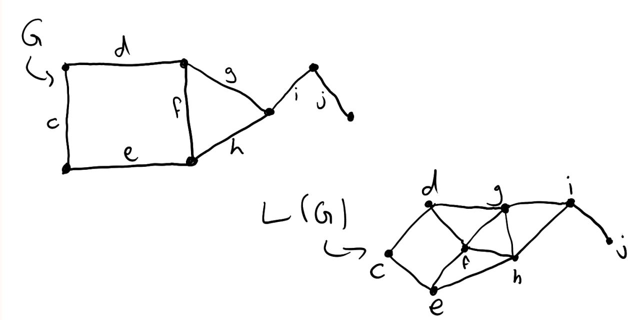
Has $\kappa(G) = 1$, $\lambda(G) = 2$.

Theorem 6 (Edge Menger)**.** Let G be a graph, a, b distinct vertices of G. Suppose that a, b in different components of $G - W \implies |W| \geq k$ (for any $W \subset E$). In other words, every $a - b$ edge-separator has size $\geq k$. Then G has k edge-disjoint $a - b$ paths.

Idea: "Replace edges of G by vertices and apply Theorem 5."

Definition (line graph)**.** For a graph G, the *line graph* $L(G)$ has vertex-set $E(G)$, with e, f joined if they share a vertex.

Example.



Start of

lecture 8

Proof of Edge Menger. Form G' from $L(G)$ by adding two points, a' and b', with a' joined to each $e \in E$ incident with a and similarly for b'. Then $\exists\ a - b$ path in $G \iff \exists\ a' - b'$ path in G', and moreover a family of k independent a'-b' paths in G' gives a family of k edge-disjoint a-b paths in G. So done by usual Vertex Menger. □

Corollary 7. Let G be a graph, $|G| > 1$. Then

$$G\ k\text{-edge-connected} \iff \forall \text{ distinct } a, b \in G,\ \exists\ k \text{ edge-disjoint } a\text{-}b \text{ paths}.$$

Proof.

$\Leftarrow$ Trivial.

$\Rightarrow$ Edge Menger □

Note. Can also prove Edge Menger by max-flow-min-cut (dual edge-capacity form).

Chapter III

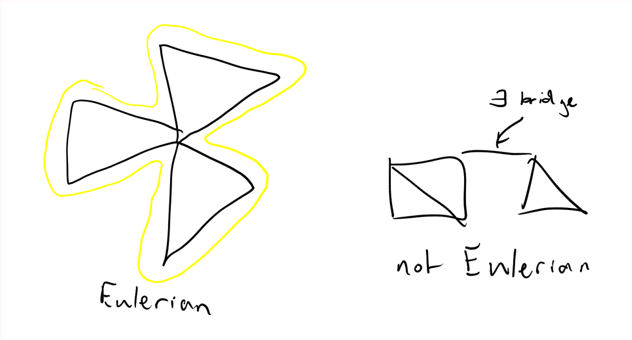
Extremal Problems

Contents

Definition (Euler circuit)**.** An *Euler circuit* in a graph G is a circuit that contains each edge of G exactly once.

Definition (Eulerian graph)**.** Say G is *Eulerian* if it has an Euler circuit.

Example.



Proposition 1. Let G be a connected graph. Then G Eulerian if and only if $d(x)$ is even for all $x \in G$ (so G Eulerian if and only if all degrees even and ≤ 1 component has an edge).

Proof.

$\Rightarrow$ If circuit goes through vertex k times then $d(x) = 2k$.

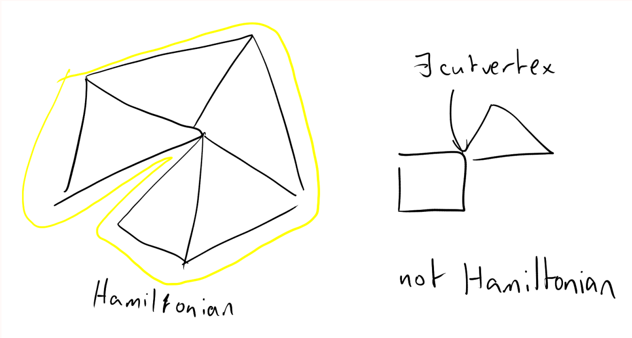
$\Rightarrow$ Induction on $e(G)$. $e(G) = 0$ trivial. Given G with $e(G) > 0$, with G connected, all degrees even: choose a longest circuit C in G not repeating any edge ($e(C) > 0$ because G has a cycle, since G not a tree since all degrees are > 2).

Suppose $E(C) \neq E(G)$. Let H be a component of $G - E(C)$ with $e(H) > 0$. Then for all $x \in H$, $d_H(x)$ even (as $d_G(x)$ even, $d_C(x)$ even). So H has an Euler circuit, C' say (induction). Since $V(H) \cap V(C) \neq \emptyset$ (G connected), can combine C and C' to find a longer circuit than C with no repeated edge. □

Definition (Hamiltonian circuit)**.** In a graph G, a *Hamiltonian cycle* is a cycle that goes through every edge of G.

Definition (Hamiltonian graph)**.** Say G is *Hamiltonian* if it contains a Hamiltonian cycle.

Example.



There is no "nice" if and only if characterisation of Hamiltonicity. No parity kind of condition since G Hamiltonian implies $G + xy$ is as well.

How "large" does a graph have to be to ensure that it is Hamiltonian? Could ask, how many edges do we need to ensure G (on n vertices) is Hamiltonian? This is a silly question, because any x with $d(x) = 1$ stops G from being Hamiltonian, so could for example take K_n with an extra vertex connected to a single vertex in K_n. This has $\binom{n}{2} - (n-2)$ edges, but it not Hamiltonian.

Sensible question: What $\delta(G)$ forces G to be Hamiltonian? If n even, then G being two disjoint $K_{n/2}$ has $\delta(G) = \frac{n}{2} - 1$, not Hamiltonian. If n odd, then G being two $K(n+1)/2$s, meeting at a point, has $\delta(G) = \frac{n-1}{2}$, not Hamiltonian.

Theorem 2. Let G be a graph on n vertices ($n \ge 2$), with $\delta(G) \ge \frac{n}{2}$. Then G is Hamiltonian.

Proof. G connected, since for any non-adjacent x, y must have $\Gamma(x) \cap \Gamma(y) \neq \emptyset$ for size reasons.

Choose a longest path $P = x_1, \dots, x_l$ in G. ($l \ge 3$ since G connected, $|G| > 3$).

Without loss of generality, G has no l-cycle, because if $l = n$ then have n-cycle, and if $l < n$ then there exists $x \notin$ cycle adjacent to cycle (as G connected), giving a path on $l + 1$ vertices.

By maximality of l, we must have $\Gamma(x_1) \subset \{x_1, \dots, x_l\}$, $\Gamma(x_l) \subset \{x_1, \dots, x_l\}$, and $x_1 x_l \notin E$. Also, cannot have i such that $x_1 x_i \in E, x_{i-1} x_i \in E$ (else have an l-cycle). So the sets $\Gamma(x_1)$ and $\Gamma^-(x_l) = \{2 \le i \le l, x_{i-1} \in \Gamma(x_l)\}$ are disjoint. But $\Gamma(x_1), \Gamma^-(x_l) \subset \{x_1, \dots, x_n\}$ and have size $\ge \frac{n}{2}$, contradiction. □

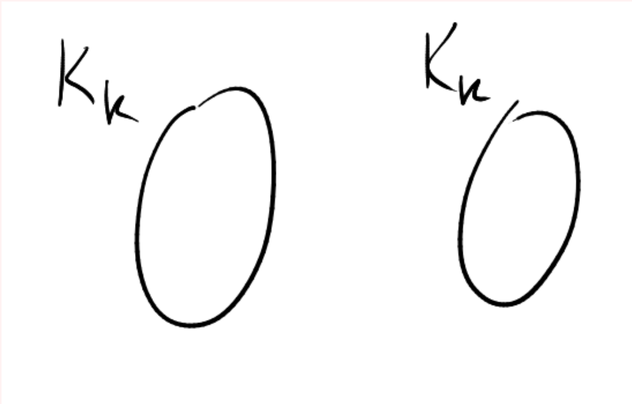
Remark. Actually, only needed $d(x) + d(y) \ge n$ $\forall x, y$ non-adjacent.

Start of lecture 9

Similarly:

Proposition 3. Let G be a connected graph on n vertices ($n \ge 3$). Then $\delta(G) \ge \frac{k}{2}$ (for some n) $\implies$ G has a path of length k.

Remark. Do need G connected – for example:



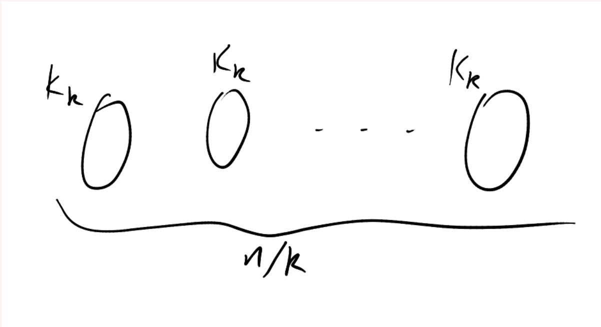
Proof. Let $x_1, \dots, x_l$ be a longest path in G ($l \ge 3$ since G connected, $|G| > 2$). Suppose $l \le k$. Then G has no l-cycle (as before), and $\Gamma(x_1), \Gamma^+(x_l) \subset \{x_2, \dots, x_l\}$ and are disjoint (as before). But $|\Gamma(x_1)|, |\Gamma^+(x_l)| \ge \frac{k}{2}$ and $|\{x_2, \dots, x_l\}| < k$, ※ . □

Theorem 4. Let G a graph on n vertices. Then $e(G) > \frac{n(k-1)}{2} \implies G \supset P_k$.

Note.

(1) Equivalently: if $G \not\supset P_k$ then $e(G) \leq \frac{n(k-1)}{2}$.

(2) Bound is best possible, e.g. if $k \mid n$:



Proof. Induction on n, $n \geq 2$.

Given G on $n \geq 3$ vertices, with $G \not\supset P_k$: want $e(G) \leq \frac{n(k-1)}{2}$. Without loss of generality, G connected: if G disconnected, with components $G_1, \dots, G_r$ on $n_1, \dots, n_r$ vertices, then $e(G_i) \leq \frac{n_i(k-1)}{2}$ (induction), so

$$e(G) \leq \sum \frac{n_i(k-2)}{2} = \frac{n(k-1)}{2}.$$

Hence, by Proposition 3, G has a vertex x of degee $\leq \frac{k-1}{2}$ ($e(G) \leq \frac{n(k-1)}{2}$ trivial if $k > n$, so without loss of generality $k < n$).

Let $G' = G - x$, then $|G'| = n - 1$, $G' \not\supset P_k$, so $e(G') \leq \frac{(n-1)(k-1)}{2}$. So $e(G) = e(G') + d(x) \leq \frac{(n-1)(k-1)}{2} + \frac{k-1}{2} = \frac{n(k-1)}{2}$. □

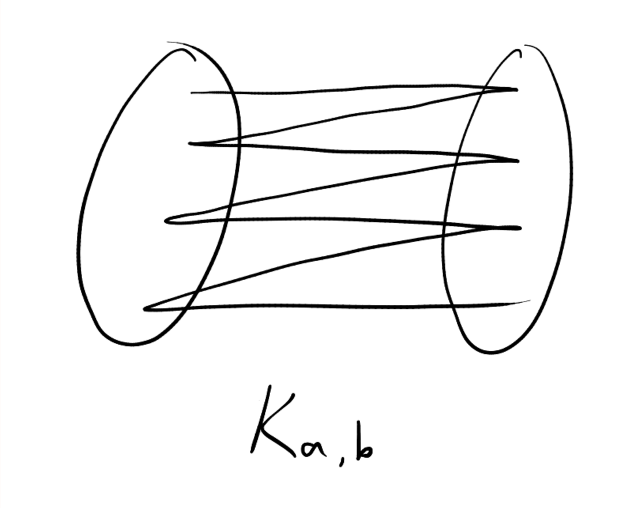
Often we ask: How "large" can a graph be with a certain property? These are called *extremal problems*. Often, this property is non-containment of a particular subgraph.

For example, Theorem 2 is about C_n, and Theorem 4. What about complete graphs?

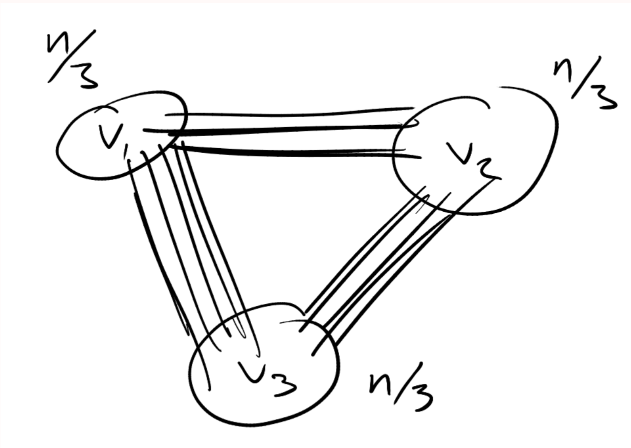
3.1. Turán's theorem

How many edges can a graph (on k vertices) have without containing a K_k?

Example. We'd try G bipartite, indeed complete bipartite, so $G = K_{a,b}$ where $a + b = n$. We'd want $a + b = \frac{n}{2}$ (n even), $a = \frac{n+1}{2}$, $b = \frac{n-1}{2}$ if n odd.



