Algebraic Topology

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Start of

lecture 1

Why study Algebraic Topology?

The main kind of problem we will study is:

Extension problem: If X is a space, $A \subset X$ a subspace and $f : A \to Y$ a continuous function, is there a continuous function $F : X \to Y$ such that $F|_A = f$.

Specific examples look like:

Theorem. There is no continuous $F : D^n \to S^{n-1}$ such that

$$S^{n-1} \stackrel{\text{incl}}{\hookrightarrow} D^n \stackrel{F}{\to} S^{n-1}$$

is the identity.

Theorem. There is no group homomorphism $F : \{0\} \to \mathbb{Z}$ such that

$$\mathbb{Z} \to \{0\} \stackrel{F}{\to} \mathbb{Z}$$

is the identity.

Theorem. $\mathbb{R}^n \cong \mathbb{R}^m \iff n = m$.

Theorem (Fundamental theorem of algebra)**.** Any non-constant polynomial over $\mathbb{C}$ has a root in $\mathbb{C}$.

Recollections

We will call a continuous function a *map*.

Lemma (Gluing lemma)**.** Let $f : X \to Y$ be a function between topological spaces, and let $C, K \subset X$ be closed sets such that $X = C \cup K$. Then f is continuous if and only if $f|_C$, $f|_K$ are continuous.

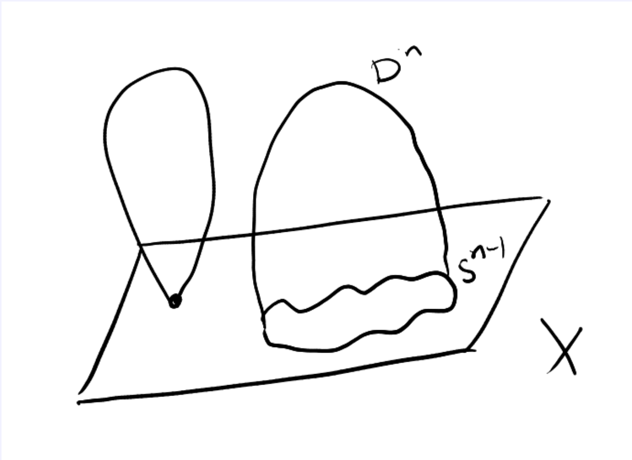
Proof. Exercise. □

Lemma (Lebesgue number lemma)**.** Let (X, d) be a metric space, and assume it is compact. For any open cover $\mathcal{U} = \{U_\alpha \subset X\}$, there is $\delta > 0$ such that each $B_\delta(x)$ is contained in some U_α.

Cell complex

Definition (Attaching an n-cell)**.** For a space X and a map $f : S^{n-1} \to X$, the space obtained by attaching an n-cell to X is

$$X \cup D^n = (X \amalg D^n)/(z \in S^{n-1} \subset D^n \tilde{f}(z) \in X.$$



Definition (Finite cell complex)**.** A (finite) *cell complex* is a space X obtained by the following recipe:

(i) Start with a finite set X^0 with the discrete topology.

(ii) If X^{n-1} has been defined, form X^n by attaching a finite collection of n-cells along some maps $\{f_\alpha : S^{n-1} \to X^{n-1}\}$. This X^n is called the n-skeleton.

(iii) Stop with $X = X^k$. k is called the dimension of X.

2.1 Homotopy

Definition (Homotopy)**.** Let $f, g : X \to Y$ be maps. A *homotopy* from f to g is a map $H : X \times I \to Y$ such that

$$\begin{aligned} H(x, 0) &= f(x) \\ H(x, 1) &= g(x) \end{aligned}$$

If such an H exists, say f *is homotopic to* g, and write $f \simeq g$.

If $A \subset X$ is a subspace, say H is a *homotopy relative to* A if $H(a, t) = H(a, 0)$ for all $t \in I$, $a \in A$. Write $f \simeq g \operatorname{rel} A$.

Proposition. Being homotopic $\operatorname{rel} A$ is an equivalence relation on the set of maps from X to Y.

Proof.

(i) $f \simeq f$ via $H(x, t) = f(x)$.

(ii) If $f \simeq g$ via H, let $H'(x, t) = H(x, 1 - t)$. This is a homotopy from g to f.

(iii) If $f \simeq g$ via H, and $g \simeq h$ via H' then let

$$H''(x, t) = \begin{cases} H(x, 2t) & 0 \le t \le \frac{1}{2} \\ H(x, 2t - 1) & \frac{1}{2} \le t \le 1 \end{cases}$$

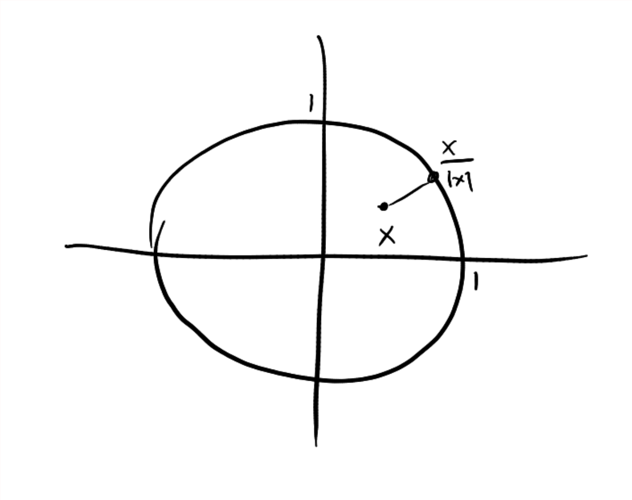
(well-defined as $H(x, 1) = g(x) = H'(x, 0)$). This is continuous on $X \times \left[0, \frac{1}{2}\right]$ and on $X \times \left[\frac{1}{2} \times 1\right]$, so is continuous on $X \times I$ by Gluing lemma. So $f \simeq h$.

□

Definition (Homotopy equivalence)**.** A map $f; X \to Y$ is a *homotopy equivalence* if there is a $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Say X is *homotopy equivalent to* Y, written $X \simeq Y$, if a homotopy equivalence $f : X \to Y$ exists.

Example. Let $X = S^1$, $Y = \mathbb{R}^2 - \{0\}$. Let $i : X \to Y$ be the inclusion. Let $r : Y \to X$, continuous, given by $x \mapsto \frac{x}{|x|}$.



Now $r \circ i = \mathrm{id}_X$. On the other hand, $i \circ r : Y \to Y$ is not the identity. But

$$H(x,t) = \frac{x}{t + |x|(1-t)}$$

is a homotopy from id_Y to $i \circ r$. So $\mathbb{R}^2 - \{b\} \simeq S^1$.

Definition (Contractible)**.** X is called *contractible* if $X \simeq \{*\}$ (here, $\{*\}$ denotes a 1-point space).

Lemma. Let $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$ be homotopic maps. Then $g_0 \circ f_0$ and $g_1 \circ f_1$ are homotopic.

Proof. Lets show $g_0 \circ f_0 \simeq g_0 \circ f_1 \simeq g_1 \circ f_1$.

To show $g_0 \circ f_0 \simeq g_0 \circ f_1$: If H is a homotopy f_0 to f_1, then $g_0 \circ H : X \times I \to Y \to Z$ is a homotopy.

To show $g_0 \circ f_1 \simeq g_1 \circ f_1$: If G is a homotopy g_0 to g_1, then $G \circ (f_1 \times \mathrm{id}_I) : X \times I \to Y \times I \to Z$ is a homotopy. □

Start of lecture 2

Proposition. We have

(i) $X \simeq X$,

(ii) If $X \simeq Y$ then $Y \simeq X$,

(iii) If $X \simeq Y$ and $Y \simeq Z$, then $X \simeq Z$.

Proof.

(i) Take $f = g = \mathrm{id}_X$, and constant N/A.

(ii) Given $f : X \to Y$, $g : Y \to X$ with $f \circ g \simeq_H \mathrm{id}_Y$, $g \circ f \simeq_G \mathrm{id}_X$, this is the same data as a $Y \simeq X$.

(iii) Suppose we have maps $f : X \to Y$, $f' : Y \to Z$, $g : Y \to X$, $g' : Z \to Y$, with $f \circ g \simeq \mathrm{id}_X$, $f' \circ g' \simeq \mathrm{id}_X$, $f' \circ g' \simeq \mathrm{id}_Z$, $g' \circ f' \simeq \mathrm{id}_Y$. Consider $f' \circ f : X \to Z$, $g \circ g' : Z \to X$. Then

$$(g \circ g') \circ (f' \circ f) = g \circ (g' \circ f') \circ f \simeq g \circ \mathrm{id}_Y \circ f = g \circ f \simeq \mathrm{id}_X,$$

and the other composition is similar. □

Definition (Retraction)**.** If $i : A \to X$ is the inclusion of a subspace, then

(i) A *retraction* is a $r : X \to A$ such that $r \circ i = \mathrm{id}_A$.

(ii) A *deformation* retraction is a retraction such that also $i \circ r \simeq \mathrm{id}_X$.

Paths

Definition (Path)**.** For a space X and points $x_0, x_1 \in X$, a path from x_0 to x_1 is a map $\gamma : I = [0,1] \to X$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$.

If $x_0 = x_1$, call γ a *loop* based at x_0.

Notation (Concatenation)**.** If γ is a path from x_0 to x_1, and γ' is a path from x_1 to x_2, then we can form the *concatenation* $\gamma \cdot \gamma' : I \to X$ via

$$(\gamma \cdot \gamma')(t) = \begin{cases} \gamma(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma'(2t+1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

(continuous by the Gluing lemma). This is a path from x_0 to x_2.

Notation (Inverse path)**.** Define the *inverse* $\gamma^{-1} : I \to X$ via

$$\gamma^{-1}(t) = \gamma(1-t).$$

This is a path from x_1 to x_0.

Notation (Constant path)**.** Define the *constant path* $c_{x_0} : I \to X$ via $c_{x_0}(t) = x_0$.

Definition (Path components)**.** Using the above, we can define an equivalence relation $\sim$ on X via

$$X_0 \sim X_1 \iff \text{there exists a path } \gamma \text{ from } x_0 \text{ to } x_1$$

The equivalence classes of $\sim$ are called *path components* of X. Say X is *path-connected* if there is only 1 equivalence class. Let $\pi_0(X) := X/\sim$.

Definition (Locally path-connected)**.** We say that a space X is *locally path-connected* if for every $x \in X$, and neighbourhood $U \ni x$, there exists a smaller neighbourhood $U \supset V \ni x$ such that V is path-connected.

Proposition. For a map $f : X \to Y$, there is a well-defined function $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ given by $\pi_0(f)([x]) = [f(x)]$. Furthermore:

(i) If $f \simeq g$ then $\pi_0(f) = \pi_0(g)$.

(ii) If $A \xrightarrow{h} B \xrightarrow{k} C$ then $\pi_0(k \circ h) = \pi_0(k) \circ \pi_0(h)$.

(iii) $\pi_0(\mathrm{id}_X) = \mathrm{id}_{\pi_0(X)}$.

Proof. To see well-defined, let $[x] = [x']$. Then there is a path γ from x to x'. Then $f \circ \gamma : I \xrightarrow{\gamma} X \xrightarrow{f} Y$ is a path for $f(x)$ to $f(x')$, so $[f(x)] = [f(x')]$. Properties (ii) and (iii) are immediate. For (i), let $H : X \times I \to Y$ be a homotopy from f to g. Now $H|_{\{x\} \times I} : \{x\} \times I \to Y$ is a path from $f(x)$ to $g(x)$, so

$$\pi_0(f)([x]) = [f(x)] = [g(x)] = \pi_0(g)([x]). \qquad \square$$

Corollary. If $f : X \to Y$ is a homotopy equivalence, then $\pi_0(f)$ is a bijection.

Proof. If $g : Y \to X$ is a homotopy inverse, then

$$\pi_0(f) \circ \pi_0(g) = \pi_0(f \circ g) = \pi_0(\mathrm{id}_Y) = \mathrm{id}_{\pi_0(Y)}$$

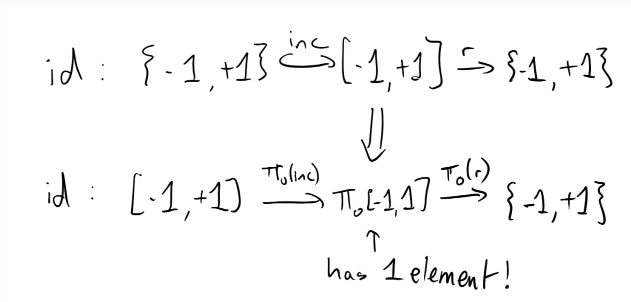
and $\pi_0(g) \circ \pi_0(f) = \mathrm{id}_{\pi_0(X)}$, so $\pi_0(f)$ is a bijection. $\square$

Example. The space $\{-1, +1\}$ with the discrete topology is *not* contractible. This is because any path in this space is constant, so

$$\pi_0(\{-1, +1\}) = \{-1, +1\}$$

of cardinality 2, so $\pi_0(\{*\}) = \{*\}$ has cardinality 1.

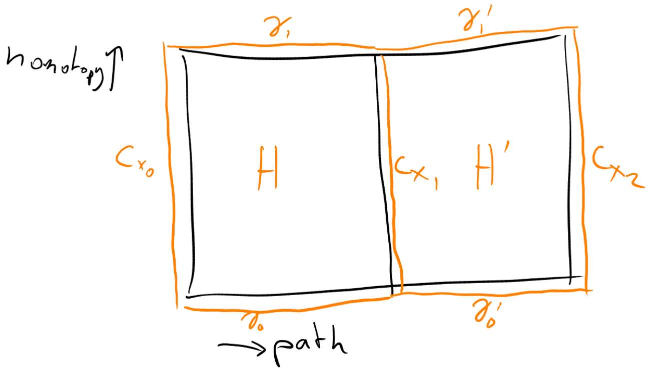
Example. The space $[-1, 1]$ does *not* retract onto $\{-1, +1\}$. Suppose it does. Then:



Definition (Homotopic paths)**.** Two paths $\gamma, \gamma' : I \to X$ both from x_0 to x_1 are called *homotopic as paths* if they are homotopic relative to $\{0, 1\} \subset I$ as in the last lecture. So $\gamma \simeq \gamma' \operatorname{rel} \{x_0, x_1\}$.

Lemma. If $\gamma_0 \simeq \gamma_1$ as paths from x_0 to x_1, and $\gamma_0' \simeq \gamma_1'$ as paths from x_1 to x_2, then $\gamma_0 \cdot \gamma_0' \simeq \gamma_1 \cdot \gamma_1'$ as paths from x_0 to x_2.

Proof. Let H be the homotopy from γ_0 to γ_1 relative to $\{x_0, x_1\}$, and H' the homotopy from γ_0' to γ_1' relative to x_1 and x_2.



$$H''(s,t) = \begin{cases} H(2s,t) & 0 \le s \le \frac{1}{2} \\ H'(2s-1,t) & \frac{1}{2} \le s \le 1 \end{cases}$$

(continuous as usual by the Gluing lemma), is a homotopy from $\gamma_0 \cdot \gamma_0'$ to $\gamma_1 \cdot \gamma_1'$ relative to x_0 and x_2. □

Start of lecture 3

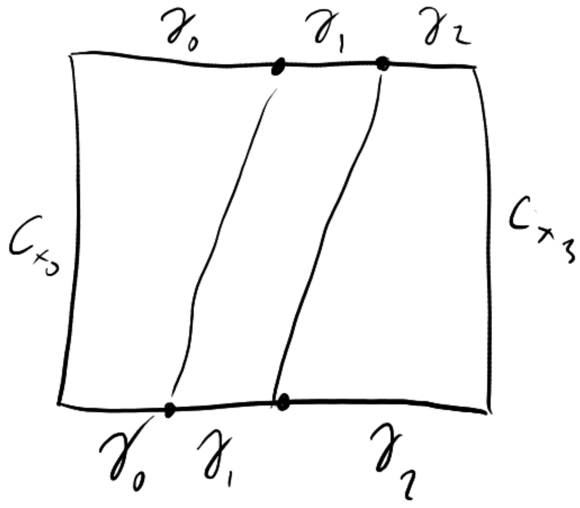
Proposition. Let γ_0 be a path from x_0 to x_1, γ_1 a path from x_1 to x_2 and γ_2 a path from x_2 to x_3. Then

(i) $\gamma_0 \cdot \gamma_1) \cdot \gamma_2 \simeq \gamma_0 \cdot (\gamma_1 \cdot \gamma_2)$ relative to x_0 and x_3.

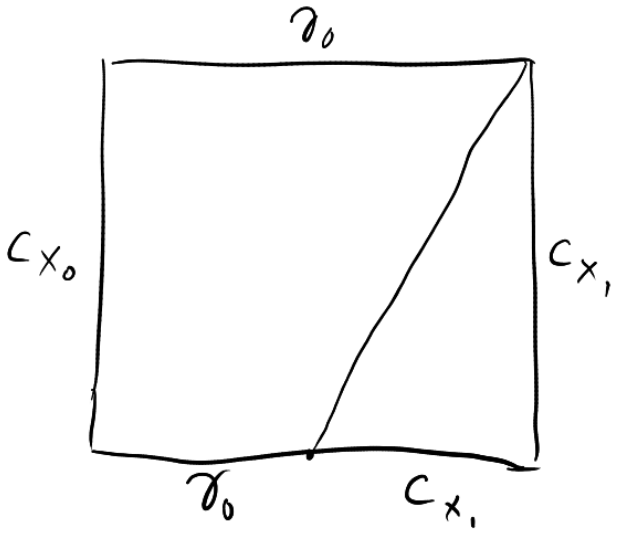
(ii) $\gamma_0 \cdot c_{x_1} \simeq \gamma_0 \simeq c_{x_0} \cdot \gamma_0$ relative to x_0 and x_1.

(iii) $\gamma_0 \cdot \gamma_0^{-1} \simeq c_{x_0}$ relative to x_0 and x_0, $\gamma_0^{-1} \cdot \gamma_0 \simeq c_{x_1}$ relative to x_1 and x_1.

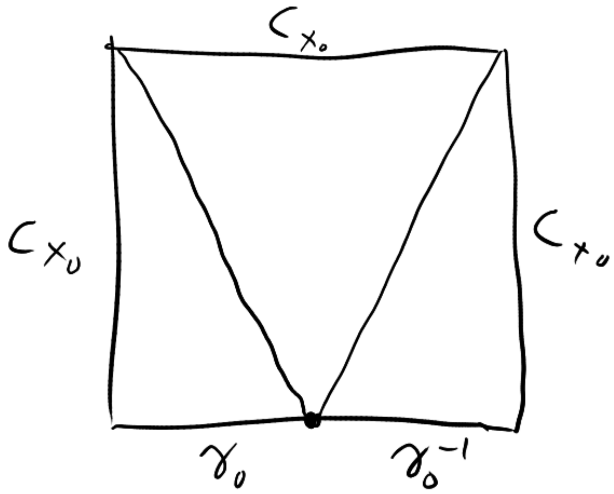
Proof. We illustrate some of the cases (other cases are similar) using diagrams and the corresponding formulae:



$$H(s,t) = \begin{cases} \gamma_0\left(\frac{4s}{t+1}\right) & 0 \leq s \leq \frac{t+1}{4} \\ \gamma_1(4s-1-t) & \frac{t+1}{4} \leq s \leq \frac{t+2}{4} \\ \gamma_2\left(1 - \frac{4(1-s)}{2-t}\right) & \frac{t+2}{4} \leq s \leq 1 \end{cases}$$



$$H(s,t) = \begin{cases} \gamma_0\left(\frac{2s}{t+1}\right) & 0 \leq s \leq \frac{t+1}{2} \\ x_1 & \frac{t+1}{2} \leq s \leq 1 \end{cases}$$



$$H(s,t) = \begin{cases} \gamma_0(2s) & 0 \le s \le \frac{1-t}{2} \\ \gamma_0(1-t) & \frac{1-t}{2} \le s \le \frac{1+t}{2} \\ \gamma_0(2-2s) & \frac{1+t}{2} \le s \le 1 \end{cases}$$

□

2.2 The fundamental group

Definition (Fundamental group)**.** Let X be a space and $x_0 \in X$. Let $\pi_1(X, x_0)$ be the set of homotopy classes of loops in X starting and ending at x_0. Then the rule

$$[\gamma] \cdot [\gamma'] = [\gamma \cdot \gamma']$$

makes $(\pi_1(X, x_0), \cdot, [c_{x_0}])$ into a group.