$k = 4$, might try:

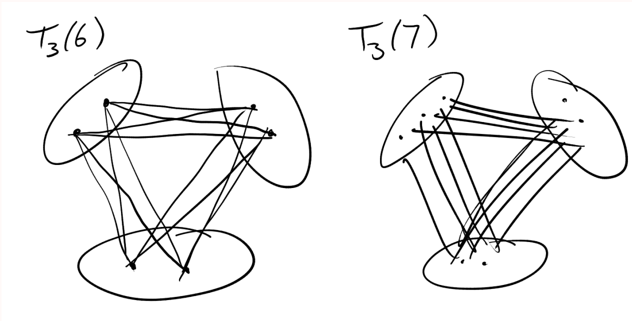


Definition (r-partite)**.** Say graph G is *r-partite* if there exists a partition $V = V_1 \cup \cdots \cup V_r$ of V such that $E(V_i) = \emptyset \ \forall n$.

It is *complete r-partite* if $E = \{xy : x \in V_i, y \in V_j, i \neq j\}$. So G is $k - 1$-partite $\implies$ G has no K_k (else 2 points of K_k in same class, ※).

Definition (Turán graph)**.** The *Turán graph* $T_r(n)$ is the complete r-partite graph on n vertices with classes as equal as possible. ($a_1, \dots, a_r$ are as equal as possible if $|a_i - a_j| \le 1 \ \forall i, j$).

Example.



Certainly $T_{k-1}(n) \not\supset K_k$. Also, $T_{k-1}(n)$ is *maximal* K_k-free: $T_{k-1}(n) + e$ always has a K_k.

If $r \mid n$: all points of $T_r(n)$ have degree $n - \frac{n}{r} = \left(1 - \frac{1}{r}\right) n$.

If not, then all points have degree $n - \left\lfloor \frac{n}{r} \right\rfloor$ or $n - \left\lceil \frac{n}{r} \right\rceil$.

Note.

(1) To obtain $T_r(n-1)$ from $T_r(n)$, remove a point from a largest class (i.e. a point of minimum degree).

(2) To obtain $T_r(n+1)$ from $T_r(n)$, add a point to a smallest class.

Theorem 5 (Turán's theorem)**.** Let G be a graph on n vertices with $e(G) > e(T_{k-1}(n))$. Then $G \supset K_k$.

Remark.

(1) Equivalently, if $G \not\supset K_k$, then $e(G) \le e(T_{k-1}(n))$.

(2) If we know G is $(k-1)$-partite, we'd be done by some form of AM-GM. But no reason why G should be $(k-1)$-partite. For example C_5 does not contain K_3, but is not bipartite.

(3) Looks like proof has to be fiddly and nasty, because $e(Tk-1(n))$ is an unpleasant formula if n is not a multiple of $k-1$.

(4) We'll actually prove a stronger result: $e(G) = e(T_{k-1}(n))$, $G \not\supset K_k \implies G \cong T_{k-1}(n)$. ("$T_{k-1}(n)$ is the unique winner"). This does imply Turán's theorem, as cannot add an edge to $T_{k-1}(n)$.

Proof of Turán's theorem. Induction on n, $n \le k-1$. Given G, $|G| = n$, $e(G) = e(T(n))$, $G \not\supset K_k$, want $G \cong T(n)$.

Claim: $\delta(G) \le \delta(T(n))$. This is because we have $\sum_{x \in G} d_G(x) = \sum_{y \in T(n)} d_{T(n)}(y)$. But the $d_{T(n)}(y)$ are as equal as possible. So $\delta(G) \le \delta(T(n))$.

Pick $x \in G$ of minimal degree, and put $G' = G - x$. $|G'| = n-1$, $G' \not\supset K_k$, and $e(G') = e(G) - \delta(G) \ge e(T(n)) - \delta(T(n)) = e(T(n-1))$. So $G' \cong T(n-1)$ (induction) with $\delta(G) = \delta(T(n))$: say classes $V_1, \dots, V_{k-1}$.

Must have, in G, that $\Gamma(x)$ misses some V_i (if $\Gamma(x)$ meets each V_i then $G \supset K_k$, ※). But $|\Gamma(x)| = n - 1 - \min |V_i|$, as this is $\delta(T(n))$.

Hence $\Gamma(x) = \bigcup_{j \neq i} V_j$ for some i such that $|V_i|$ is of minimum size.

So G is complete $(k-1)$-partite, vertex classes V_j (each $j \neq i$) and $V_i \cup \{x\}$, so $G \cong T(n)$. □

Start of lecture 10

Note that there exist many other proofs of Turán's theorem (which are non-trivially different, and many of which are as beautiful as this one).

3.2. The problem of Zarankiewicz

This is the "bipartite version" of Turán's theorem: How many edges can a bipartite graph (n vertices in each class) have if it does not contain $K_{t,t}$?

Example. $t = 2$, graph has no $K_{2,2} \cong C_4$, so could take $G = C_{24}$.

Write $Z(n,t)$ for the maximum. Does $Z(n,t)$ grow quadratically (t fixed, $n \to \infty$)?

Theorem 6. Let $t \geq 2$. Then $Z(n,t) \leq 2n^{2-\frac{1}{t}}$ for n sufficiently large.

Proof. Let G be bipartite, vertex classes X, Y. ($|X| = |Y| = n$), $G \not\supset K_{t,t}$. Let the degrees in X be $d_1, \dots, d_n$. Without loss of generality, $d_i \geq t-1$ $\forall i$ (if $d_i \leq t-2$ then add edges to that vertex to make $d_i = t-1$).

For given $A \subset Y$, $|A| = t$, how many $x \in X$ have $\Gamma(X) \supset A$? Must be $\leq t-1$, else $G \supset K_{t,t}$.

Thus, the number of (x, A) with $x \in X$, $A \subset Y$, $|A| = t$, $A \subset \Gamma(x)$ is $\leq (t-1)\binom{n}{t}$. But each $x \in X$ belongs to exactly $\binom{d(x)}{t}$ such (x, A). So $\sum \binom{d_i}{t} \leq (t-1)\binom{n}{t}$. Now, the function $\binom{x}{t} = \frac{x(x-1)\cdots(x-t+1)}{t!}$ is convex for $x \geq t-1$. This is because if we let $y = x - t + 1$, then this fraction is $\frac{(y+t-1)\cdots y}{t!}$, a non-negative linear combination of powers of y.

So $\sum \binom{d_i}{t} \geq n\binom{d}{t}$, where d is the average of the d_i (noting $e(G) = nd$). Whence $n\binom{d}{t} \leq (t-1)\binom{n}{t}$. Thus

$$\frac{n(d-t+1)^t}{t!} \leq \frac{(t-1)n^2}{t!},$$

which can be manipulated to give

$$d \leq (t-1)^{\frac{1}{t}} n^{1-\frac{1}{t}} + t - 1$$

so $d \leq 2n^{1-\frac{1}{t}}$ (for n large), as required. □

Does $Z(n,t)$ actually grow at rate $n^{2-\frac{1}{t}}$ (fixed t, $n \to \infty$)?

Example. $t = 2$. Our upper bound is $Z(n,2) \leq cn^{3/2}$. Lower bound? Certainly $Z(n,2) \geq cn^1$ (for example by taking $2n$-cycle). What about $Z(n,2) \geq n^{1.01}$? This is not at all obvious. In fact, we do have $Z(n,2) \geq cn^{3/2}$ (graphs) from algebra-projective planes).

Example. $t = 3$. Our upper bound is $Z(n,3) \leq cn^{5/3}$. In fact, it turns out that $\frac{5}{3}$ is correct (but even harder than the $t = 2$ case).

Example. $t = 4$. Noone knows!

3.3. The Erdős-Stone Theorem (non-examinable)

Note that this subsection is entirely non-examinable.

For a fixed graph H, write $EX(n, H)$ for the maximum value of $e(G)$ where $|G| = n$, $G \not\supset H$.

Example. Turán's theorem says

$$EX(n, K_k) \sim \left(1 - \frac{1}{k-1}\right)\binom{n}{2}$$

or more precisely,

$$\frac{EX(n, K_k)}{\binom{n}{2}} \to 1 - \frac{1}{k-1}$$

as $n \to \infty$. We call the left hand side of the above limit the *density* of G.

Theorem 2 says that

$$EX(n, P_k) \sim \frac{n(k-1)}{2},$$

so

$$\frac{EX(n, P_k)}{\binom{n}{2}} \to 0$$

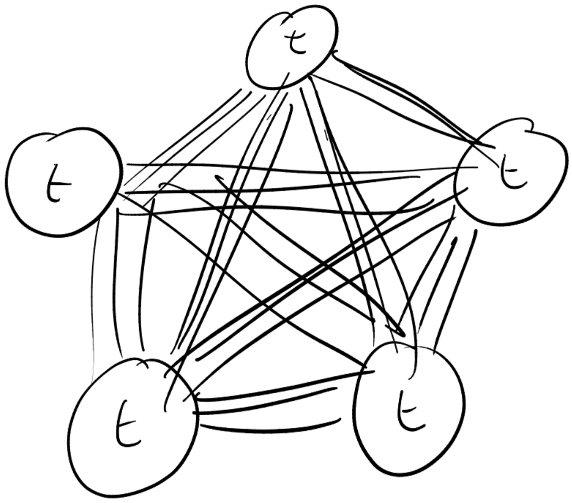
as $n \to \infty$.

General question: How does $EX(n, H)$ behave as $n \to \infty$?

For given H, let r be the least integer such that H is r-partite. For example H bipartite has $r = 2$. C_7 has $r = 3$ (it is 3-partite but not bipartite).

Start of lecture 11

Write $K_r(t) = K_r(rt)$ (K_r but with each point 'blown up' to a set of size t).



Turán's theorem says

$$\frac{e(G)}{\binom{n}{2}} > 1 - \frac{1}{r-1} \implies G \supset K_r$$

What if $\frac{e(G)}{\binom{n}{2}} > 1 - \frac{1}{r-1} + 0.01$? Remarkably, this implies that $G \supset K_r(1000)$ (n large).

Theorem (Erdős-Stone Theorem)**.** For all integers r, t, and $\varepsilon > 0$, there exists $n_0(r, \varepsilon, t)$ such that for all $n \geq n_0$,

$$|G| = n, \frac{e(G)}{\binom{n}{2}} > 1 - \frac{1}{r-1} + \varepsilon \implies G \supset K_r(t).$$

Sketch proof. Have G, $|G| = n$, average degree $> \left(1 - \frac{1}{r-1} + \varepsilon\right) n$.

(1) Pass to a subgraph H on n' vertices (n' still large), with $\delta(H) > \left(1 - \frac{1}{r-1} + \varepsilon\right) n'$ (similar to Example Sheet 1, Question 7).

(2) H contains a $K_{r-1}(t')$ (t' large), $= K$ say (induction on r).

(3) Each $x \in H - K$ has $\geq t$ neighbours in each class of K (by $\delta(H)$).

(4) Get t points $x_1, \dots, x_t \in H - K$, each joined to same t-set in each class of K (by pigeonhole). So have $K_r(t)$. □

Given H, let r be the least such that H is r-partite. Then $H \not\supset T_{r-1}(n)$ (as $T_{r-1}(n)$ is

$(r-1)$-partite), so

$$\frac{EX(n,H)}{\binom{n}{2}} \ge 1 - \frac{1}{r-1}.$$

However, $H \subset K_r(t)$, some t (as H is r-partite), so Erdős-Stone Theorem tells us that

$$\frac{e(G)}{\binom{n}{2}} > 1 - \frac{1}{r-1} + \varepsilon \implies G > H.$$

Conclusion:

$$\frac{EX(n,H)}{\binom{n}{2}} \to 1 - \frac{1}{r-1}$$

as $n \to \infty$.

Remark. If H bipartite, this says that $\frac{EX(n,H)}{\binom{n}{2}} \to 0$. How fast does $EX(n,H)$ grow? This is unknown even for many very simple H, for example C_{2k}, $k \ge 6$.

This marks the end of this subsection of non-examinable content.

Chapter IV

Colourings

Contents

Definition (r-colouring)**.** An r-*clouring* of a graph G is a function $c : V(G) \to [r] :== \{1, \dots, r\}$ such that $xy \in E(G) \implies c(x) \neq c(y)$.

The *chromatic number* $\chi(G)$ of G is the least r for which G has an r-colouring.

Example.

(1) $\chi(P_n) = 2$.

(2) $\chi(C_n) = \begin{cases} 2 & n \text{ even} \\ 3 & n \text{ odd} \end{cases}$.

(3) $\chi(K_n) = n$ (each point gets a different colour).

(4) T a tree $\implies \chi(T) = 2$ (for example remove a leaf + induction).

(5) $\chi(K_{m,n}) = 2$.

Note. G bipartite $\implies$ G is 2-colouring. Conversely, G 2-colourable $\implies$ G bipartite (as can take $X = \{x : c(x) = 1\}$, $Y = \{x : c(x) = 2\}$). So, G bipartite $\iff \chi(G) \leq 2$.

In general, G r-partite $\iff$ $\chi(G) \leq r$. So $\chi(G)$ is the least r such that G is r-partite. Thus Erdős-Stone Corollary says

$$\frac{EX(n, H)}{\binom{n}{2}} \to 1 - \frac{1}{\chi(H) - 1}$$

For any G on n vertices, have $\chi(G) \leq n$. But can often improve this:

Proposition 1. For any graph G, have $\chi(G) \leq \Delta(G) + 1$.

Proof. Order $V(G)$ as $x_1, \dots, x_n$, and colour each x_i in turn. When we come to colour x_i, it has $\leq \Delta(G)$ neighbours, so we have $\leq \Delta(G)$ colours to avoid. □

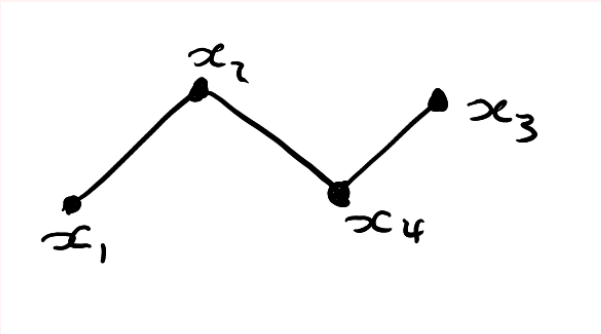
Remark.