Claim: This definition does actually make $\pi_1(X, x_0)$ into a group.

Proof. Lemma from last lecture shows it is well-defined. Previous proposition gives the group axioms. □

Definition (Based space)**.** A *based space* is a space X with a chosen point $x_0 \in X$ called the base point, (X, x_0).

A *map of based spaces* $f : (X, x_0) \to (Y, y_0)$ is a map $X \xrightarrow{f} Y$ such that $f(x_0) = y_0$. A *based homotopy* is a homotopy relative to $[x_0] \subset X$.

Proposition. To a based map $f : (X, x_0) \to (Y, y_0)$ there is associated a function $\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $\pi_1(f)([\gamma]) = [f \circ \gamma]$. It satisfies:

(i) It is a group homomorphism.

(ii) If f is based homotopic to f', $\pi_1(f) = \pi_1(f')$.

(iii) If $(A, a) \xrightarrow{h} (B, b) \xrightarrow{k} (C, c)$ are based maps, then $\pi_1(k \circ h) = \pi_1(k) \circ \pi_1(h)$.

(iv) $\pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X, x_0)}$.

Proof. (i) $f \circ c_{x_0} = c_{y_0}$ so $\pi_1(f)$ preserves identity element. $f \circ (\gamma \cdot \gamma') = (f \circ \gamma) \cdot (f \circ \gamma')$, hence $\pi_1(f)$ is a homomorphism.

(ii) – (iv) Elementary.

□

Notation. We will write $\pi_1(f) =: f_*$.

Proposition. Let u be a path from x_0 to x_1 in X. It induces a group isomorphism

$$\begin{aligned} u_\# : \pi_1(X, x_0) &\to \pi_1(X, x_1) \\ [\gamma] &\mapsto [u^{-1} \cdot \gamma \cdot u] \end{aligned}$$

satisfying

(i) If $u \simeq u'$ as paths then $u_\# = u'_\#$.

(ii) $(c_{x_0})_\# = \mathrm{id}_{\pi_1(X,x_0)}$.

(iii) If v is a path from x_1 to x_2 then $(u \cdot v)_\# = v_\# \cdot u_\#$.

(iv) If $f : X \to Y$ sends x_0 to y_0 and x_1 to y_1, then

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f} & \pi_1(Y, y_0) \\ \big\downarrow u_\# & & \big\downarrow (f \circ u)_\# \\ \pi_1(X, x_1) & \xrightarrow{f_*} & \pi_1(Y, y_1) \end{array}$$

"this diagram commutes", i.e.

$$(f \circ u)_\# \circ f_* = f_* \circ u_\#$$

(v) If u is a path from x_0 to x_0, then $u_\#$ is conjugation by $[u] \in \pi_1(X, x_0)$.

Proof. $u_\#$ is a group homomorphism via

$$\begin{aligned} u_\#([\gamma]) \cdot u_\#([\gamma']) &= [a^{-1} \cdot \gamma \cdot a] \cdot [a^{-1} \cdot \gamma' \cdot u] \\ &= [u^{-1} \cdot \gamma \cdot u \cdot u^{-1} \cdot \gamma' \cdot u] \\ &= [u^{-1} \cdot \gamma \cdot \gamma' \cdot u] \\ &= u_\#([\gamma] \cdot [\gamma']) \end{aligned}$$

It is an isomorphism as $(u^{-1})_\#$ is an inverse. For (iv),

$$\begin{aligned} ((f \circ u)_\# \circ f_*)([\gamma]) &= (f \circ u)_\#([f \circ \gamma]) \\ &= [(f \circ u)^{-1} \cdot f \circ \gamma \cdot f \circ u] \\ &= [f \circ (u^{-1} \cdot \gamma \cdot u)] \\ &= f_*([u^{-1} \cdot \gamma \cdot u]) \\ &= (f_* \circ u_\#)([\gamma]) \end{aligned}$$

□

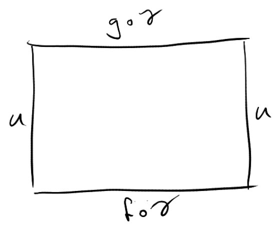
Lemma. If $H : X \times I \to Y$ is a homotopy fomr f to g and $x_0 \in X$ is a base point base point, then $u = H(x_0, \bullet) : I \to Y$ is a path from $f(x_0)$ to $g(x_0)$.

Then

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, f(x_0)) \\ \downarrow{\scriptstyle g_*} & \swarrow{\scriptstyle u_\#} & \\ \pi_1(Y, g(x_0)) & & \end{array}$$

commutes (i.e. $u_\# \circ f_* = g_*$).

Proof. $I \times I \overset{\gamma \times \mathrm{id}}{\to} X \times I \overset{H}{\to} Y$.



Want $g \circ \gamma \simeq u^{-1} \cdot (f \circ \gamma) \cdot u$ as loops. The top edge in the square is homotopic to the concatenation of the other three edge. Applying $H \circ (\gamma \times \mathrm{id}_I)$ gives the required homotopy. □

Start of lecture 4

Theorem. If $f : X \to Y$ is a homotopy equivalence, $x_0 \in X$, then

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

is an isomorphism.

Proof. Let $g; Y \to X$ be a homotopy inverse, $f \circ g \simeq_H \mathrm{id}_Y$, $g \circ f \simeq_{H'} \mathrm{id}_X$. Let $u' : I \to X$ be

$$u'(t) = H'(x_0, 1 - t),$$

a path from x_0 to $g \circ f(x_0)$. The previous lemma gives an isomorphism

$$u'_\# = (g \circ f)_* \cdot \pi_1(X, x_0) \overset{f_*}{\hookrightarrow} \pi_1(Y, f(x_0)) \overset{g_*}{\twoheadrightarrow} \pi_1(X_1, gf(x_0))$$

We want to show ↔ is an isomorphism, and to do this it is enough to show ↠ is injective. To show g_* is injective, consider $u(t) = H(g(x_0), 1 - t)$ and get an isomorphism

$$u_\# = (f \circ g)_* \cdot \pi_1(Y, f(x_0)) \overset{g_*}{\hookrightarrow} \pi_1(X, gf(x_0)) \overset{f_*}{\twoheadrightarrow} \pi_1(Y, fgf(x_0))$$

So g_* is injective, so the f_* above is $g^{-1}{}_* \circ u'_\#$, an isomorphism. □

Definition (Simply connected)**.** A space X is *simply-connected* if it is path-connected and $\pi_1(X, x_0) = \{e\}$ for some (hence all) $x_0 \in X$.

Example. A contractible space is simply-connected. $X \simeq \{*\}$, so $\pi_0(X)$ and $\pi_1(X, x_0)$ are trivial.

Lemma. X is simply-connected if and only if $\forall x_0, x_1 \in X$, there is a unique homotopy class of paths form x_0 to x_1.

Proof. Let X be simply-connected, and $x_0, x_1 \in X$. As X is path-connected, there *exists* a path from x_0 to x_1. If γ, γ' are two such paths, then $\gamma^{-1} \cdot \gamma$ is a loop based at x_1, so $[\gamma^{-1}\gamma'] \in \pi_1(X, x_1) = \{e\}$, so $\gamma^{-1} \cdot \gamma' \simeq c_{x_1}$ relative to x_1. So $\gamma' \simeq \gamma \cdot \gamma^{-1} \cdot \gamma' \simeq \gamma \cdot c_{x_1} \simeq \gamma$ relative to end points.

Conversely, if X has the stated property then:

(i) It is path-connected.

(ii) Any loop based at x_0 is homotopic to c_{x_0} as loops.

Hence $\pi_1(X, x_0) = \{e\}$. □

Covering Spaces

Definition (Covering map)**.** A *covering map* $p : \tilde{X} \to X$ is a continuous map such that for any $x \in X$ there exists an open neighbourhood $U \ni x$ such that

$$p^{-1}(U) = \coprod_{\alpha \in I} V_\alpha$$

and $p|_{V_\alpha} \to U$ is a homeomorphism.

Example. Let $S^1 \subset \mathbb{C}$ be the unit complex numbers, and

$$\begin{aligned} p : \mathbb{R} &\to S^1 \\ t &\mapsto e^{2\pi i t} \qquad = (\cos(2\pi t), \sin(2\pi t)) \end{aligned}$$

Let $U_{y>0} = \{x + iy \in S^1 \mid y > 0\}$. Then

$$p^{-1}(U_{y>0}) = \coprod_{j \in \mathbb{Z}} \left(j, j + \frac{1}{2}\right)$$

Now

$$\begin{aligned} p|_{(j, j+\frac{1}{2})} : \left(j, j + \frac{1}{2}\right) &\to U_{y>0} \\ j + \frac{\arccos(x)}{2\pi} &\leftarrow\!\shortmid x + iy \end{aligned}$$

Similarly for $U_{y<0}$, $U_{x<0}$, $U_{x>0}$. So p is a covering map.

Example. For some $n > 0$, let

$$\begin{aligned} p : S^1 &\to S^1 \\ z &\mapsto z^n \end{aligned}$$

Let $y \in S^1$ and consider $p^{-1}(y) = \{n\text{-th roots of } y\}$. Choosing a root ξ and letting $\eta = e^{\frac{2\pi i}{n}}$, this can be written as

$$p^{-1}(y) = \{\xi, \eta\xi, \eta^2\xi, \ldots, \eta^{n-1}\xi\}$$

Then $S^1 - \{y\}$ is open and

$$p^{-1}(S^1 - \{y\}) = S^1 - \{\xi, \eta\xi, \eta^2\xi, \ldots, \eta^{n-1}\xi\}$$

$V_0 = \left\{z \in S^1 \mid 0 < \arg\left(\frac{z}{\xi}\right) < \frac{2\pi}{n}\right\}$ and $V_i = \eta^i \cdot V_0$. Each $x \neq y \in S^1$ has a unique n-th root in each V_i, so $p|_{V_i} : V_i \to S^1 - \{y\}$ is a bijection: in fact a homeomorphism.

Example. $S^2 \subset \mathbb{R}^3$ be the unit vectors. Let

$$\mathbb{RP}^2 = S^2 / x \sim -x.$$

Have

$$\begin{aligned} p : S^2 &\to \mathbb{RP}^2 \\ x &\mapsto [x] \end{aligned}$$

Let $V = \{(x, y, z) \in S^2 \mid z \neq 0\}$, and $U = p(V)$. Then $p^{-1}(U) = V$ is open, so U is open in $\mathbb{RP}^2$. Now $p^{-1}(U) = V = V_{z>0} \amalg V_{z<0}$.

Claim: $p|_{V_{z>0}} : V_{z>0} \to U$ and $p|_{V_{z>0}}$ are homeomorphisms.

Proof. To construct an inverse $g : U \to V_{z>0}$, use definition of quotient topology. Consider

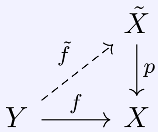
$$\begin{aligned} t : V &\to V_{z>0} \\ (x, y, z) &\mapsto (x, y, z) \text{ if } z \geq 0 \\ &\mapsto (x, y, -z) \text{ if } z < 0 \end{aligned}$$

a continuous map. Note t descends to $\bar{t} : U \to V_{z>0}$ as a map of sets, so a continuous map by definition of quotient toplogy. It is inverse to $p|_{V_{z>0}}$. □

Do same with x-coordinate and y-coordinate to show that this is a covering map of $\mathbb{RP}^2$.

Start of lecture 5

Definition (Lift)**.** Let $p : \tilde{X} \to X$ be a covering map, $f : Y \to X$ be a map. A *lift* of f along p is a $\tilde{F} : Y \to \tilde{X}$ such that $p \circ \tilde{f} = f$.



Definition (Evenly-covered)**.** Given $p : \tilde{X} \to X$ a covering map, we say that $U \subset X$ is evenly-covered by $\{V_\alpha : \alpha \in I\}$ if $p^{-1}(U) = \coprod_{\alpha \in I} V_\alpha$ and $p|_{V_\alpha} : V_\alpha \xrightarrow{\cong} U_\alpha$.

Lemma (Uniqueness of Lifts)**.** If $\tilde{f}_0$ and $\tilde{f}_1$ are lifts of $f : Y \to X$ along a covering map $p : \tilde{X} \to X$ then

$$S := \{y \in Y \mid \tilde{f}_0(y) = \tilde{f}_1(y)\}$$

is both open and closed. In particular, if Y is connected, then $S = \emptyset$ or $S = Y$.

Proof. First show S is open. Let $s \in S$ and let $U \ni f(s)$ be an open neighbourhood which is evenly-covered ($p^{-1}(U) = \bigsqcup V_\alpha$). Now $\tilde{f}_0(s)$ and $\tilde{f}_1(s)$ agree so live in the same V_α. Then on $N = \tilde{f}_0^{-1}(V_\alpha) \cap \tilde{f}_1^{-1}(V_\alpha)$ we have

$$p|_{V_\alpha} \circ \tilde{f}_0|_N = f|_N = p|_{V_\alpha} \circ \tilde{f}_1|_N,$$

but $p|_{V_\alpha}$ is a homeomorphism, so $\tilde{f}_0|_N = \tilde{f}_1|_N$. So $s \in N \subset S$, so S is open.

Now we show S is closed. Let $y \in \overline{S}$ and $\tilde{f}_0(y) \neq \tilde{f}_1(y)$. Let $U \ni f(y)$ be an open neighbourhood that is evenly-covered. Then $\tilde{f}_0(y) \in V_\beta$ and $\tilde{f}_1(y) \in V_\gamma$ with $\beta \neq \gamma$ (as $\tilde{f}_0(y) \neq \tilde{f}_1(y)$). So $\tilde{f}_0^{-1}(V_\beta) \cap \tilde{f}_1^{-1}(V_\gamma)$ is an open set, containing $y \in \overline{S}$. By definition of closure, it intersects S. But then V_β and V_γ must intersect, contradiction. $\square$

Lemma (Homotopy lifting lemma)**.** Let $p : \tilde{X} \to X$ be a covering map, $H : Y \times I \to X$ from f_0 to f_1, and $\tilde{f}_0$ be a lift of f_0. Then there exists a *unique* homotopy $\tilde{H} : Y \times I \to \tilde{X}$ such that:

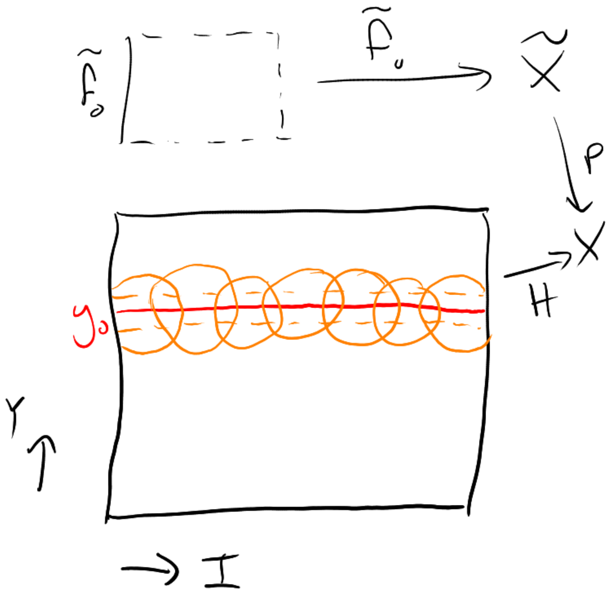
(i) $\tilde{H}(\bullet, 0) = \tilde{f}_0(\bullet)$.

(ii) $p \circ \tilde{H} = H$.

Proof. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X by sets which are evenly-covered, i.e. $p^{-1}(U_\alpha) = \coprod_{\beta \in I_\alpha} V_\beta$ and $p|_{V_\beta} : V_\beta \xrightarrow{\cong} U_\alpha$. Now $\{H^{-1}(U_\alpha)\}$ is an open cover of $Y \times I$, and for each $y_0 \in Y$ it gives an open cover of $\{y_0\} \times I$. By the Lebesgue number lemma, there is a $N = N(y_0)$ such that each path

$$H|_{\{y_0\} \times \left[\frac{i}{N}, \frac{i+1}{N}\right]} : \{y_0\} \times \left[\frac{i}{N}, \frac{i+1}{N}\right] \to X$$

has image inside some U_α.



In fact, as $\{y_0\}\times I$ is compact, there is an open $W_{y_0} \ni y_0$ such that each $H\left(W_{y_0} \times \left[\frac{i}{N}, \frac{i+1}{N}\right]\right)$ lies in some U_α. We can construct a lift $\tilde{H}|_{W_{y_0}\times I}$ as follows:

(i) $H|_{W_{y_0}\times[0,\frac{1}{N}]} \to U_\alpha \subset X$ and have $\tilde{f}_0|_{W_{y_0}} : W_{y_0} \to \tilde{X}$ with image in some V_β lying in U_α.

Define $\tilde{H}|_{W_{y_0}\times[0,\frac{1}{N}]} \overset{H|}{\to} U_\alpha \underset{\cong}{\overset{p|_{V_\beta}^{-1}}{\to}} V_\beta \subset \tilde{X}$.

(ii) Proceed in the same way, lifting $H|_{W_{y_0}\times[\frac{1}{N},\frac{2}{N}]}$ starting at $\tilde{H}|_{W_{y_0}\times\{\frac{1}{N}\}}$ etc.

At the end of this process, we get a $\tilde{H}|_{W_{y_0}\times I}$ lifting $H|_{W_{y_0}\times I}$ and extending $\tilde{f}_0$ at time 0. We can do this for each $y_0 \in Y$, so it is enough to check that on

$$(W_{y_0} \times I) \cap (W_{y_1} \times I) = (W_{y_0} \cap W_{y_1}) \times I,$$

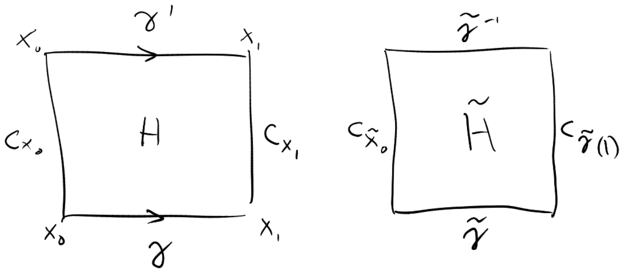
the two lifts constructed agree.

For a $y_2 \in W_{y_0} \cap W_{y_1}$, the two choices give lifts of $H|_{\{y_2\}\times I}$ which agree with $\tilde{f}_0(y_2)$ at time 0. By the Uniqueness of Lifts, these lifts must agree on the whole of $\{y_2\} \times i$. So they agree. □

Corollary (Path lifting)**.** If $p : \tilde{X} \to X$ is a covering map, then $\gamma : I \to X$ is a path from x_0 to x_1, and $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$. Then there is a unique path $\tilde{\gamma} : I \to \tilde{X}$ such that $\tilde{\gamma}(0) = \tilde{x}_0$ and $p \circ \tilde{\gamma} = \gamma$.

Corollary. Let $p : \tilde{X} \to X$ be a covering map, $\gamma, \gamma' : I \to X$ be paths from x_0 to x_1, and $\tilde{\gamma}, \tilde{\gamma}' : I \to \tilde{X}$ be their lifts starting at $\tilde{x}_0 \in p^{-1}(x_0)$. If $\gamma \simeq \gamma'$ as paths, then $\tilde{\gamma} \simeq \tilde{\gamma}'$ as paths. In particular, $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$.

Proof. Let $H : I \times I \to X$ be a homotopy from γ to γ' relative to endpoints.



Homotopy lifting lemma gives a $\tilde{H} : I \times I \to \tilde{X}$, which is a homotopy of paths. □

Corollary. Let $p : \tilde{X} \to X$ be a covering map and X be path-connected. Then the sets $p^{-1}(x)$ are all in bijection with each other.

Proof. Let $\gamma : I \to X$ be a path from x_0 to x_1. Define

$$\begin{aligned} \gamma_* : p^{-1}(x_0) &\to p^{-1}(x_0) \\ y_0 &\mapsto \tilde{\gamma}(1) \end{aligned}$$

for $\tilde{\gamma}$ the lift of γ starting at y_0. Similarly $(\gamma^{-1})_* : p^{-1}(x_1) \to p^{-1}(x_0)$. Then

$$\begin{aligned} (\gamma^{-1})_* \gamma_*(y_0) &= \text{end point of the lift of } \gamma^{-1} \cdot \gamma \text{ which starts at } y_0 \\ &= \text{end point of } c_{y_0} \\ &= y_0 \end{aligned}$$

□

Start of lecture 6

Lemma. Let $p : \tilde{X} \to X$ be a covering map, $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0) \subset \tilde{X}$. Then

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$$

is an injective homomorphism.