(1) Can have equality, for example complete graph or odd cycle.

(2) Often $\chi(G)$ is much less than $\Delta(G)$, for example $K_{1,n-1}$.

(3) Could view Proposition 1 as an application of the *greedy algorithm* on ordering $x_1, \ldots, x_n$: when we came to colour vertex x_i, we use least colour available.

(4) Greedy might use $> \chi(G)$ colours, for example



(5) No 'formula' for $\chi(G)$.

4. Colouring Planar Graphs

Proposition 2 (6-colour Theorem)**.** Let G be planar. Then $\chi(G) \leq 6$.

Proof. Clearly this works if $|G| \leq 6$. Hence consider $|G| = n \geq 7$.

We claim $\delta(G) \leq 5$. Indeed, we have $e(G) \leq 3n - 6$, so $\sum_{x \in G} d(x) \leq 6n - 12$, so there is an x with $d(x) \leq 5$.

Choose $x \in G$ with $d(x) \leq 5$, and let $G' = G - x$. We can 6-colour G' by induction on n. Now, $|\Gamma(x)| \leq 5$, so $\Gamma(x)$ has at most 5 colours. Hence we can colour x with a 6-th colour. □

How about 5 colours? If G planar $\implies$ $\delta(G) \leq 4$, then the same proof would work. However this is not true, for example the icosahedron has $\delta(G) = 5$ (or a football).

Start of lecture 12

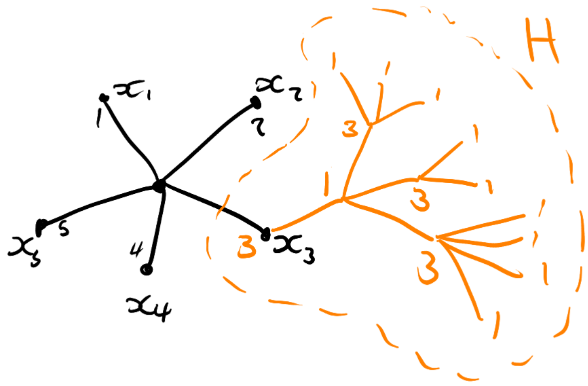
Theorem 3 (5-colour Theorem)**.** G planar $\implies$ G 5-colourable.

Proof. Induction on $|G|$. $|G| \leq 5$ trivial.

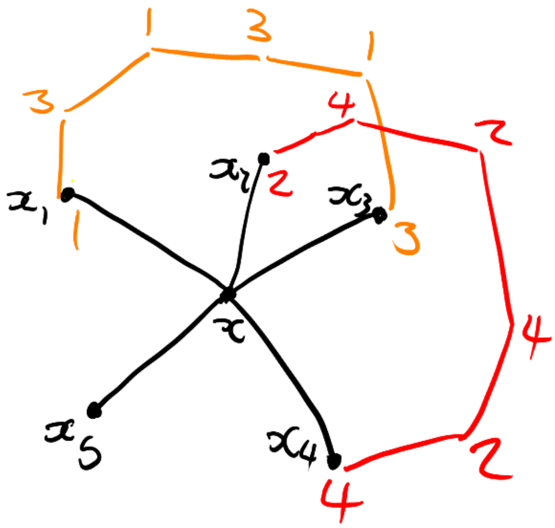
Given planar G, $|G| \geq 6$: have $x \in G$ with $d(x) \leq 5$ (as before). So remove x. By induction, we can 5-colour $G - x$. Can now colour x, unless $d(x) = 5$ and all colours in appear in $\Gamma(x)$. Say $\Gamma(x) = \{x_1, x_2, \ldots, x_3\}$ (clockwise) with x_i of colour i for all i.

Question: Is there a $1-3$ path from x_3 to x_1 (a $1-3$ path is a path that alternates between colours 1 and 3)?

If no, let H be the $1-3$ component of x_3 (all the vertices that are part of a $1-3$ path starting at x_3). Then $x_1 \notin H$. So flip colours 1 and 3 on H. Still a legal colouring of $G - x$, but now can use colour 3 for x.



If yes, then there is no $2-4$ path from x_2 to x_4 (else it would meet the $1-3$ path from x_1 to x_3 as our drawing is planar).



(even if our paths go the other way round, we can see that in any of the 4 cases, they must intersect). So done as before: swap 2 and 4 on the $2-4$ component of x_2, leaving 2 as a colour for x. □

Remark.

(1) These $i-j$ paths are called 'Kempe chains'.

(2) What if we wanted to colour the faces of G (so that faces sharing an edge get different colours)? Called 'colouring a plane map'.

Definition (Dual graph)**.** For a plane graph G, the *dual graph* G' has vertices being the faces of G, with two joined if they share an edge. Clearly G' planar.

Then a colouring of the faces of G is precisely a colouring of G'. So Theorem 3 tells us that every plane map is 5-colourable.

(3) Ofcourse, can have planar G with $\chi(G) = 4$. For example, K_4.

The following theorem is non-examinable.

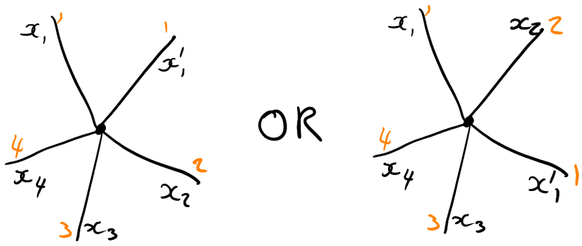
Theorem (4-colour Theorem)**.** G planar $\implies$ G 4-colourable.

"Proof". Induction on $|G|$: $|G| \le 4$ trivial.

Given planar G, $|G| \ge 5$: have $x \in G$ with $d(x) \le 5$ (as before). So remove x: can 4-colour $G - x$ (induction). So done unless all 4 colours appear in $\Gamma(x)$.

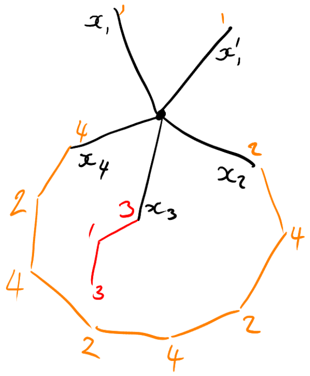
If $d(x) = 4$: Have say $\Gamma(x) = \{x_1, \dots, x_4\}$ (clockwise), with $c(x_i) = i$ for all i. If no $1-3$ path from x_1 to x_3, swap 1 and 3 on the $1-3$ component of x_3 (leaving colour 3 for x). If there exists $1-3$ path from x_1 to x_3, then no $2-4$ path from x_4 to x_2 (as G planar), so done as before.

If $d(x) = 5$, could have:

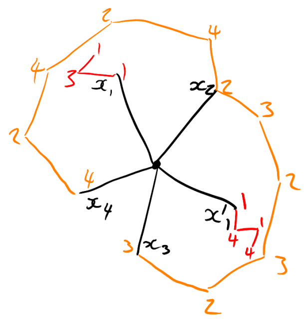


First case: if no $2-4$ path from x_2 to x_4, then done as usual. If there exists a $2-4$

path from x_2 to x_4, then no $1-3$ path from x_3 to x_1 or x_1'. So swap 1 and 3 on $1-3$ component of x_3, leaving colour 3 available for x.



Second case: without loss of generality $\exists\ 2-3$ path from x_2 to x_3 (else done), so no $1-4$ path from x_1' to x_4. Also without loss of generality $\exists\ 2-4$ path from x_2 to x_4 (else done), so there is no $1-3$ path from x_1 to x_3.



So done: swap 1 and 4 on the $1-4$ component of x_1' and 1 and 3 on the $1-3$ component of x_1. Then we can use colour 1 for x. □

This "Proof" was given by Kempe in 1879. In 1890, he found a mistake in the proof. Where is the mistake?

The 4-colour Theorem was proved in 1976 by Appel and Haken.

In the proof of Theorem 3, we showed that "a vertex of degree 5 or 4 or ...or 0". Formed an unavoidable set of reducible configurations.

Appel and Haken found such a set of about 1900 configurations (computers used).

The following content is now examinable again.

We know any graph G has $\chi(G) \le \Delta(G) + 1$. Can have equality, for example K_n, or C_{odd}.

Aim: $\chi(G) \le \Delta(G)$ unless $G = K_n$ or C_{odd} (for G connected).

Remark. If (connected) G not regular, then can always colour in $\Delta(G)$ colours. Indeed, choose $x_n \in G$ with $d(x) \le \Delta(G) - 1$. Then choose x_{n-1} with $x_n \in \Gamma(x_{n-1})$ (G connected), choose x_{n-2} with $\Gamma(x_{n-2})$ meeting $\{x_{n-1}, x_n\}$ (G connected)...Keep going. We obtain $x_n, \dots, x_1$ such that every x_i ($i \le n$) has a 'forward edge', i.e. there exists $j > i$ with $x_i x_j \in E$. Now just run greedy on $x_1, \dots, x_n$. At x_i we have $\le \Delta(G) - 1$ neighbours coloured (all i), so greedy uses $\le \Delta(G)$ colours.

Start of lecture 13

Theorem 4 (Brooks' Theorem)**.** Let G be a connected graph, not a complete graph or an odd length cycle. Then $\chi(G) \le \Delta(G)$.

Proof. Without loss of generality, assume G is Δ-regular (by remark above), and $\Delta(G) \ge 3$ ($\Delta(G) = 1$ is trivial, and $\Delta(G) = 2$ implies G is a cycle, so trivial).

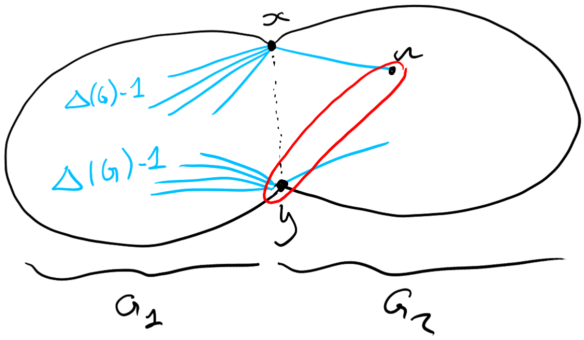
If not true, let G be a smallest counterexample (e.g. $|G|$ minimal). Without loss of generality, G has no cutvertex. Indeed, suppose x is a cutvertex. Let components of $G - x$, together with x, be $G_1, \dots, G_k$. Then each G_i is $\Delta(G)$-colourable (since each G_i is not $\Delta(G)$-regular, as $d_{G_i}(x) \le \Delta(G) - 1 \ \forall n$) Hence G is $\Delta(G)$-colour.

Case 1: G is 3-connected: Want an ordering of the vertices as $x_1, \dots, x_n$ such that for all $i < n$ there exists $j > i$ with $x_i x_j \in E$ and two members of $\Gamma(x_n)$ get the same colour – then done (as before, by running greedy). Pick any vertex x_n. Cannot have $\Gamma(x_n)$ a complete graph, elese $\Gamma(x_n) \cup \{x_n\}$ is a $K_{\Delta(G)+1}$, a contradiction as G is not a complete graph (and is connected), so some vertex in this $K_{\Delta(G)+1}$ is joined to some point outside it ※.

Choose $x_1, x_2 \in \Gamma(x_n)$ with $x_1 x_2 \notin E$. Then $G - \{x_1, x_2\}$ connected (as G is 3-connected), so can order it as $x_3, \dots, x_n$ such that each x_i has a forward edge (as before). Now run greedy algorithm on $x_1, \dots, x_n$. This completes the first case.

Case 2: G not connected: Let $\{x, y\}$ be a separator of size 2 (i.e. $G - \{x, y\}$ disconnected). Let $G_1, \dots, G_k$ be components of $G - \{x, y\}$ with x, y added. Then each G_i is $\Delta(G)$-colour (since G_i not $\Delta(G)$-regular, as $d_{G_i}(x), d_{G_i}(y) \le \Delta(G) - 1$ for all i). If

$xy \in E$, then each of these colourings give x, y distinct colours, so can recolour them and combine to obtain a $\Delta(G)$-colouring of G. So may assume $xy \notin E$. Now, if each G_i has $d_{G_k}(x) \leq \Delta(G)-2$ or $d_{G_i}(y) \leq \Delta(G)-2$. Then by recolouring we may assume that the $\Delta(G)$-colouring of G_i gives x, y distinct colours – so done as above. Hence some G_i has $d_{G_i}(x) = d_{G_i}(y) = \Delta(G) - 1$ – say $i = 1$. So $k = 2$, and $d_{G_2}(x) = d_{G_2}(y) = 1$. Let $\Gamma_{G_2}(x) = \{u\}$. Then $\{y, u\}$ is a separator, not of this form.



□

5. Chromatic Polynomials

Definition (Chromatic polynomial)**.** For a graph G, and for $t = 1, 2, 3, \dots$, write $P_G(t)$ for the number of t-colourings of G. (So $\chi(G)$ is the least t such that $P_G(t) \neq 0$). P_G is *chromatic polynomial* of G.

Why is it a polynomial?

Example.

(1) $P_{K_n}(t) = t(t-1)(t-2)\cdots(t-n+1)$.

(2) $P_{E_n}(t) = t^n$.

(3) $P_{P_n}(t) = t(t-1)^n$. Similarly, for T a tree on n vertices, $P_T(t) = t(t-1)^{n-1}$ (remove a leaf and induct).

(4) $P_{C_n}(t) =?$. For the first vertex, we have t choices, then $t-1$ for the next, then $t-1$ for the next and so on, ...until we get to the final vertex. Then the number of choices depends on whether the two neighbours are coloured the same colour. It turns out that it is a bit complicated to work out the chromatic polynomial in this case.