Proof. Let $\gamma : I \to \tilde{X}$ be a loop based at $\tilde{x}_0$, and suppose $p_*[\gamma] = [p \circ \gamma] = [c_{x_0}]$ so $p \circ \gamma \simeq_H c_{x_0}$ as loops. Now lift H to a homotopy $\tilde{H}$ starting at γ (by Path lifting): then $\tilde{H}$ is a homotopy of paths from γ to a lift of c_{x_0}, which must be $c_{\tilde{x}_0}$ (by uniqueness). So $[\gamma] = [c_{\tilde{x}_0}] = e \in \pi_1(\tilde{X}, \tilde{x}_0)$. □

In the proof of the previous Corollary we constructed for a path $\gamma : I \to X$ from x_0 to x_1 a bijection $\gamma_* : p^{-1}(x_0) \to p^{-1}(x_1)$. It only depended on the homotopy class of the path γ. This defines a (right) action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$, via

$$y_0 \cdot [\gamma] := \tilde{\gamma}(1)$$

for $\tilde{\gamma}$ *the* lift of γ starting at y_0.

Lemma. Let $p : \tilde{X} \to X$ be a covering map, X path-connected and $x_0 \in X$. Then:

(i) $\pi_1(X, x_0)$ acts transitively on $p^{-1}(x_0)$ if and only if $\tilde{X}$ is path-connected.

(ii) The stabiliser of $y_0 \in p^{-1}(x_0)$ is $\mathrm{Im}(\pi_1(\tilde{X}, y_0) \xrightarrow{p_*} \pi_1(X, x_0)) \leq \pi_1(X, x_0)$.

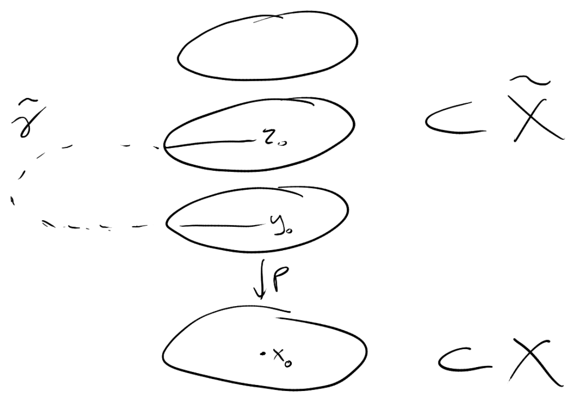
(iii) If $\tilde{X}$ is path-connected then there is a bijection

$$\frac{\pi_1(X, x_0)}{p_*\pi_1(\tilde{X}, y_0)} \to p^{-1}(x_0)$$

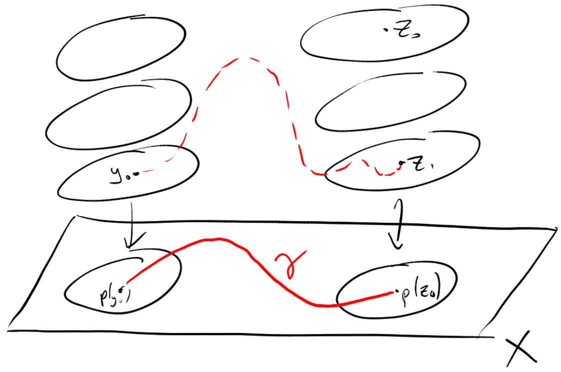
induced by acting on $y_0 \in p^{-1}(x_0)$.

Proof.

(i) Let $\tilde{X}$ be path-connected, and $y_0, z_0 \in p^{-1}(x_0)$. Let $\tilde{\gamma} : I \to \tilde{X}$ be a path from y_0 to z_0, so $\gamma = p \circ \tilde{\gamma} : I \to X$ is a loop based at x_0. *The* lift of γ starting at y_0 is $\tilde{\gamma}$, and ends at z_0, so $y_0 \cdot [\gamma] = z_0$, so transitive.



Conversely, suppose the action is transitive, let $y_0, z_0 \in \tilde{X}$. Choose a path γ from $p(y_0)$ to $p(z_0)$, lift it starting at y_0, then it ends at z_1, lying in the same fibre as z_0.



Suppose y_0 and z_0 are *not* in the same path-component. Then z_1 and z_0 are also not. But taking $x_0' = p(z_0)$, $\pi_1(X, x_0')$ acts transitively on $p^{-1}(x_0') \ni z_0, z_1$. So there exiss a loop γ in X whose lift starting at z_0 ends at z_1, i.e. z_0 and z_1 are in the same path-component.

(ii) Suppose $y_0 \cdot [\gamma] = y_0$, then the lift $\tilde{\gamma}$ of γ starting at y_0 also ends at y_0, so is a loop. Then $[\gamma] = p_*[\tilde{\gamma}]$, so

$$\mathrm{Stab}_{\pi_1(X,x_0)}(p^{-1}(x_0)) \le \mathrm{Im}(p_*).$$

If $[\gamma] = p_*[\gamma']$ then γ' is the lift of γ starting at y_0, but also ends at y_0 so $y_0 \cdot [\gamma] = y_0$.

(iii) Orbit-stabiliser. □

Definition (n-sheeted)**.** If $p : \tilde{X} \to X$ is a covering map we say that it is *n-sheeted* if $p^{-1}(x_0)$ all have the same cardinality $n \in \mathbb{N} \cup \{\infty\}$.

Definition (Universal cover)**.** If $p : \tilde{X} \to X$ is a covering map we say that it is a *universal cover* if $\tilde{X}$ is simply-connected.

Corollary. If $p : \tilde{X} \to X$ is a universal cover, then each $\tilde{x}_0 \in p^{-1}(x_0)$ determines a bijection

$$l : \pi_1(X, x_0) \xrightarrow{\sim} p^{-1}(x_0)$$
$$[\gamma] \mapsto \tilde{\gamma}(1)$$

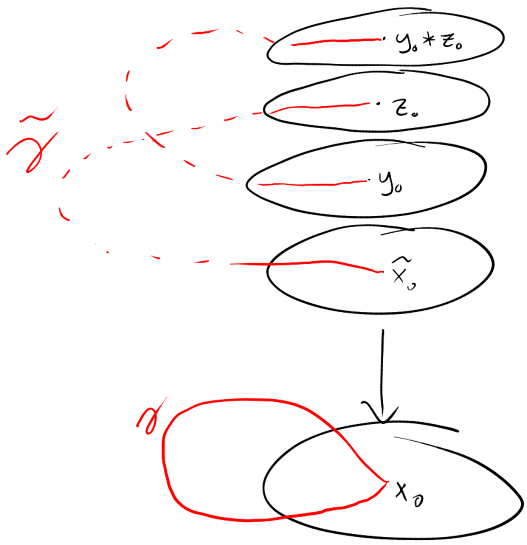
for $\tilde{\gamma}$ the lift of γ starting at $\tilde{x}_0$.

This induces a group-law on $p^{-1}(x_0)$ via

$$y_0 * z_0 := l(l^{-1}(y_0) \cdot l^{-1}(z_0))$$

Spelling out, this is given by:

- Choose a path $\tilde{\gamma} : I \to \tilde{X}$ from $\tilde{x}_0$ to z_0,
- Let γ be the lift of $p \circ \tilde{\gamma}$ starting at y_0,
- $y_1 * z_1 = \gamma(1)$



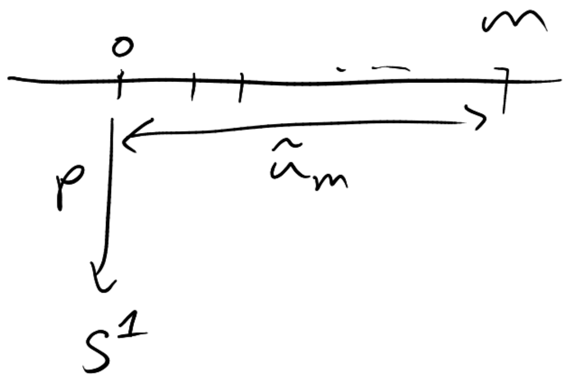
3.1 Fundamental group of S^1

Theorem. Let $u : I \to S^1$ be $u(s) = e^{2\pi i s}$, which is based at $1 \in S^1 \subset \mathbb{C}$. Then there is an isomorphism $\pi_1(S^1, 1) \cong (\mathbb{Z}, +, 0)$ sending u to $1 \in \mathbb{Z}$.

Proof. We know $p : \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$ is a covering map. $\mathbb{R}$ is contractible so simply-connected. So this is a universal cover. Hence

$$l : \pi_1(S^1, 1) \xrightarrow{\sim} p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$$

is a bijection. To compute $l^{-1}(m)$ we can take $\tilde{u}_m : I \to \mathbb{R}$, $t \mapsto mt$, so $u_m = p \circ \tilde{u}_m$ is a loop in S^1. Take $\tilde{x}_0 = 0 \in \mathbb{Z}$.



So

$$\begin{aligned} n * m &= \text{the end point of the lift of } u_m \text{ starting at } n \\ &= (t \mapsto n + mt)|_{t=1} \end{aligned}$$
$$n + m$$

So $* = +$. □

Theorem. The disc D^2 does not retract to its boundary S^1.