Definition (Contraction)**.** For a graph G with edge $e = xy$, the *contraction* G/e is obtained by replacing the vertices x and y by a single vertex e, joined to each neighbour of x or y.

Lemma 5 (deletion-contraction)**.** Let G be a graph, e an edge. Then

$$P_G = P_{G-e} - P_{G/e}.$$

(This relation is sometimes called *cut-fuse* relation).

Proof. Colourings of $G - e$ with endpoints of e different colours corresponds exactly to colourings of G. Also, colourings of $G-e$ with endpoints of e the same colour correspond to colourings of G/e. Hence $\forall t$,

$$P_{G-e}(t) = P_G(t) + P_{G/e}(t)$$ □

Note. We cannot use deletion-contraction (together with $P_{E_n}(t) = t^n$) to define P_G, as it might not be well-defined.

Proposition 6. Let G be a graph on n vertices, with m edges. Then P_G is a polynomial in t, of degree n, leading terms $t^n - mt^{n-1} + \cdots$.

Proof. Induction on $e(G)$. $e(G) = 0$ trivial ($P_{E_n}(t) = t^n$).

Given G, $e(G) > 0$, pick $e \in E$. Have for all t, (deletion-contraction)

$$P_G(t) = P_{G-e}(t) - P_{G/e}(t).$$

By induction,

$$\begin{aligned} P_{G-e}(t) &= t^n - (m-1)t^{n-1} + \cdots \\ P_{G/e} &= t^{n-1} - \cdots \end{aligned}$$

so

$$P_G(t) = t^n - mt^{n-1} + \cdots$$

□

Start of lecture 14

Remark.

(1) $P_G(t)$ does carry other information about G. For example, it turns out that

$$P_G(t) = t^n - mt^{n-1} + \left(\binom{n}{2} - \#\text{ triangles in } G\right) t^{n-2} - \cdots$$

(2) As P_G is a polynomial, we can talk about $P_G(t)$ for any real (or complex) t.

(3) The 4-colour Theorem says: G planar $\implies$ P_G does not have a root at 4. No such 'polynomial' proof of the 4-colour Theorem is known. It *is* known that $P_G(2+\phi) \neq 0$, where $\phi = \frac{1+\sqrt{2}}{2}$ ($2+\phi$ is approximately 3.6).

5.1. Edge Colourings

Definition (Edge colouring)**.** A *k-edge-colouring* of a graph G is a function $c : E(G) \to [k]$ such that $c(e) \neq c(e')$ whenever $e \neq e'$ share a vertex.

Definition (Edge-chromatic number)**.** The *edge-chromatic number* of *chromatic index* of G, written $\chi'(G)$, is the least k such that G has a k-edge-colouring.

Example.

$$\chi'(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}$$

Note that $\chi'(G) = \chi(L(G))$ (but sadly this is not much help).

Can have χ', χ far apart, for example $K_{1,n}$ has $\chi(K_{1,n}) = 2$, $\chi'(K_{1,n}) = n$.

Always have $\chi'(G) \ge \Delta(G)$ (and not always equal, for example C_{odd}). Also, $\chi'(G) \le 2\Delta(G) - 1$ (run greedy on any ordering).

Calculating $\chi'(K_n)$ is an exercise on Example Sheet 3. This graph is interesting because it is not entirely obvious how to calculate the edge-chromatic number, despite the fact that the graph has such a nice structure.

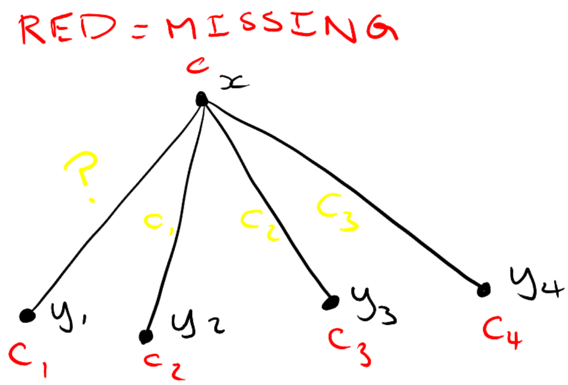
Remarkably:

Theorem 7 (Vizing's Theorem)**.** For any graph G, $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G)$.

Proof. We'll show that every graph has a $(\Delta + 1)$-edge-colouring.

Induction on $e(G)$, $e(G) = 0$ trivial.

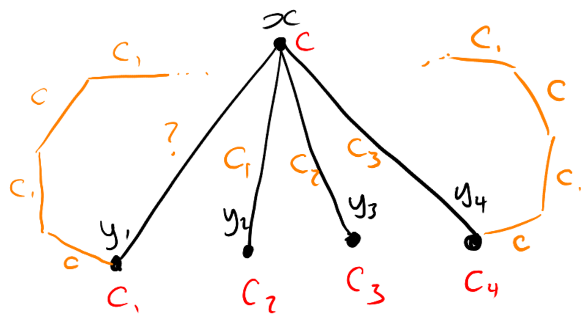
Given G, $e(G) > 0$: choose any $e \in E$, and have $(\Delta + 1)$-edge-colouring of $G - e$ (induction), say $e = xy$. Note that every vertex has an unused colour (as number of colours is greater than max degree): say colour c at x and c_1 at y. Without loss of generality $c \ne c$, else done (use colour c for x).



Choose a maximal sequence $y_1, \ldots, y_k$ of distinct neighbours of x such that some colour c_i is missing at y_i and xy_i has colour c_{i-1} (each $2 \leq i \leq k$). (Exists since the graph is finite). Must have stopped at y_k because either c_k does not occur at x or $c_k = c_j$ for some $j < k$.

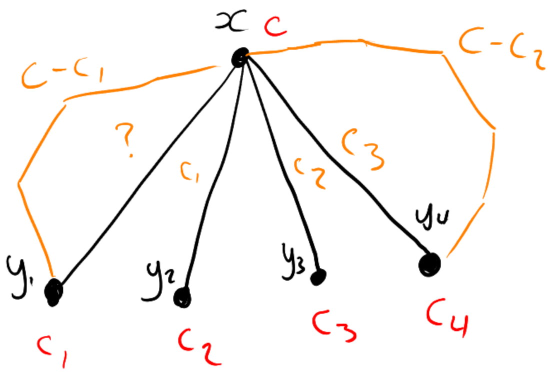
If c_k does not occur at x, can give xy_i colour c_i (all $1 \leq i \leq k$), giving a legal colouring of G.

If $c_k = c_j$ for some $j < k$ then without loss of generality $j = 1$, by giving edge xy_i (each $i < j$) colour c_i – leaving xy_j as the uncoloured edge.



If no $c - c_1$ path from y_1 to x: swap colours c and c_1 on the $c - c_1$ component of y_1. Now c is missing at y_1 (by our swap) and c is still missing at x_1, so can colour xy_1 with colour c. So may assume there exists $c - c_1$ path from y_1 to x.

If no $c - c_1$ path from y_k to x: swap colours c and c_1 on the $c - c_1$ component of y_k. Now c is missing at y_k and at x. So done by giving xy_i colour c_i $(1 \leq i \leq k-1)$ and give xy_k colour c. So may assume that there also exists $c - c_1$ path from y_k to x.

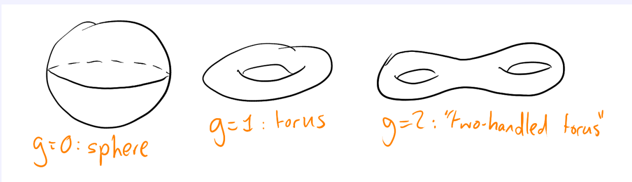


Let H be the $c - c_1$ component of x. Then $\Delta(H) \leq 2$, and H connected, so H is a path or a cycle. But $d_H(x) = d_H(y_1) = d_H(y_k) = 1$, contradiction. □

5.2. Graphs on Surfaces

We know that $\chi(G) \leq 5$ (actually ≤ 4) for any graph drawn in the plane, or equivalently on a sphere. What happens on other surfaces?

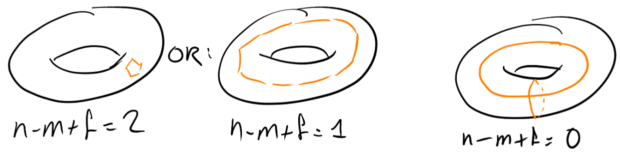
Definition (Surface of genus g)**.** For $g = 0, 1, 2, \ldots$ the *surface of genus* g (or the *compact orientable surface of genus* g) consists of the sphere, with g handles attached:



Known that for planar G:

$$n - m + f = 2 \qquad (G \text{ connected})$$
$$n - m + f \geq 2 \qquad (\text{general } G \text{ – add edges to make } G \text{ connected})$$

What about on a torus?



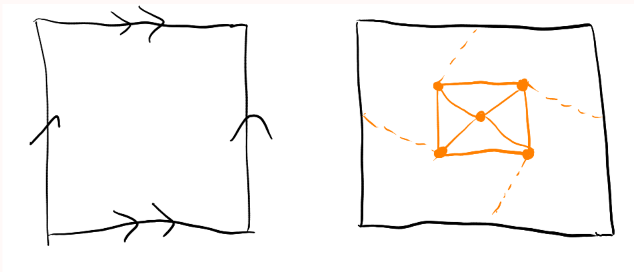
Fact: for any graph on surface of genus g, $n - m + f \geq 2 - 2g$. We call $2 - 2g$ the *euler characteristic* of the surface, written E.

Start of lecture 15

For G drawn on the surface of Euler characteristic E, have $n - m + f \geq E$.

We know $3f \leq 2m$ (as usual by counting edges via faces / vertices), for $m \geq 3$. So $n - m + \frac{2m}{3} \geq E$, i.e. $n - \frac{m}{3} \geq E$, so $m \leq (n - E)$.

Example. We can draw K_5 on a torus:



Exercise: draw K_6 and K_7 on a torus.

What can we say about $\chi(G)$?

Theorem 8 (Heawood's Theorem)**.** Let G be a graph drawn on a surface of Euler characteristic $E \leq 0$. Then

$$\chi(G) \leq H(E) \leq \left\lfloor \frac{7 + \sqrt{49 - 24E}}{2} \right\rfloor$$

Remark.

(1) $H(0) = 7$. So our K_7 cannot be improved, and nor can the bound $\chi(G) \leq 7$.

(2) Amusingly, $H(2) = 4$.

Proof. Let G have $\chi(G) = k$. Need to show $k \leq H(E)$. Pick G minimal (least n) with $\chi(G) = k$. So certainly $n \geq k$. Also, $\delta(G) \geq k - 1$ (by minimality of G). Now, from $m \leq 3(n - E)$ we have $2m \leq 6(n - E)$, so average degree $\leq 6 - \frac{E}{2}$, and in particular $\delta(G) \leq 6 - \frac{6E}{2}$. Hence $k - 1 \leq 6 - \frac{E}{2} \leq 6 - \frac{6E}{k}$ (as $n \geq k$ and $E \leq 0$). So $k^2 - k \leq 6k - 6E$, i.e. $k^2 - 7k + 6E \leq 0$ – whence bound (solve quadratic). $\square$

Remark. Heawood's Theorem is best possible – can draw $K_{H(E)}$ on surface (this is the "Map Color Theorem", 1960s).

Chapter V

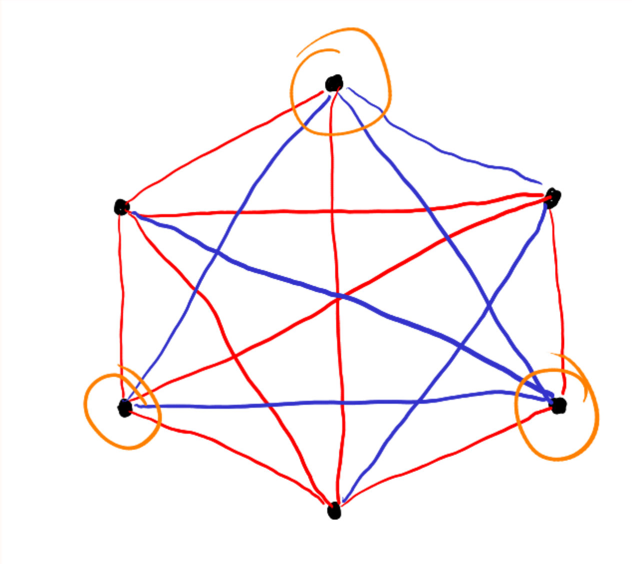
Ramsey Theory

Contents

The philosophical question guiding this chapter is:

"Can we find some order in enough disorder?"

Example. Suppose we have $c : E(K_6) \to \{1, 2\}$ (2-coloured K_6). Can we find a triangle, all edges red or all edges blue (a *monochromatic* triangle)?



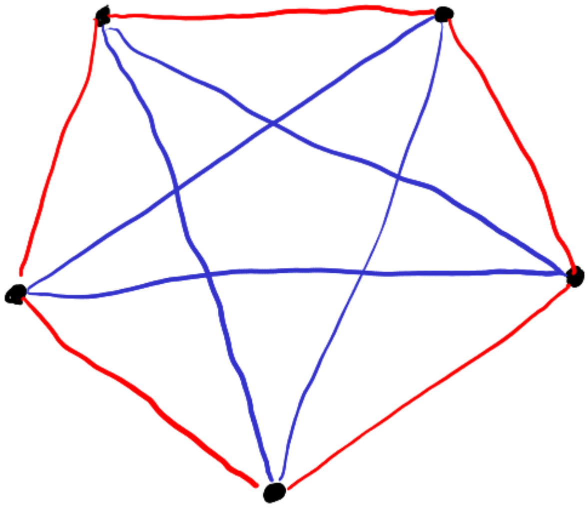
Answer: yes. Pick vertex x. Have $d(x) = 5$, so there exists ≥ 3 edges out of x of the same colour: say xy_1, xy_2, xy_3 red. If any y_iy_j red then we have found a red triangle (x, y_i, y_j). If all y_iy_j blue then we have a blue triangle (y_1, y_2, y_3).

How about a monochromatic K_4?

Definition. Write $R(s)$ for the least n (if it exists) such that whenever K_n is 2-coloured, there exists a monochromatic K_s.

(Equivalently, least n such that every graph G on n vertices has $K_s \subset G$ or $K_s \subset \overline{G}$).

So the above proof shows $R(3) \leq 6$. In fact, $R(3) = 6$, since there is a colouring of K_5 which has no monochromatic triangle:



What about $R(4)$? We'll go for a 'halfway house' of finding a red K_3 or a blue K_4.

Definition. For $s, t \geq 2$ write $R(s,t)$ for for the least n (if it exists) such that whenever K_n is 2-coloured, there exists red K_s or blue K_t.

Example.

(1) $R(s,s) = R(d)$.

(2) $R(s,t) = R(t,s)$.

(3) $R(s,2) = s$.

Theorem 1 (Ramsey's Theorem)**.** $R(s,t)$ exists $\forall s,t$. Moreover, $R(s,t) \leq R(s-1,t) + R(s,t-1)$ $\forall s,t \geq 3$.

Proof. Enough to show that, if $R(s-1,t)$ and $R(s,t-1)$ are finite, then

$$R(s,t) \leq R(s-1,t) + R(s,t-1)$$

As then $R(s,t)$ finite for all s,t (induction on $s+t$)!

Let $a = R(s-1,t)$, $b = R(s,t-1)$. Given a 2-colouring of K_{a+b}: pick a vertex x. Have $d(x) = a+b-1$. So there exists a red edges or there exists b blue edges from x.

If a red edges: say $xy_1, \dots, xy_a$ are all red. In the K_a given by vertices $y_1, \dots, y_a$. Have red K_{s-1} or blue K_t. If we have a blue K_t, then done. If we have a red K_{s-1}, then when combined with x, we have a K_s.

If b blue edges, similar. □

Remark. Very few of these 'Ramsey numbers' $R(s,t)$ are known exactly (see later). Question: how fast does $R(s)$ grow?

Start of lecture 16

Corollary 2. $R(s,t) \le \binom{s+t-2}{s-1}$ for all $s,t \ge 2$. In particular, $R(s) \le 2^{2s}$.

Proof. Induction on $s+t$. True if $s=2$ or $t=2$. Given $s,t \ge 3$:

$$R(s,t) \le R(s-1,t) + R(s,t-1) \le \binom{s+t-3}{s-1} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}.$$ □

What about more colours?

Definition. For $k \ge 1$ and $s_1, \dots, s_k \ge 2$, write $R_k(s_1, \dots, s_k)$ for the least n (if it exists) such that whenever K_n is k-coloured, there exists a monochromatic K_{s_i} for some $1 \le i \le k$.

Theorem 3 (Ramsey for k colours)**.** $R_k(s_1, \dots, s_k)$ exists for all k and $s_1, \dots, s_k$.

Proof. "Turquoise spectacles":

Induction on k. $k=1$ is trivial (or alternatively can start with $k=2$ by using Theorem 1). Given $k>1$, we'll show that $R_k(s_1, \dots, s_k) \le R(s_1, R_{k-1}(s_1, R_{k-1}(s_2, \dots, s_k))$. Indeed, given a k-colouring of K_n, where $n = R(s_1, R_{k-1}(s_2, \dots, s_k))$: view the colours as '1' and '2 or 3 or ...or k'. By definition of n, we obtain either a K_{s_1} coloured 1 (done) or a $K_{R_{k-1}(s_2,\dots,s_k)}$ coloured with $k-1$ colours (so done by definition of $R_{k-1}(s_2, \dots, s_k)$). □

Remark. Or could redo the proof of Theorem 1.

What about r-sets? Suppose we colour each *triangle* red or blue – do we get a 4-set all of whose triangles are the same colour?

This is asking (in general) for a much *denser* monochromatic structure. For a set X and $r = 1, 2, 3, \dots$, write $X^{(r)} = \{A \subset X : |A| = r\}$ – the collection of all r-sets in X.

Unless otherwise stated, $X = [n] = \{1, \dots, n\}$.

Notation. Write $R^{(r)}(s,t)$ for the least n (if it exists) such that whenever $X^{(r)}$ is 2-coloured ($c : X^{(r)} \to \{1,2\}$), then there exists a red s-set (an s-set, all of whose r-sets are colour 1) or a blue t-set (for each $r \geq 1$ and $s, t \geq r$).

Remark.

(1) $R^{(2)}(s,t) = R(s,t)$

(2) $R^{(1)}(s,t) = s + t - 1$ (pigeonhole)

(3) $R^r(s,t) = R^r(t,s)$

(4) $R^r(s,r) = s$

Theorem 4 (Ramsey for r-sets)**.** $R^r(s,t)$ exists for all $r \geq 1$ and $s, t \geq r$.

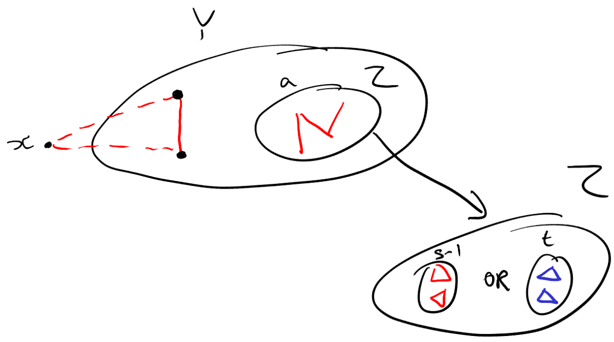
Idea: in proof of $r = 2$ (Ramsey's Theorem), we used $r = 1$ (i.e. pigeonhole).

Proof. Induction on r: $r = 1$ trivial (alternatively start at $r = 2$ using Theorem 1). Given $r > 1$. Induct on $s + t$ ($s = r$ or $t = r$ is straightforward).

So, given $r > 1$ and $s, t > 1$, we'll show

$$R^{(r)}(s,t) \leq R^{(r-1)}(R^{(r)}(s-1,t), R^{(r)}(s,t-1)) + 1.$$

Let $a = R^{(r)}(s-1,t)$, $b = R^{(r)}(s,t-1)$, $n = R^{(r-1)}(a,b) + 1$. Given a 2-colouring c of $X^{(r)} = [n]^{(r)}$:



Pick $x \in X$, and put $Y = X \setminus \{x\}$. Then have induced colouring c' of $Y^{(r)}$: $c'(A) = c(A \cup \{x\})$, for each $A \in Y^{(r-1)}$. By definition of n, we have a red a-set or blue b-set for c'.

If a-set: have $Z \subset Y$, $|Z| = a$ such that $\forall A \in Z^{(r-1)}$ have $c(A \cup \{x\})$ is red. Inside Z, by definition of a, there exists a red $s-1$ set for c or a blue $t-1$ set for c. But a blue t-set for c is done, and a red $(s-1)$-set for c forms, with x, a red s-set for c.

If blue b-set: same argument but swapping colours. □

Remark. ALso works for k colours (e.g. by turquoise spectacles).

What bounds do we get on $R^{(r)}(s,t)$?

Define functions $f_1, f_2, \dots$ as follows:

- $f_1(x) = 2x$
- $f_r(x) = \underbrace{f_{r-1}(f_{r-1}(\dots f_{r-1}}_{x \text{ times}}(1)\dots))$ for each $r > 1$.

So $f_2(x) = 2^x$, $f_3(x) = \overbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}^{x}$. What about f_4?

$$f_4(1), \quad f_4(2) = 2^2 = 4, \quad f_4(3) = 2^{2^{2^2}} = 65536, \quad f_4(4) = \overbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}^{65536}$$

Our bound for $R(s,t)$ is of the form $f_2(s+t)$ and for $R^{(r)}(s+t)$ of form $f_r(s+t)$. These very large upper bounds are often a feature of such 'double inductions'. (Lower bounds, e.g. on $R(s,t)$? See later.)

Start of

lecture 17

5.3. Infinite Ramsey Theory

Suppose we have a 2-colouring of $\mathbb{N}^{(2)}$ (edges of complete graph on $\mathbb{N}$). Can we always find an infinite monochromatic subset?

Example.

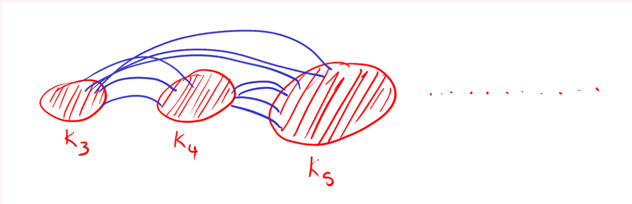
(1) Give edge ij colour red if $i+j$ is even, and colour blue if $i+j$ is odd. Then we could take $M = \{\text{evens}\}$.

(2) Give edge ij colour red if $\max(n = 2^k \text{ dividing } i+j)$ is even, and colour blue otherwise. Could take $M = \{\text{powers of } 4\}$.

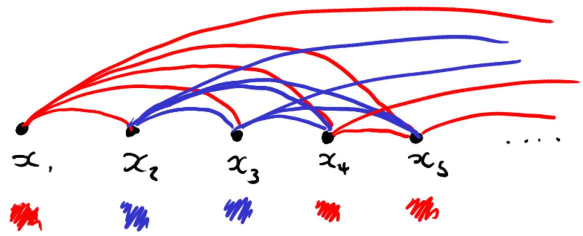
(3) Give edge ij colour red if the number of primes dividing $i+j$ is even, and colour blue otherwise. $M = \ ?$.

Theorem 5 (Infinite Ramsey)**.** Whenever $\mathbb{N}^{(2)}$ is 2-coloured, there exists an infinite monochromatic set.

Note. Much more than asking for arbitrarilty large finite monohromatic subsets. For example:



Proof. Pick $x_1 \in \mathbb{N}$. We have infinitely many edges from x_1. So infinitely many are the same colour – say all edges $x_1 y$, for each $y \in A_1$ have colour c_1. Now choose $x_2 \in A_1$. Again, there exists infinitely many $A_2 \subset A_1$ such that all edges from x_2 to A_2 have same colour, say coloud c_2. Continue inductively. We obtain distinct points $x_1, x_2, \ldots$ and colours $c_1, c_2, \ldots$ such that $x_i x_j$ $(i < j)$ has colour c_i.



But must have $i_1, i_2, \ldots$ with $c_{i_1} = c_{i_2} = \cdots$, and now see that $\{x_{i_1}, x_{i_2}, \ldots\}$ is monochromatic. □

Remark.

(1) Called a '2-pass' proof.

(2) In Example 3 earlier in this subsection, no explicit M is known.

(3) Same proof works for k colours (or 'turquoise spectacles').

Example. Any sequence $x_1, x_2, \ldots$ of reals has a monotone subsequence. Indeed, 2-colour $\mathbb{N}^{(2)}$ by giving ij $(i < j)$ colour *up* if $x_i < x_j$, and colour *down* if $x_i \geq x_k$. Now apply Infinite Ramsey.

How about 2-colouring $\mathbb{N}^{(r)}$? For example, $r = 3$. Give ijk $(i < j < k)$ colour red if i divides $j + k$, and blue if not. Could take $M = \{\text{powers of } 2\}$.

Theorem 6 (Infinite Ramsey for r-sets)**.** Given $r = 1, 2, \ldots$: whenever $\mathbb{N}^{(r)}$ 2-coloured, there exists a monochromatic set.

Proof. Induction on r. $r = 1$ is just pigeonhole (or could start with $r = 2$ using Theorem 5). Given $r > 1$, and a 2-colouring c of $\mathbb{N}^{(r)}$: fix $x_1 \in \mathbb{N}$. Then c induces a 2-colouring c' of $(\mathbb{N} \setminus \{x_1\})^{(r-1)}$. By $c'(A) = c(A \cup \{x_1\})$ for each A. So (by induction) there exists an infinite $A_1 \subset \mathbb{N} \setminus \{x_1\}$ and colour c_1 such that for each $(r-1)$-set $A \subset A_1$ we have $A \cup \{x_1\}$ gets colour c_1. Pick $x_2 \in A_2$ and continue as before. We obtain distinct points $x_1, x_2, \ldots$ and colours $c_1, c_2, \ldots$ such that the colour of $x_{i_1} \cdots x_{i_r}$ $(i_1 < i_2 < \cdots < i_r)$ is c_{i_1} for all $i_1 < \cdots < i_r$. But must have $i_1 < i_2 < \cdots$ with $c_{i_1} = c_{i_2} = \cdots$, and now $\{x_{i_1}, x_{i_2}, \ldots\}$ is monochromatic. □

Example. We say what, given points $(1, x_1), (2, x_2), (3, x_3)$ in $\mathbb{R}^2$, can find an infinite subset whose induced graph (piecewise-linear, through those points) is increasing or decreasing. In fact, can also insist that the induced graph is convex or concave. Indeed, colour $\mathbb{N}^{(3)}$ by giving ijk colour *convex* if (j, x_j) lies below the line through (i, x_i) and (k, x_k), and give ijk the colour *concave* otherwise. Then apply Theorem 6 ($r = 3$).

Exact Ramsey Numbers

Very few of the (non-trivial) $R(s, t)$ are known exactly. We know:

$R(3,3) = 6$	$R(3,4) = 9$	$R(3,5) = 14$	$R(3,6) = 18$
$R(3,7) = 23$	$R(3,8) = 28$	$R(3,9) = 36$	$R(4,4) = 18$
$R(4,5) = 25$			

$R(5) = R(5,5)$ is unknown! We do know that $43 \le R(5) \le 48$.

For more colours, only known (non-trivial) case is $R_3(3,3,3) = 17$.

For r-sets, only known case is $R^{(3)}(4,4) = 13$.

This is because we are asking "exactly how much disorder do we need to guarantee a certain amount of order?"