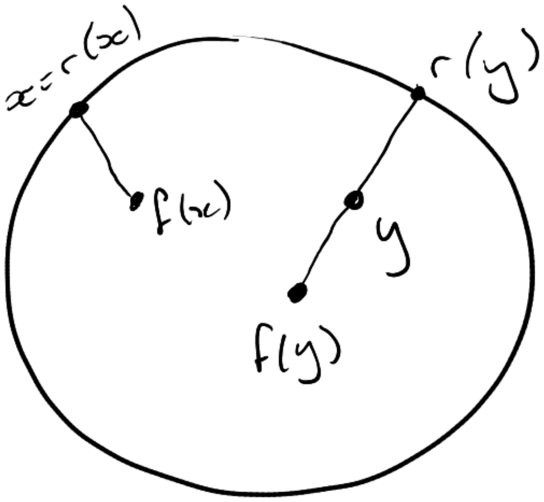
Proof. Suppose $r : D^2 \to S^1$ is a retraction, $i : S^1 \hookrightarrow D^2$, so $r \circ i = \mathrm{id}_{S^1}$:

$$\mathrm{id} : \underbrace{\pi_1(S^1, 1)}_{\cong \mathbb{Z}} \xrightarrow{i_*} \underbrace{\pi_1(D^2, 1)}_{\cong \{e\}} \xrightarrow{r_*} \underbrace{\pi_1(S^1, 1)}_{\cong \mathbb{Z}}$$

This is clearly not possible, so r does not exist. □

> **Corollary** (Brouwer fixed point theorem)**.** Any map $f : D^2 \to D^2$ has a fixed point.

Proof. Suppose not. Define $r : D^2 \to S^1$ as shown:



This would be a retraction onto the boundary, contradictin the previous theorem. □

Start of lecture 7

> **Theorem** (Fundamental Theorem of Algebra)**.** Any non-constant polynomial over $\mathbb{C}$ has a root in $\mathbb{C}$.

Proof. Let $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ a monic polynomial over $\mathbb{C}$. Choose

$$r > \max(|a_1| + |a_2| + \cdots + |a_n|, 1)$$

On the circle $|z| = r$, we have

$$\begin{aligned} |z^n| &= |z^{n-1}|r \\ &> |z^{n-1}|(|a_1| + \cdots + |a_n|) \\ &> |a_1 z^{n-1} + \cdots + a_n| \end{aligned}$$

Therefore for any $t \in [0, 1]$, the polynomial

$$p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$$

does not have a root on the circle $|z| = r$. Consider the homotopy of loops in $S^1 \subset \mathbb{C}$:

$$F(s,t) = \frac{p_t(r \cdot e^{2\pi i s})/p_t(r)}{|p_t(r \cdot e^{2\pi i s})/p_t(r)|}.$$

The divisions is allowed since we know that at all times t, the polynomial does not have a root on the circle $|z| = r$. At $t = 0$, it is the loop $s \mapsto e^{2\pi i s n}$. This is $n \in \mathbb{Z} \cong \pi_1(S^1, 1)$. At $t = 1$, it is the loop

$$s \overset{f_r}{\to} \frac{p(r \cdot e^{2\pi i s})/p(r)}{|p(r \cdot e^{2\pi i s})/p(r)|}.$$

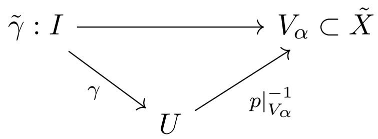
This also represents $n \in \mathbb{Z} \cong \pi_1(S^1, 1)$. Suppose p has no roots. Then f_n is a continuous loop for all $r \in [0, \infty)$ and varying r gives a homotopy from f_r to f_0. Note that the loop f_0 is:

$$s \mapsto f_0(s) = \frac{p(r)/p(r)}{|p(r)/p(r)|} = 1$$

This corresponds to the constant loop, i.e. $0 \in \mathbb{Z} \cong \pi_1(S^1, 1)$. Hence we must have $n = 0 \in \mathbb{Z}$, so p was constant. □

3.2 Construction of universal covers

Observation 1: Let $p : \tilde{X} \to X$ a universal cover, $x \in X$. Let $U \ni x$ a neighbourhood which is evenly-covered (i.e. $p^{-1}(U) = \coprod V_\alpha$). Let $\gamma : I \to U$ be a loop in U based at x_0. This lifts to a



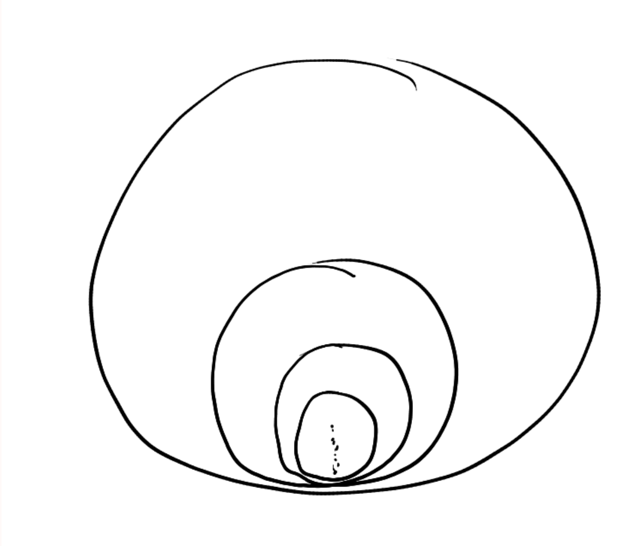
This is homotopic to a constant loop in $\tilde{X}$, as $\tilde{X}$ is simply-connected. Applying p gives $\gamma \simeq c_{x_0}$ in X, i.e. "every $x \in X$ has a neighbourhood $U \ni x$ such that the map $\pi_1(U, x) \overset{(\text{inc})_*}{\to} \pi_1(X, x)$ is trivial". We call this property "semi-locally simply connected".

Definition (Semi-locally simply-connected)**.** We say a space X is *semi-locally simply-connected* if every $x \in X$ has a neighbourhood $U \ni x$ such that

$$\pi_1(U, x) \overset{(\text{inc})_*}{\to} \pi_1(X, x)$$

is trivial.

Example. The "Hawaiian earring" is not semi-locally simply-connected:



The circles all share a common tangency point, and we have a circle of radius $\frac{1}{n}$ for each n.

Observation 2: Suppose $p : \tilde{X} \to X$ is a universal cover and $x_0 = p(\tilde{x}_0)$ is a base point. Any $y \in \tilde{X}$ has a *unique* path α from $\tilde{x}_0$ to y (up to homotopy). Then

$$y = \text{the end point of the lift of } p \circ \alpha \text{ starting at } \tilde{x}_0.,$$

i.e. there is a *bijection*

$$\begin{aligned} \tilde{X} &\to \left\{\begin{matrix}\text{homotopy classes of}\\ \text{paths in } X \text{ starting}\\ \text{at } x_0\end{matrix}\right\} \\ y &\mapsto p \circ \alpha \\ \tilde{\gamma}(1) &\leftarrow\!\shortmid [\gamma] \end{aligned}$$

Theorem (Existence of universal covers)**.** Let X be path-connected, locally path-connected, and semi-locally simply-connected. Then it has a universal cover.

Proof (non-examinable). As a set let

$$\tilde{X} := \left\{\begin{matrix}\text{homotopy classes of}\\ \text{paths in } X \text{ starting}\\ \text{at } x_0\end{matrix}\right\},$$

and

$$\begin{aligned} p : \tilde{X} &\to X \\ [\gamma] &\mapsto \gamma(1) \end{aligned}$$

Want:

(i) Make a topology on $\tilde{X}$.

(ii) Show p is continuous.

(iii) Show p is a covering map.

(iv) Show $\tilde{X}$ is simply-connected.

Consider

$$\mathcal{U} = \{U \subset X \mid U \text{ open, path-connected and } \pi_1(U, x) \to \pi_1(X, x) \text{ is trivial } \forall x \in U\}$$

Claim: This is a basis for the topology on X.

Proof: Let $V \ni x$ be an open neighbourhood. Then:

(i) X semi-locally simply-connected $\implies \exists U' \ni x$ such that $\pi_1(U', x) \to \pi_1(X, x)$ is trivial.

(ii) As X is locally path-connected, can fine $V \cap U' \supset U \ni x$ which is path-connected.

(iii) The map

$$\begin{array}{ccc} \pi_1(U, x) & \longrightarrow & \pi_1(X, x) \\ & & \nearrow_{\text{triv}} \\ & \pi_1(U', x) & \end{array}$$

is trivial.

(iv) Let $y \in U$ be another point, and $u : I \to U$ be a path from x to y. Then the following diagram commutes

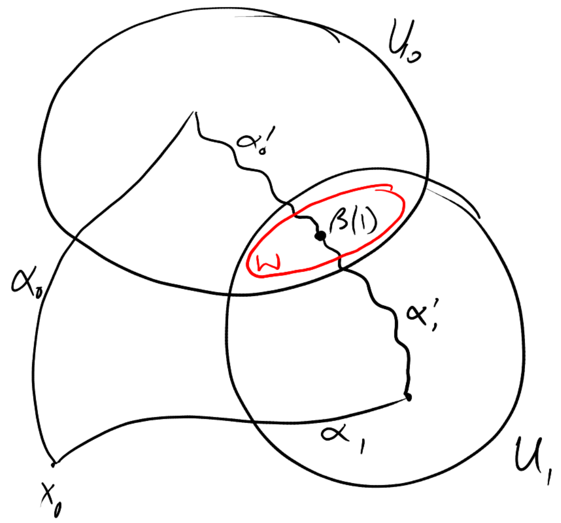
$$\begin{CD} \pi_1(U, y) @>>> \pi_1(X, y) \\ @A{u_\#}A{\simeq}A @A{u_\#}A{\simeq}A \\ \pi_1(U, x) @>{\text{trivial}}>> \pi_1(X, x) \end{CD}$$

which shows the top map is trivial.

This completes the proof of the claim.

For $[\alpha] \in \tilde{X}$ and a $U \in \mathcal{U}$ such that $\alpha(1) \in U$, define

$$([\alpha], U) = \{[\beta] \in \tilde{X} \mid [\beta] = [\alpha \cdot \alpha'] \text{ for some path } \alpha' \text{ in } U\}.$$



Claim: These sets form a basis for a topology on $\tilde{X}$.

Proof: Let $[\beta] \in ([\alpha_0], U_0) \cap (\alpha_1, U_1)$. As in the figure, there are α_0', α_1' with $[\alpha \cdot \alpha_0'] = [\beta] = [\alpha_1 \cdot \alpha_1']$. Let $\beta(1) \in W \subset U_0 \cap U_1$ with $W \in \mathcal{U}$. Want to show

$$([\beta], W) \subset ([\alpha_0], U_0) \cap ([\alpha_1], U_1)$$

If $[\gamma] \in ([\beta], W)$, then there is a path δ in W with

$$[\gamma] = [\beta \cdot \delta] = [\alpha_0 \cdot \underbrace{\alpha_0' \cdot \delta}_{\in U_0}] \in ([\alpha_0], U_0).$$

Similarly

$$[\gamma] = [\beta \cdot \delta] = [\alpha_1 \cdot \underbrace{\alpha_1' \cdot \delta}_{\in U_1}] \in ([\alpha_1], U_1).$$

This finishes the proof of the claim.