"Put it on a computer?"

To show $R(5) \le 4$, to show $R(5) \le 48$, we need to look at $2^{\binom{47}{2}} > 2^{1000} > 10^{300}$ colourings – no chance. Even if we use a clever symmetry argument to divide the work by 48, and then use another clever argument to square root the amount of work required, we would still need to look at over 10^{100} colourings, so still no chance.

Start of

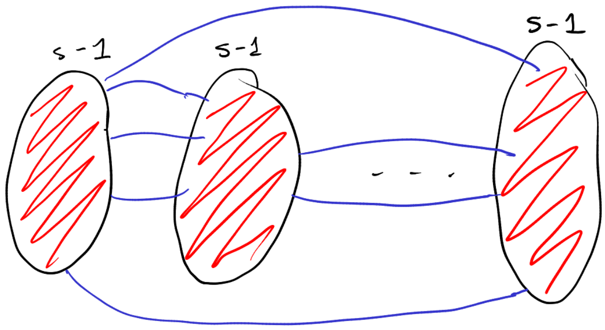
lecture 18

Chapter VI

Random Graphs

Contents

We know $R(s) \leq 4^s$. How fast does $R(s)$ grow? Easy to see that $R(s) > (s-1)^2$:



It was generally believed (in the 19030s and 1940s), that $R(s) \sim cs^2$. But remarkably:

Theorem 1 (Erdős, 1947)**.** $R(s) \geq (\sqrt{2})^s$ for all $s \geq 3$.

Proof. Choose a colouring of K_n at random – each edge coloured red or blue, with probability $\frac{1}{2}$, independently.

Then $\mathbb{P}(\text{a fixed } s\text{-set is monochromatic}) = 2 \cdot \left(\frac{1}{2}\right)^{\binom{s}{2}}$ (the probability that it is coloured entirely red is $\left(\frac{1}{2}\right)^{\binom{s}{2}}$, and then we double since it could be coloured either all red or all blue). Also, the number of s-sets is $\binom{n}{s}$, so certainly

$$\mathbb{P}(\text{exists a monochromatic } s\text{-set}) \leq \binom{n}{s} 2^{1-\binom{s}{2}}.$$

Hence $R(s) > n$ (i.e. there exists a 2-colouring of K_n with no monochromatic s-set) if $\binom{n}{s}2^{1-\binom{s}{2}} < 1$, i.e. if $\binom{n}{2} < 2^{\binom{s}{2}-1}$. Now, $\binom{n}{s} \leq \frac{n^s}{s!}$ and $s! \geq 2^{\frac{s}{2}+1}$ for all $s \geq 3$ (induction on s). So

$$\binom{n}{s} \leq \frac{n^s}{2^{\frac{s}{2}+1}}.$$

So $R(s) > n$ if $n^s < 2^{\frac{s^2}{2}}$, i.e. if $n < 2^{\frac{s}{2}}$. □

Remark.

(1) The above is a ‘random graphs’ argument.

(2) Could rephrase as: $\#\text{colourings} = 2^{\binom{n}{2}}$ and

$$\#\text{colourings making a fixed } s\text{-set monochromatic} = 2 \cdot 2^{\binom{n}{2} - \binom{s}{2}},$$

so done if $\binom{n}{2} 2^{\binom{n}{2} - \binom{s}{2} + 1} < 2^{\binom{n}{2}}$. But this is not a helpful viewpoint – for example, later we will pick edges with probability $\neq \frac{1}{2}$.

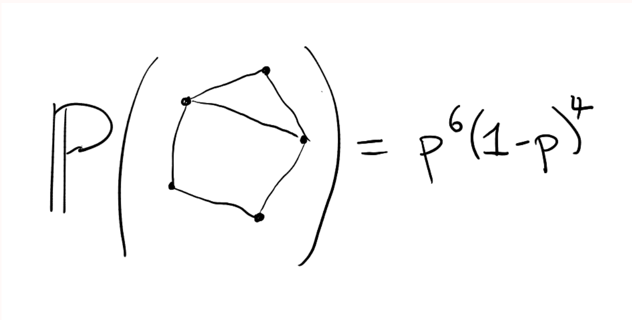
(3) Proof gives no hint of how to *construct* a bad colouring.

(4) In fact, no *construction* is known giving $R(s)$ exponential in s.

We have $(\sqrt{2})^2 \le R(s) \le 4^s$. The right growth speed is unknown. No lower bound of the form $(\sqrt{2} + \varepsilon)^s$ is known, but it was proved in 2023 that $R(s) \le (4 - \varepsilon)^s$ (for some small ε).

Definition (Probability space on a graph)**.** For $0 < p < 1$, the probability space $G(n, p)$ is defined as follows: we form a graph on n vertices by selecting each edge, independently, with probability p (so in the proof of Theorem 1, we worked in $G\left(n, \frac{1}{2}\right)$).

Example. In $G(5, p)$,



Can be useful to consider $p \neq \frac{1}{2}$...

Recall that for the ‘problem of Zarankiewicz’ (see Section 3.2), we had $Z(n, t) \le 2n^{2 - \frac{1}{t}}$. How about a lower bound – preferably not of the form $c \cdot n^1$ (preferably non-trivial).

We could do the following:

Form a random bipartite graph (vertex classes X, Y, each size n) by choosing edges independent with probability p. Then

$$\mathbb{P}(\text{a fixed } K_{t,t} \subset G) = p^{t^2}.$$

Also, $\#K_{t,t} = \binom{n}{t}^2$. So

$$\mathbb{E}(\#K_{t,t} \text{ in } G) = \binom{n}{t}^2 p^{t^2} \le \frac{1}{4} n^{2t} p^{t^2}.$$

So take $p = n^{-\frac{2}{t}}$ – gives $\mathbb{E}(\#K_{t,t}) \le \frac{1}{4}$. So $\mathbb{P}(\text{no } K_{t,t} \text{ in } G) \ge \frac{3}{4}$. Also, $\mathbb{E}(\#\text{edges of } G) = pn^2$. So

$$\mathbb{P}\left(\#\text{edges} \ge \frac{1}{2}pn^2\right) \ge \frac{1}{2}.$$

Hence there exists a graph G with no $K_{t,t}$ and $e(G) \ge \frac{1}{2}pn^2 = \frac{1}{2}n^{2-\frac{2}{t}}$. Thus $Z(n,t) \ge \frac{1}{2}n^{2-\frac{2}{t}}$.

But we can do better.

Theorem 2. $Z(n,t) \ge \frac{1}{2}n^{2-\frac{2}{t+1}}$.

Idea: If G has n edges and r $K_{t,t}$s, remove an edge from each $K_{t,t}$ to obtain a graph with $m - r$ edges and no $K_{t,t}$.

Proof. Form a random bipartite graph (vertex classes size n), by choosing each edge with probability p independently. Let $M = \#\text{edges}$, $R = \#K_{t,t}$. Then

$$\mathbb{E}(M) = pn^2 \qquad \text{and} \qquad \mathbb{E}(R) = \binom{n}{t}^2 p^{t^2} \le \frac{1}{2}n^{2t}p^{t^2},$$

so

$$\mathbb{E}(M - R) \ge pn^2 - \frac{1}{2}n^{2t}p^{t^2}.$$

Pick $p = n^{-\frac{2}{t+1}}$: so $pn^2 = n^{2-\frac{2}{t+1}}$ and $n^{2t}p^{t^2} = n^{2t-\frac{2t^2}{t+1}} = n^{2-\frac{2}{t+1}}$. So

$$\mathbb{E}(M - R) \ge \frac{1}{2}n^{2-\frac{2}{t+1}}.$$

Hence there exists G with m edges, r $K_{t,t}$s and $m - r \ge \frac{1}{2}n^{2-\frac{2}{t+1}}$. Whence $Z(n,t) \ge \frac{1}{2}n^{2-\frac{2}{t+1}}$. □

This method is called 'modifying a random graph'.

Start of

lecture 19

5.4. Graphs with high chromatic number

To ensure $\chi(G) \geq k$, could just have $G \supset K_k$. Not necessary to have $G \supset K_k$ – for example C_5 has no K_3, but $\chi(C_5) = 3$.

Can have $\chi(G)$ much greater than $\mathrm{CL}(G)$, the *clique number* of G, namely $\max\{k : G \supset K_k\}$.

Example.

(1) The G of Theorem 1: G on $2^{s/2}$ vertices, no $K_s \subset G$, so $\mathrm{CL}(G) \leq s-1$. Also, each *independent set* (set with no edges) has size $\leq s-1$. But in any vertex-colouring of a graph, each colour class is an independent set. Hence $\chi(G) \geq \frac{2^{s/2}}{s-1}$ – much more than $\mathrm{CL}(G)$. Better:

(2) Can find G which is triangle-free ($\mathrm{CL}(G) = 2$), but $\chi(G)$ arbitrarily large – quite hard (on Example Sheet 3).

Could we even ask for girth ≥ 5? Or more – like G with girth ≥ 10, $\chi(G) \geq 100$? Sounds unlikely, but…

Theorem 3. $\forall k, g$ there exists graph G with girth $\geq g$ and $\chi(G) \geq k$.

Idea: Find G on n vertices such that the number of short cycles is $\leq \frac{n}{2}$ and each independent set has size $\leq \frac{n}{2k}$ – then done, by removing a vertex from each short cycle to obtain a graph H with girth $\geq g$ and $\chi(G) \geq \frac{n/2}{n/2k} = k$.

Proof. For large n, choose $G \in G(n,p)$ where $p = n^{-1+\frac{1}{g}}$. For $i = 3, 4, \dots, g-1$, let $X_i = \#i$ cycles in G. Let $X = X_3 + \cdots + X_{g-1}$ be the number of cycles in G of length $< g$. Then

$$\begin{aligned}\mathbb{E}(X_i) &\leq (\#\text{possible } i\text{-cycles})\mathbb{P}(\text{given } i\text{-cycle} \subset G) \\ &\leq n^i p^i \\ &= n^{i/g} \\ &\leq n^{\frac{g-1}{g}}\end{aligned}$$

Hence $\mathbb{E}(X) \leq g \cdot \frac{n}{n^{1/g}} < \frac{n}{4}$ for n large (as $n^{1/g} \to 0$ as $n \to \infty$). So $\mathbb{P}\left(X > \frac{n}{2}\right) < \frac{1}{2}$ (else $\mathbb{E}(X) \geq \frac{n}{2} \cdot \frac{1}{2} = \frac{n}{4}$ ※). Write $t = \frac{n}{2k}$ (n a multiple of $2k$), and let $Y =$

#independent t-sets in G. Then

$$\begin{aligned}
\mathbb{E}(Y) &\le n^t (1-p)^{\binom{t}{2}} \\
&\le n^t e^{-p\binom{t}{2}} && \text{(using } 1-x \le e^{-x}\text{)} \\
&\le \exp\left(\frac{n}{2k}\log n - n^{-1+\frac{1}{g}} \cdot \frac{n^2}{16k^2}\right) \\
&\to 0
\end{aligned}$$

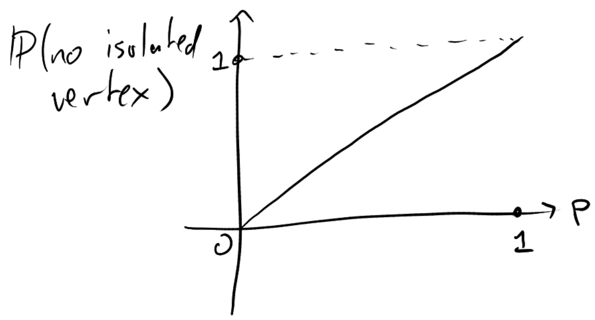
as $n \to \infty$ ($n^{1+\frac{1}{g}}$ grows faster than $n \log n$). So $\mathbb{P}(X = 0) > \frac{1}{2}$ if n large enough. Hence there exists G on n vertices with $\le \frac{n}{2}$ short cycles and no independent set of $\frac{n}{2k}$. □

5.5. The structure of a random graph

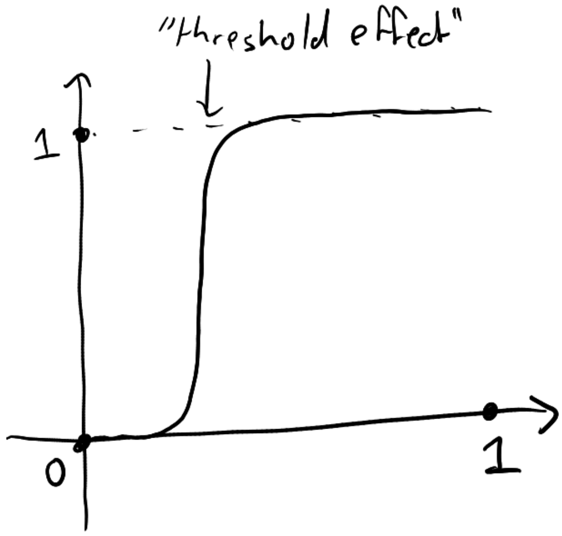
What does $G \in G(n, p)$ look like? How do the properties of G vary as p invreases? For example, how to $\mathbb{P}$(no isolated vertex) behave?

(A vertex is *isolated* if it has no neighbours).

We might guess:



But in fact:



Why does this happen? Where is the threshold?

Probability Digression / Reminder

Let X be a random variable taking values in $0, 1, 2, \ldots$.

To show $\mathbb{P}(X = 0)$ **is big:** enough to show $\mu = \mathbb{E}(X)$ is small. Indeed, for any $t > 0$ have $\mathbb{P}(X \geq t)t \leq \mu$. So $\mathbb{P}(X \geq t) \leq \frac{\mu}{t}$ (Markov). In particular, $\mathbb{P}(X \geq 1) \leq \mu$, so $\mathbb{P}(X = 0) \geq 1 - \mu$.