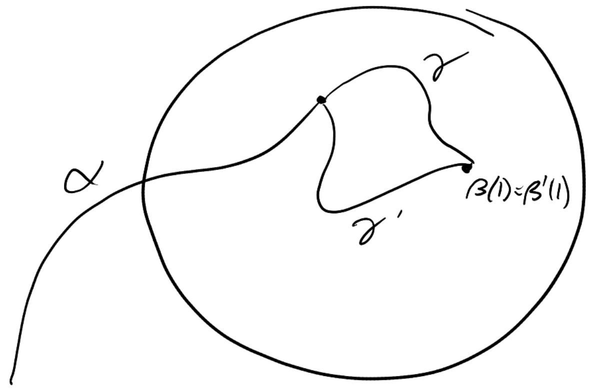
Start of lecture 8

We now check that p is continuous. It is enough to check that $p^{-1}(U)$ is open for $U \in \mathcal{U}$. If $[\alpha] \in p^{-1}(U)$, then $[\alpha] \in ([\alpha], U) \subset p^{-1}(U)$, so $p^{-1}(U)$ is open.

To see that p is a covering map, we first show that each

$$p|_{([\alpha],U)} : ([\alpha], U) \to U$$

is a homeomorphism. As U is path-connected, for any $y \in U$ there is a path γ in U from $\alpha(1)$ to y, so $p([\alpha \cdot \gamma]) = y$, so the map is surjective. If $[\beta], [\beta'] \in ([\alpha], U)$ maps to the same thing under $p|_{([\alpha],U)}$, then β and β' end at the same point. So there exist paths γ, γ' in U such that $[\beta] = [\alpha \cdot \gamma]$, $[\beta'] = [\alpha \cdot \gamma']$.



So

$$[\beta'] = [\alpha \cdot \gamma \cdot \gamma^{-1} \cdot \gamma'] = [\alpha \cdot \gamma] = [\beta]$$

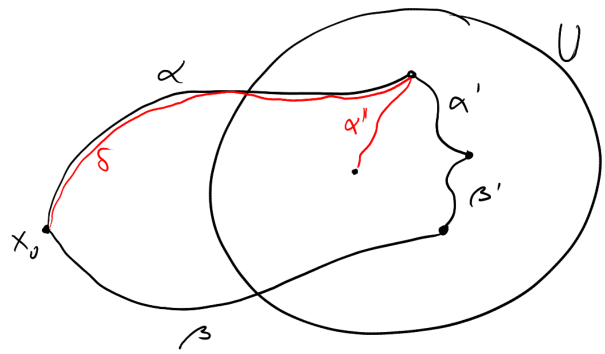
where the first equality comes from the fact that $\gamma^{-1} \cdot \gamma'$ is a loop in U, so homotopic to a constant loop in X. So $p|_{([\alpha],U)}$ is injective. So $p|_{([\alpha],U)}$ is a bijection, and continuous. It is also open as

$$p(([\beta], V)) = V,$$

so $p|_{([\alpha],U)}$ is a homeomorphism. We now claim that $p^{-1}(U)$ is partitioned into $([\alpha], U)$s. When checking that p is continuous, we saw that they cover, so need to show that if they intersect then they are equal. So let $[\gamma] \in ([\alpha], U) \cap ([\beta], U)$, i.e. there are paths α', β' in U such that $[\gamma] = [\alpha \cdot \alpha'] = [\beta \cdot \beta']$. Let $[\delta] \in ([\alpha], U)$ So

$$[\delta] = [\alpha \cdot \alpha''] = [\alpha \cdot \alpha' \cdot (\alpha')^{-1} \cdot \alpha''] = [\beta \cdot \underbrace{\beta' \cdot (\alpha')^{-1} \cdot \alpha''}_{\subset U}]$$

so $[\delta] \in ([\beta], U)$



i.e. $([\alpha], U) \subseteq ([\beta], U)$, similarly for the opposite containment.

Finally, we need to show that $\tilde{X}$ is simply-connected. Note: if $\gamma : I \to X$ is a path, then its lift $\tilde{\gamma} : I \to \tilde{X}$ startng at $[c_{x_0}]$ ends at $[\gamma]$, because

$$s \mapsto [t \mapsto \gamma(st)] : I \to \tilde{X}$$

is the lift. So if a loop γ in X lifts to a loop in $\tilde{X}$ based at $[c_{x_0}]$, so

$$[\gamma] = [c_{x_0}] \implies p_* \pi_1(\tilde{X}, [c_{x_0}]) = \{e\} \subseteq \pi_1(X, x_0).$$

But p_* is injective, so

$$\pi_1(\tilde{X}, [c_{x_0}]) = \{e\}$$

□

3.3 The Galois Correspondence

If $p : \tilde{X} \to X$ is a covering map, $\tilde{X}$ path-connected, $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$, then

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$$

is injective, giving a subgroup of $\pi_1(X, x_0)$.

If $\tilde{x}_0' \in p^{-1}(x_0)$ is another basepoint, let γ be a path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_0'$. Then $p \circ \gamma$ is a loop based at x_0, and we have

$$[p \circ \gamma]^{-1} \cdot p_* \pi_1(\tilde{X}, \tilde{x}_0) \cdot [p \circ \gamma] = p_* \pi_1(\tilde{X}, \tilde{x}_0') \leq \pi_1(X, x_0)$$

So fixing a path-connected based space (X, x_0) we get

$$\left\{\begin{matrix}\text{based path-connected covering maps} \\ p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)\end{matrix}\right\} \xrightarrow{p \mapsto \mathrm{Im}(p_*)} \left\{\text{subgroups of } \pi_1(X, x_0)\right\}$$

$$\left\{\begin{matrix}\text{path-connected covering map} \\ p : \tilde{X} \to X\end{matrix}\right\} \xrightarrow{p \mapsto \mathrm{Im}(p_*)} \left\{\begin{matrix}\text{conjugacy classes of} \\ \text{subgroups of } \pi_1(X, x_0)\end{matrix}\right\}$$

Proposition (Surjectivity of Galois correspondence)**.** Let X be path-connected, locally path-connected, and semi-locally simply-connected. Then for any $H \le \pi_1(X, x_0)$ there is a $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ with $p_* \pi_1(\tilde{X}, \tilde{x}_0) = H$.

Proof. Let $\overline{X} \xrightarrow{q} X$ be the universal cover we have constructed in Existence of universal covers. Define $\sim_H$ on $\overline{X}$ by $[\gamma] \sim_H [\gamma'] \iff \gamma(1) = \gamma'(1)$ and $[\gamma \cdot (\gamma')^{-1}] \in H \le \pi_1(X, x_0)$. So:

(i) $[\gamma] \sim_H [\gamma]$.

(ii) If $[\gamma] \sim_H [\gamma']$ then $[\gamma \cdot (\gamma')^{-1}] \in H$, so $[\gamma' \cdot (\gamma')^{-1}] \in H$. So $[\gamma'] \sim_H [\gamma]$.

(iii) If $[\gamma] \sim_H [\gamma']$, $[\gamma'] \sim_H [\gamma'']$, then $\gamma(1) = \gamma'(1) = \gamma''(1)$, and
$$[\gamma \cdot (\gamma'')^{-1}] = [\gamma \cdot (\gamma')^{-1} \cdot \gamma' \cdot (\gamma'')^{-1}] = \underbrace{[\gamma \cdot (\gamma')^{-1}]}_{\in H} \underbrace{[\gamma' \cdot (\gamma'')^{-1}]}_{\in H} \in H.$$
So $[\gamma] \sim_H [\gamma'']$.

So $\sim_H$ is an equivalence relation. Define $\overline{X}_H = \overline{X} / \sim_H$, the quotient space, and $p_H : \overline{X}_H \to X$ to be the induced map. If $[\gamma] \in ([\alpha], U)$, $[\gamma'] \in ([\beta], U)$ satisfy $[\gamma] \sim_H [\gamma']$ then $([\alpha], U)$ and $[\beta], U)$ are identified by $\sim_H$, as $[\gamma \cdot \eta] \sim_H [\gamma' \cdot \eta]$ for any path η in U. So p_H is a covering map.

It remains to show that $(p_H)_* \pi_1(\overline{X}_H, [[c_{x_0}]]) = H \le \pi_1(X, x_0)$. If $[\gamma] \in H$ then the lift of γ to $\overline{X}$ starting at $[c_{x_0}]$ ends at $[\gamma]$, so the lift to $\overline{X}_H$ ends at $[[\gamma]] = [[c_{x_0}]]$, so is a loop, i.e.
$$H \subseteq (p_H)_* \pi_1(\overline{X}_H, [[c_{x_0}]]).$$
If $[\gamma] \in (p_H)_* \pi_1(\overline{X}_H, [[c_{x_0}]])$ then the lift $\overline{\gamma}$ of γ to $\overline{X}$ starting at $[c_{x_0}]$ ends at $[\gamma]$, so $[\gamma] \sim_H [c_{x_0}]$, as it becomes a loop in $\overline{X}_H$ by assumption. So $[\gamma] \in H$. □

Start of lecture 9

Proposition (Based uniqueness)**.** Let (X, x_0) satisfy the conditions for Existence of universal covers. If $p_1 : (\tilde{X}_1, \tilde{x}_1) \to (X, x_0)$, $p_2 : (\tilde{X}_2, \tilde{x}_2) \to (X, x_0)$ are path-connected covering spaces, then the following are equivalent:

- there exists homeomorphism $h : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ such that $p_2 \circ h = p_1$
- $(p_1)_* \pi_1(\tilde{X}_1, \tilde{x}_1) = (p_2)_* \pi_1(\tilde{X}_1, \tilde{x}_1)$

Proof. If h exists, then
$$\mathrm{Im}((p_1)_*) = \mathrm{Im}((p_2)_* \circ h_*)$$

as h_* is an isomorphism.