To show $\mathbb{P}(X = 0)$ **is small:** not enough to have μ large, for example

$$X = \begin{cases} 0 & \text{probability } \frac{99}{100} \\ 10^{10} & \text{probability } \frac{1}{100} \end{cases}$$

So instead we look at the variance $V = \mathrm{Var}(X) = \mathbb{E}((X - \mu)^2) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$. Then

$$\mathbb{P}(|X - \mu| \geq t) = \mathbb{P}(|X - \mu|^2 \geq t^2) \leq \frac{V}{t^2}$$

by Markov (this is known as Chebyshev inequality). So $\mathbb{P}(|X - \mu| \geq \mu) \leq \frac{V}{\mu^2}$, whence $\mathbb{P}(X = 0) \leq \frac{V}{\mu^2}$. Conclusion: to show $\mathbb{P}(X = 0)$ is small, check $\frac{V}{\mu^2}$ is small.

Suppose X counts the number of some events A that occur. Then $\mu = \mathbb{E}(X) = \sum_A \mathbb{P}(A)$.

Variance? Have

$$\mathbb{E}(X)^2 = \sum_A \sum_B \mathbb{P}(A)\mathbb{P}(B)$$

$$\begin{aligned}\mathbb{E}(X^2) &= \mathbb{E}\left(\left(\sum_A \mathbb{1}_A\right)^2\right) \\ &= \mathbb{E}\left(\sum_A \sum_B \mathbb{1}_A \mathbb{1}_B\right) \\ &= \sum_A \sum_B \mathbb{E}(\mathbb{1}_A \mathbb{1}_B) \\ &= \sum_A \sum_B \mathbb{P}(A \cap B) \\ &= \sum_A \sum_B \mathbb{P}(A) PP(B \mid A)\end{aligned}$$

So $\mathrm{Var}(X) = \sum_A \mathbb{P}(A) \sum_B (\mathbb{P}(B \mid A) - \mathbb{P}(B))$. Key: $\mathbb{P}(B \mid A) - \P(B)$ is 0 if A and B are independent.

Start of lecture 20

Theorem 4. Let λ be fixed. Then:

(i) If $\lambda < 1$ then almost surely $G \in G\left(n, \lambda \frac{\log n}{n}\right)$ has an isolated vertex.

(ii) If $\lambda > 1$ then almost surely $G \in G\left(n, \lambda \frac{\log n}{n}\right)$ has no isolated vertex.

('almost surely' means: with probability tending towards 1 as $n \to \infty$).

"$\mathbb{P} = \frac{\log n}{n}$ is a threshold for existence of an isolated vertex".

Proof. Let $X = \#$isolated vertices in $G(n, p)$. Then

$$\mu = \mathbb{E}(X) = n(1-p)^{n-1} = \frac{n}{1-p}(1-p)^n.$$

(ii) Have $p = \lambda \frac{\log n}{n}$, where $\lambda > 1$. So

$$\mu \le \frac{n}{1-p} e^{-pn} = \frac{n}{1-p} e^{-\log n} = \frac{n^{1-\lambda}}{1-p} \to 0$$

as $n \to \infty$. So almost surely $X = 0$.

(i) Have $p = \lambda\frac{\log n}{n}$, where $\lambda < 1$. So $\mu \geq \frac{n}{1-p}e^{-p(1+\delta)n}$, any $\delta > 0$ (p small) – as $1 - x \geq e^{-(1+\delta)x}$ for x small, whence

$$\mu \geq \frac{n}{1-p}n^{-x(1+\delta)} = \frac{n^{1-\lambda(1+\delta)}}{1-p}.$$

So pick fixed $\delta > 0$ with $\lambda(1+\delta) < 1$. Then

$$\mu \geq \frac{n^{\text{positive number}}}{1-p} \to \infty.$$

Also,

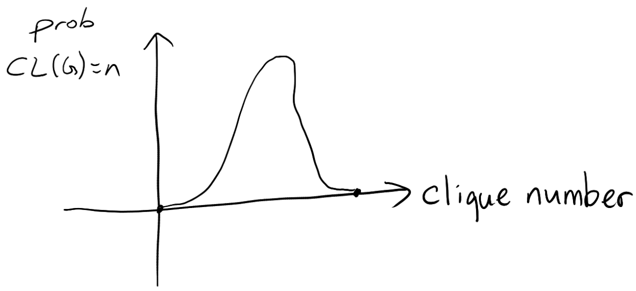
$$\begin{aligned} V &= \sum_A \mathbb{P}(A) \sum_B (\mathbb{P}(B \mid A) - \mathbb{P}(B)) \\ &= \underbrace{n(1-p)^{n-1}(1-(1-p)^{n-1})}_{A=B} + \underbrace{n(n-1)(1-p)^{n-1}((1-p)^{n-2} - (1-p)^{n-1})}_{A \neq B} \\ &\leq \mu + n^2(1-p)^{n-1}p(1-p)^{n-2} \\ &= \mu + \mu^2 \frac{p}{1-p} \end{aligned}$$

So $\frac{V}{\mu^2} \leq \frac{1}{\mu} + \frac{p}{1-p} \to 0$ as $n \to \infty$. □

A different kind of 'threshold effect' comes from graph parameters, for example $\mathrm{CL}(G)$ (clique number).

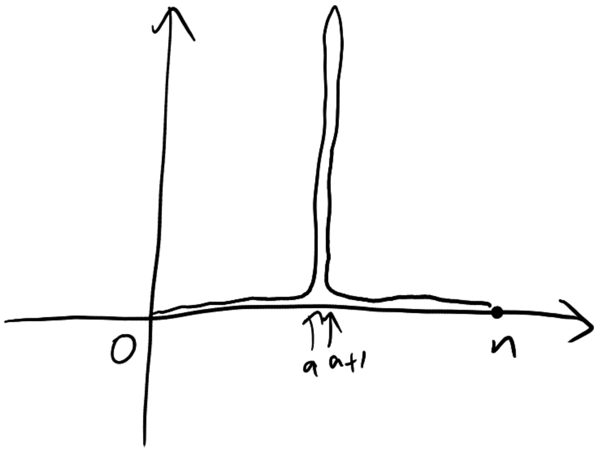
So fix $0 < p < 1$, and we ask: how is the clique number of $G \in G(n,p)$ distributed?

We'd expect



Width of hump? Might guess about $\sqrt{n}$, or maybe $\log n$.

But in fact:



i.e. there exists a such that the clique number of $G \in G(n,p)$ is a or $a+1$ almost surely.

Theorem 5. Let $0 < p < 1$ be fixed, and let d be a real with $\binom{n}{d} p^{\binom{d}{2}} = 1$. Then $G \in G(n,p)$ has clique number $\lceil d \rceil$ or $\lfloor d \rfloor$ or $\lfloor d \rfloor - 1$ almost surely.

Remark. With more work, could get down to only 2 values.

Proof. Let $X = \# K_K$ in G. We'll show that if $k \ge d+1$ then $X = 0$ almost surely, and if $k \le d-1$ then $X \ge 1$ almost surely. W'll show that if $k \ge d+!$ then $X = 0$ almost surely, and if $k \le d-1$ then $X \ge 1$ almost surely. Have

$$\mu = \mathbb{E}(X) = \binom{n}{k} p^{\binom{k}{2}}.$$

So $k \ge d+1 \implies \mu \to 0$ as $n \to \infty$ (check), so almost surely $X = 0$. Now, for $k \le d-1$, have $\mu \to \infty$ as $n \to \infty$ (check). Also

$$V = \underbrace{\binom{n}{k}}_{\# A} \underbrace{p^{\binom{k}{2}}}_{\mathbb{P}(A)} \sum_{s=2}^{k} \underbrace{\binom{k}{s}\binom{n-k}{k-s}}_{\# B \text{ with } \#(A \cap B) = s} \left(p^{\binom{k}{2} - \binom{s}{2}} - p^{\binom{k}{2}}\right)$$

In the sum:

- First term is $\binom{k}{2}\binom{n-k}{k-2} p^{\binom{k}{2}} \left(\frac{1}{p} - 1\right) \le \binom{k}{2}\binom{n-k}{k-2} p^{\binom{k}{2}} \frac{1}{p}$.

- Last term is $1 - p^{\binom{k}{2}} \leq 1$.

In fact, sum is bounded by first + last: more precisely, the sum is $\leq C(\text{first} + \text{last})$, for some C (check). So

$$V \leq \mu C(\binom{k}{2}\binom{n-k}{k-2}p^{\binom{k}{2}}\frac{1}{p} + 1),$$

whence $\frac{V}{\mu^2}$ (check), so that $X \neq 0$ almost surely. □

Start of

lecture 21

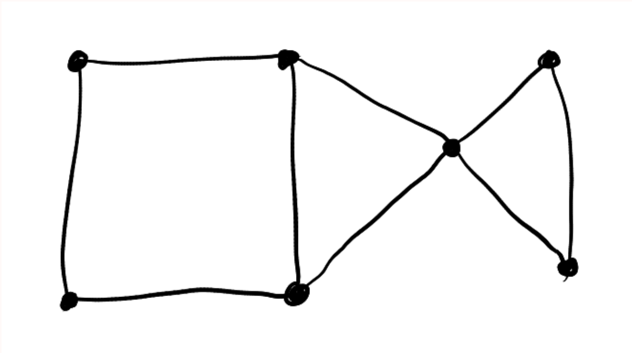
Chapter VII

Algebraic Methods

Contents

Definition (Diameter)**.** The *diameter* of G is $\max\{d(x,y) : x,y \in G\}$.

Example.



has diameter 3.

Thus G has diameter 1 $\iff$ G is a complete graph.

What about diameter 2? How large can $n = |G|$ be in terms of $\Delta(G)$? 'Expanding out' from a vertex x, we see

$$V(G) = \{x\} \cup \Gamma(x) \cup \Gamma(\Gamma(x)),$$

whence

$$n \le 1 + \Delta + \Delta(\Delta - 1) = 1 + \Delta^2.$$

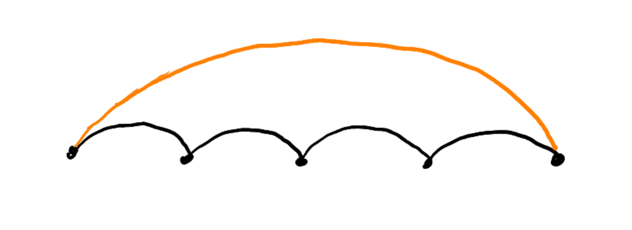
So if $n = \Delta^2 + 1$, then G is Δ-regular (as x is arbitrary).

Definition (Moore graph)**.** A k-regular graph is a *Moore graph* or a *Moore graph of diameter* 2 if G has diameter 2 and $|G| = k^2 + 1$.

Equivalently, k-regular G is a Moore graph $\iff$ any distinct x, y joined by a *unique* path of length ≤ 2 ($k \ne 1$).

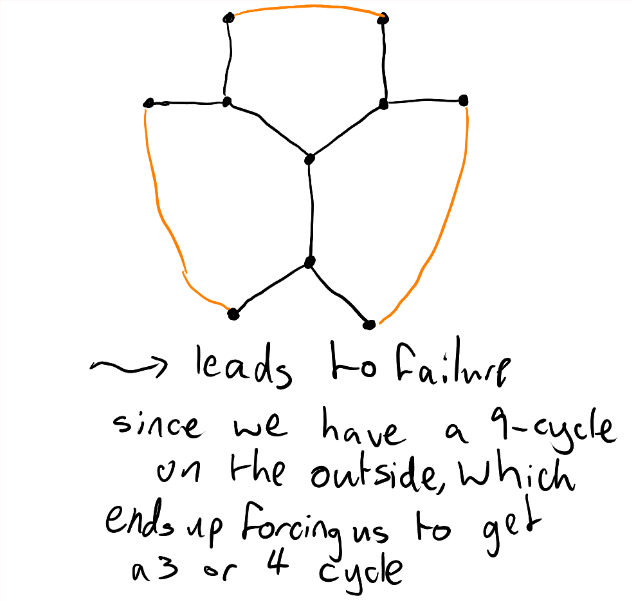
Equivalently: diameter 2 and no C_3 or C_4 in G (in other words, girth ≥ 5).

Example. $k = 2$:

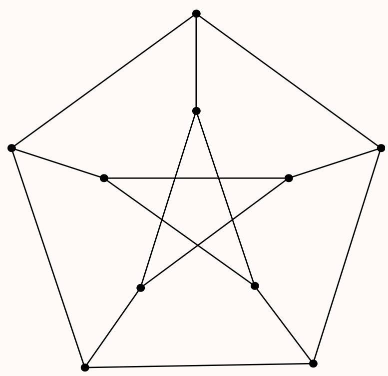


Has $2^2 + 1 = 5$ vertices (this graph is C_5).

Example. $k = 3$: Want $3^2 + 1 = 10$ vertices:



but the Petersen graph works:



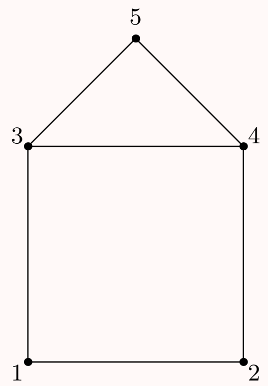
Example. $k = 4$: no such example exists.

Definition (Adjacency matrix)**.** For a graph G on vertex-set $[n] = \{1, \ldots, n\}$, the *adjacency matrix* of G is the $n \times n$ matrix A with

$$A_{ij} = \begin{cases} 1 & ij \in E \\ 0 & ij \notin E \end{cases}.$$

So A is a real symmetric matrix.

Example. If G is



Then

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

So A contains all the information of G. For example what is A^2?

$$(A^2)_{ij} = \sum_k A_{ik} A_{kj} = \#\text{walks of length 2 from } i \text{ to } j.$$

Similarly

$$(A^3)_{ij} = \#\text{walks of length 3 from } i \text{ to } j$$

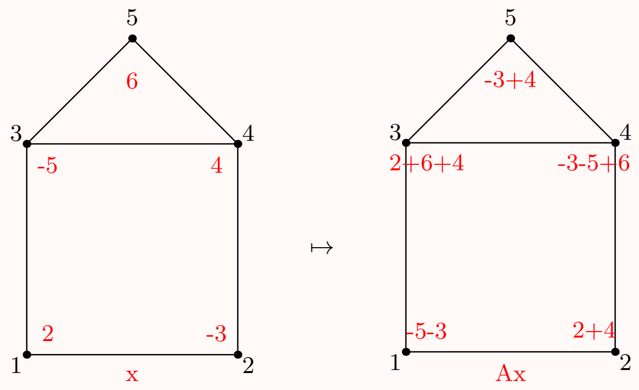
and in general with 3 replaced with n.

Have a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ given by A, namely

$$(Ax)_i = \sum_j A_{ij} x_j$$

$(x \mapsto Ax)$.

Example. Using same example as before:



"Add up neighbouring values." i.e. if $x = (2, -3, -5, 4, 6)$ then

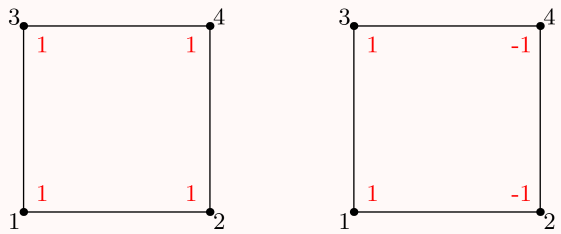
$$Ax = (-5 - 3, 2 + 4, 2 + 6 + 4, -3 - 5 + 6, -5 + 4)$$

Since A is real symmetric, A is diagonalisable, i.e. there exists basis of eigenvalues, say $e_1, \dots, e_n$ – may assume this is is an orthonormal basis (eash e_i length 1, orthogonal to each other). Write eigenvalues (real) as $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. Have $\sum \lambda_i = 0$ (as $\mathrm{Tr}(A) = 0$), so $\lambda_1 > 0$ and $\lambda_n < 0$ (unless $G = E_n$). Often, it is easy to work out the eigenvalues of G.

Example. Eigenvalues of of C_4:

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Have $\text{rank}(A) = 2$, so only 2 non-zero eigenvalues. Other two? We can spot that the following are eigenvectors:



So the eigenvalues are $0, 0, 2, -2$ (or find $0, 0, 2$ and use that the sum is 0).

Write $\lambda_{\max} = \lambda_1$, $\lambda\text{min} = \lambda_n$ – eg C_4 has $\lambda_{\max} = 2$, $\lambda_{\min} = -2$.

Know $\lambda_{\max} > 0$, $\lambda_{\min} < 0$ (if $G \neq E_n$).

Have out of eigenvectors $e_1, \dots, e_n$, eigenvalues $\lambda_1, \dots, \lambda_n$, given any $x \in \mathbb{R}^n$, write $x = \sum_i c_i e_i$ for some $c_1, \dots, c_n \in \mathbb{R}$. FIx $\|x\| = 1$, ie $\sum_i c_i^2 = 1$.

Then $Ax = \sum_i \lambda_i c_i e_i$, so $\langle Ax, x \rangle = \sum_i \lambda_i c_i^2$. Hence

$$\lambda_{\min} \le \langle Ax, x \rangle \le \lambda_{\max} \quad \forall x \in \mathbb{R}^n, \|x\| = 1.$$

Bounds are attained – $\langle Ax, x \rangle = \lambda_{\max}$ if $c_1 = 1$, rest $= 0$. $\langle Ax, x \rangle = \lambda_{\min}$ if $c_n = 1$, rest $= 0$.

Conclusion:

$$\lambda_{\max} = \max_{\|x\|=1} \langle Ax, x \rangle, \qquad \text{and} \qquad \lambda_{\min} = \min_{\|x\|=1} \langle Ax, x \rangle. \tag{$*$}$$

Start of

lecture 22

Proposition 1 (Eigenvalue facts)**.** Let G be a graph. Then:

(i) λ is an eigenvalue $\implies |\lambda| \le \Delta$.

(ii) For G connected: Δ is an eigenvalue $\iff G$ regular.

(iii) For G connected: $-\Delta$ is an eigenvalue $\iff G$ regular and bipartite.

(iv) $\lambda_{\text{max}} \ge \delta$.

Proof.

(i) Let λ be an eigenvalue with eigenvector x. Choose i with $|x_i|$ maximal – without loss of generality $x_i = 1$. Then $(Ax)_i = \sum_{j \in \Gamma(i)} x_j$, so

$$|(Ax)_i| \le \sum_{j \in \Gamma(i)} |x_j| \, le \Delta \cdot 1.$$

Then $|\lambda| \le \Delta$.

(ii) $\Leftarrow$ Let $x = (1, \dots, 1)$ – then $Ax = (\Delta, \Delta, \dots, \Delta)$.

$\Rightarrow$ Choose i with $|x_i|$ maximal. Without loss of generality $x_i = 1$. Then as we have equality in the inequality of (i) above, we must have $d(i) = \Delta$ and $x_j = 1$ for all $j \in \Gamma(i)$. We can repeat at each $j \in \Gamma(i)$ (as $x_j = 1$) to obtain $d(j) = \Delta$ and $x_{j'=1}$ for all $j' \in \Gamma(j)$. Continue: we obtain $d(j) = 1$ for all j (as G connected).

(iii) $\Leftarrow$ Let $x = 1$ on X, -1 on Y (X, Y our bipartition). Then $Ax = (-\Delta)x$.

$\Rightarrow$ Choose i with $|x_i|$ maximal. Without loss of generality $x_i = 1$. Then since we have equality in the inequality of (i), we must have $d(i) = \Delta$ and $x_j = -1$ for all $j \in \Gamma(i)$. So can repeat at each $j \in \Gamma(i)$ to obtain $d(j) = \Delta$ and $x_{j'} = 1$ for all $j' \in \Gamma(j)$. Continue: we get that $x_j = \pm 1$ for all j, and $jj' \in E$ implies $x_j = 1$, $x_{j'} = -1$ or vice versa (as G connected). So G has no odd cycle, hence is bipartite (and we've already seen that it is Δ-regular).

(iv) By ($*$), enough to find x, $\|x\| = 1$, with $\langle Ax, x \rangle \ge \delta$. Let $x = (1, 1, \dots, 1)$. Then $(Ax)_i \ge \delta$ for all i. So $\langle Ax, x \rangle \ge \delta n$, with $\langle x, x \rangle = n$. □

Remark. Proof of (ii) actually gives that if G is connected then eigenvalue k has multiplicity 1.

Eigenvalues can relate to other graph parameters. For example, we know $\chi(G) \le \Delta + 1$. Can strengthen this to:

Proposition 2. For any graph G, have $\chi(G) \le \lambda_{\max} + 1$.

Proof. Without loss of generality $|G| \ge 2$ ($|G| = 1$ trivial). Choose $v \in [n] = V(G)$ with $d(v) = \delta$, and let $G' = G - v$.

Claim: $\lambda_{\max}(G') \le \lambda_{\max}$.

Once we prove this we're done, because we can colour G' in $\lambda_{\max} + 1$ colours using induction, and now $d(v) = \delta \le \lambda_{\max}$ (by Proposition 1(iv)), so can colour v as well.

Proof of claim: Adjacency matrix of G', B say, is formed from A by deleting v-th row and column: say n-th row and column. By $(*)$, enough to show that

$$\max_{\substack{x \in \mathbb{R}^{n-1} \\ \|x\|=1}} \langle Bx, x\rangle \le \max_{\substack{x \in \mathbb{R}^{n} \\ \|x\|=1}} \langle Ax\rangle x.$$

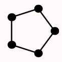
But if $x \in \mathbb{R}^{n-1}$, say $x = (x_1, \dots, x_{n-1})$, then $y = (x_i, \dots, x_{n-1}, 0) \in \mathbb{R}^n$ with $\|y\| = \|x\|$ and $\langle Bx, x\rangle = \langle Ay, y\rangle$. □

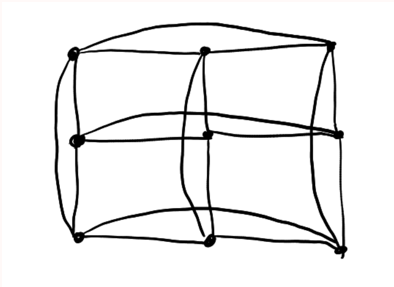
5.6. Towards Moore Graphs

Definition (Strongly regular)**.** Say a graph G is *strongly regular* with *parameters* (k, a, b) if G is k-regular, with any two adjacent vertices having a common neighbours and any two non-adjacent vertices having b common **??**.

"One step up from being regular".

Example.

1. C_4:  has $(2, 0, -2)$.
2. C_5:  has $(2, 0, 1)$.
3. And in general, if G is a Moore graph of degree k, then G is strongly regular, with parameters $(k, 0, 1)$.
4. Triangle $\times$ triangle:



has $(4, 1, 2)$.

Seems that 'strongly regular' is a bit more than regular, but...

Start of lecture 23

Remark (Silly remark)**.** If G is a complete graph, then b is not well-defined – K_t is $(t-1, t-2, b)$ is strongly regular for any b. Also, if $b = 0$, then G need not be connected – for example two copies of K_t is $(t-1, t-2, 0)$.

Being strongly regular is actually *extremely* restrictive:

Theorem 3 (Rationality criterion for strongly regular graphs)**.** Let G be a strongly regular graph on n vertices, with parameters (k, a, b). (G not complete, $b \ge 1$). Then the numbers $\frac{1}{2}\left(k - 1 \pm \frac{(n-1)(b-a)-2k}{\sqrt{(a-b)^2+4(k-b)}}\right)$ are integers.

Note. Denominator $\sqrt{(a-b)^2 + 4(k-b)} \neq 0$ – else $a = b$, $b = k$, contradicting $a \le k - 1$.

Proof. Have G connected (since $b \ge 1$). We also have that G is k-regular, so k is an eigenvalue with multiplicity 1 and eigenvector $(1, 1, \ldots, 1)$.

Consider matrix A^2. We know

$$(A^2)_{ij} = \#\text{walks of length 2 from } i \text{ to } j$$

so

$$(A^2)_{ij} = \begin{cases} k & \text{if } i = j \\ a & \text{if } i, j \text{ adjacent} \\ b & \text{if } i, j \text{ non-adjacent} \end{cases}$$

and thus $A^2 = kI + aA + b(J - I - A)$, where J is the matrix with all entries being 1. So

$$A^2 + (b - a)A + (bI - bJ - kI) = 0,$$

which is not quite quadratic, as there exists a J-term. So for eigenvector x, eigenvalue $\lambda \neq k$: have $\langle x, (1, \ldots, 1)\rangle = 0$, so $Jx = 0$. Hence

$$\begin{aligned} 0x &= A^2x + (b - a)Ax + (b - k)Ix \\ &= \lambda^2 x + (b - a)\lambda x + (b - k)x \\ &= (\lambda^2 + (b - a)\lambda + (b - k))x \end{aligned}$$

Thus $\lambda^2 + (b - a)\lambda + (b - k) = 0$. Hence the eigenvalues $\neq k$ are λ, μ given by $\frac{1}{2}(a - b \pm \sqrt{(b - a)^2 + 4(k - b)})$. Let their multiplicities be r, s respectively. Then $r + s = -1$ (as A is diagonalisable), and $r\lambda + s\mu + k = 0$ (as $\operatorname{Tr} A = 0$). Solving for r, s gives the numbers in the theorem. □

Back to Moore Graphs

Theorem 4. Let G be a Moore graph of degree k. Then $k \in \{2, 3, 7, 57\}$.

Proof. Have G strongly regular, parameters (k, a, b). Hence by Theorem 3, either the number $(n - 1)(b - a) - 2k = k^2 - 2k$ is 0 or $\sqrt{(a - b)^2 + 4(k - b)} = \sqrt{4k - 3}$ is an integer.

If $k^2 - 2k = 0$: then we have $k = 2$.

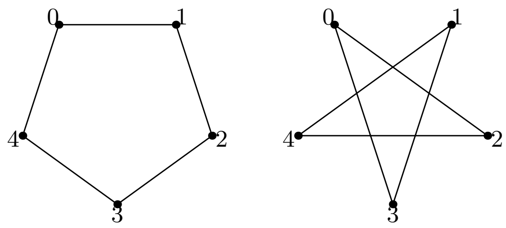
If not: write $t = \sqrt{4k - 3}$. t divides $k^2 - 2k = \left(\frac{t^2+3}{4}\right)^2 - 2\left(\frac{t^2+3}{4}\right)$. So t divides

$$(t^2 + 3)^2 - 8(t^2 + 3) = t^4 - 2t^2 - 15.$$

Hence t divides 15, so $t = 1, 3, 5$ or 15. These give $k = \frac{t^2+3}{4}$ gives 1 (not allowed) or 3 or 7 or 57. □

Which of these can occur?

We saw earlier that for $k = 2$, C_5 works, and for $k = 3$ the Petersen graph works. It helps to think of the drawing of the Petersen graph drawn like this:



where we join t in the pentagon to t in the pentagram.

Now for $k = 7$: The *Hoffman-Singleton graph*: 7-regular, 50 vertices, diameter 2. Take 5 pentagons $P_0, \dots, P_4$ and 5 pentagrams $Q_0, \dots, Q_4$, and join vertex t in P_i to vertex $t + ij$ in Q_j. This works!

$k = 57$: graph, 57-regular, on 3250 vertices, diameter 2. It is unknown whether this graph exists. Sometimes called the “missing Moore graph”. If it exists, turns out that it cannot be transitive (meaning the automorphism group is transitive, or more intuitively “all vertices look the same”).

Index