For the other direction, let $H \leq \pi_1(X, x_0)$ be the common image. Will show that $\tilde{X}_1$ and $\tilde{X}_2$ are homeomorphic to $\overline{X}_H$ (over X). Consider

$$\begin{aligned} r : \overline{X} &\to \tilde{X}_1 \\ [\gamma] &\mapsto \tilde{\gamma}(1) \end{aligned}$$

end point of the lift $\tilde{\gamma}$ of γ to $\tilde{X}_1$, starting at $\tilde{x}_1$. Notice

$$\begin{aligned} r([\gamma]) = r([\gamma']) &\iff \tilde{\gamma} \text{ and } \tilde{\gamma}' \text{ end at the same point of } \tilde{X}_1 \\ \iff [\gamma' \cdot \gamma^{-1}] \in (p_1)_* \pi_1(\tilde{X}_1, \tilde{x}_1) = H \\ &\iff [\gamma] \sim_H [\gamma'] \end{aligned}$$

So r descends to a map

$$q : (\overline{X}_H, [[c_{x_0}]]) \to (\tilde{X}_1, \tilde{x}_1),$$

a bijection. It is also an open map, as $\overline{X}_H$ and $\tilde{X}_1$ are both locally homeomorphic to X. So q is a homeomorphism. □

Corollary (Unbased uniqueness)**.** Let X satisfy the conditions from Existence of universal covers. If $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are path-connected covering spaces, then the following are equivalent:

- there exists homeomorphism $h : \tilde{X}_1 \to \tilde{X}_2$ such that $p_2 \circ h = p_1$
- $(p_1)_* \pi_1(\tilde{X}_1, \tilde{x}_1)$ and $(p_2)_* \pi_1(\tilde{X}_2, \tilde{x}_2)$ are conjugate in $\pi_1(X, x_0)$ for any $\tilde{x}_1 \in p_1^{-1}(x_0)$, $\tilde{x}_2 \in p_2^{-1}(x_0)$.

Proof. If h exists, choose $\overline{x}_1 \in p_1^{-1}(x_0)$ and $\tilde{x}_2 = h(\tilde{x}_1)$. Based uniqueness applies to see that the groups are determined.

Conversely, suppose $[\gamma] \in \pi_1(X, x_0)$ is such that

$$[\gamma]^{-1} \cdot (p_1)_* \pi_1(\tilde{X}_1, \tilde{x}_1) \cdot [\gamma] = (p_2)_* \pi_1(\tilde{X}_2, \tilde{x}_2).$$

Lif γ to $\tilde{X}_1$ starting $\tilde{x}_1$, and say it ends at $\tilde{x}_1'$. Then

$$\text{LHS} = (p_1)_* \pi_1(\tilde{X}_1, \tilde{x}_1').$$

Now by Based uniqueness, we can find a homeomorphism $h : (\tilde{X}_1, \tilde{x}_1') \to (\tilde{X}_2, \tilde{x}_2)$, which can be viewed as an unbased homeomorphism. □

So we now have bijections:

$$\left\{\begin{matrix}\text{based path-connected}\\ \text{covering maps of } (X,x_0)\end{matrix}\right\} \Big/ \begin{matrix}\text{based homeomorphism}\\ \text{over } X\end{matrix} \leftrightarrow \left\{\begin{matrix}\text{subgroups of}\\ \pi_1(X,x_0)\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{path-connected}\\ \text{covering maps of } X\end{matrix}\right\} \Big/ \begin{matrix}\text{homeomorphism}\\ \text{over } X\end{matrix} \leftrightarrow \left\{\begin{matrix}\text{conjugacy classes of}\\ \text{subgroups of } \pi_1(X,x_0)\end{matrix}\right\}$$

4.1 Free groups and presentations

Definition (Alphabet and words)**.** Let $S = \{s_\alpha\}_{\alpha\in I}$ be a set, called the *alphabet*, $S^{-1} = \{s_\alpha^{-1}\}_{\alpha\in I}$ (suppose $S \cap S^{-1} = \emptyset$.

A *word* in the alphabet S is a (possibly empty) finite sequence $(x_1, x_2, \dots, x_n)$ of elements in $S \cup S^{-1}$. A word is *reduced* if it does not contain $(s_\alpha, s_\alpha^{-1})$ or $(s_\alpha^{-1}, s_\alpha)$ as a subword. An *elementary reduction* consists of removing such a subword from a word.

Definition (Free group)**.** The *free group* on the alphabet S, denoted $F(S)$, is the set of reduced (possibly empty) words in this alphabet. The group operation is given by concatenation, and doing elementary reductions until it is reduced.

Note that the group operation here is not obviously well-defined. We will not prove here that it is well-defined. See the lecturer's notes for a proof.

Clearly we have that the empty string is the identity, and for any word $(x_1, \dots, x_n)$, it has inverse $(x_n^{-1}, \dots, x_1^{-1})$.

There is a function $i : S \to F(S)$, $s \mapsto (s)$.

Lemma (Universal property of free groups)**.** For any group H, the function

$$\left\{\begin{matrix}\text{homomorphisms}\\ \phi{:}F(S)\to H\end{matrix}\right\} \overset{-\circ i}{\to} \{functions \phi : S \to H\}$$

is a bijection.

Proof. Given $\phi : S \to H$, want a $\phi : F(S) \to H$ such that $\phi((s)) = \phi(s)$. Let, on a not-necessarily-reduced word $(s_{\alpha_1}^{\varepsilon_1}, \dots, s_{\alpha_n}^{\varepsilon_n})$,

$$\phi((s_{\alpha_1}^{\varepsilon_1}, \dots, s_{\alpha_n}^{\varepsilon_n})) = \phi(s_{\alpha_1})^{\varepsilon_1} \cdots \phi(s_{\alpha_n}^{\varepsilon_n}) \in H.$$

If the word contained $(s_\alpha, s_\alpha^{-1})$, then the result contains $\phi(s_\alpha)\phi(s_\alpha)^{-1} = e \in H$. So ϕ is well-defined. As the group operation on $F(S)$ is by concatenation, we see ϕ is a homomorphism. □

Definition (Group with relations)**.** Let S be a set and $R \subseteq F(S)$. Then

$$\langle S \mid R \rangle = F(S)/\langle\!\langle R \rangle\!\rangle$$

with

$$\langle\!\langle R \rangle\!\rangle = \{(r_1^{\varepsilon_1})^{g_1} \cdots (r_n^{\varepsilon_n})^{g_n} \mid r_i \in R, \varepsilon_i \in \{\pm 1\}, g \in F(S)\} \trianglelefteq F(S),$$

where h^g denotes $g^{-1}hg$.

Call this (S and R) a *presentation* of the group $\langle S \mid R \rangle$. If S and R are finite, call it a *finite presentation*.

Start of lecture 10

Lemma (Universal property of group presentations)**.** For any group H, the function

$$\left\{ \substack{\text{group homomorphisms} \\ \psi : \langle S \mid R \rangle \to H} \right\} \to \left\{ \substack{\text{functions } \phi : S \to H \\ \text{such that } \phi(r) = e \ \forall r \in R} \right\}$$
$$\psi \mapsto [S \xrightarrow{\text{inc}} F(S) \xrightarrow{\text{quot}} \langle S \mid R \rangle \xrightarrow{\psi} H]$$

is a bijection.

Proof. Suppose ψ, ψ' determine functions $\phi = \phi' : S \to H$. Then

$$F(S) \xrightarrow{\text{quot}} \langle S \mid R \rangle \underset{\psi'}{\overset{\psi}{\rightrightarrows}} H$$

are equal by the universal property of free groups. As "quot" is onto, $\psi = \psi'$.

Conversely, given a $\phi : S \to H$ such that $\phi(r) = r \ \forall r \in R$, consider

$$\varphi : F(S) \to H$$

Now $R \subset \ker(\varphi)$, so as $\ker(\phi)$ is normal, $\langle\!\langle R \rangle\!\rangle \leq \ker(\varphi)$. This φ extends to a homomorphism

$$\langle S \mid R \rangle = \frac{F(S)}{\langle\!\langle R \rangle\!\rangle} \to H$$

as required. □

Example. If G is a group, the function $\phi = \text{id} : G \to G$ gives a homomorphism $\varphi : F(G) \to G$, which is onto. Let $R = \ker(\varphi)$, so $\langle G \mid R \rangle = \frac{F(G)}{\langle\!\langle R \rangle\!\rangle} \cong G$.

Example. $G = \langle a, b \mid a \rangle$, $H = \langle t \mid \rangle$. Consider

$$\begin{aligned} \phi : \{a, b\} &\to H \\ a &\mapsto e \\ b &\mapsto t \\ \phi' : \{t\} &\to G \\ t &\mapsto b \end{aligned}$$

Then $\psi(a) = e$, so we get a homomorphism

$$\psi : \langle a, b \mid a \rangle \to \langle t \mid \rangle,$$

by the Universal property of group presentations. We also get

$$\psi' : \langle t \mid \rangle \to \langle a, b \mid a \rangle.$$

Note that

$$\psi' \circ \psi([a]) = [e] = [a]$$
$$\psi' \circ \psi([b]) = [b]$$

so as $[a], [b]$ generate, we have $\psi' \circ \psi = \mathrm{id}$. Similarly $\psi \circ \psi' = \mathrm{id}$. So

$$\langle a, b \mid a \rangle \cong \langle t \mid \rangle.$$

Example. Let $G = \langle a, b \mid ab^{-3}, ba^{-2} \rangle$. Have

$$[a][b]^{-3} = e, \qquad [b][a]^{-2} = e,$$

so

$$[a] = [b]^3, \qquad [b] = [a]^2,$$

so $[a] = [a]^6$, so $\boxed{e = [a]^5}$. These show that every element is equal to one of $e, [a], [a]^2, [a]^3, [a]^4$. Consider:

$$\begin{aligned} \phi : \{a, b\} &\to \mathbb{Z}/5\mathbb{Z} \\ a &\mapsto 1 \\ b &\mapsto z \end{aligned}$$

Then $\varphi(ab^{-3}) = e = \varphi(ba^{-2})$, so we get a homomorphism $\psi : \langle a, b \mid ab^{-3}, ba^{-2} \rangle \to \mathbb{Z}/5\mathbb{Z}$, which is an isomorphism since $[a] \mapsto 1$.

4.2 Free products with amalgamations

Definition (Free product)**.** Consider group homomorphisms

$$G_1 \overset{i_1}{\leftarrow} H \overset{i_2}{\rightarrow} G_2$$

and suppose $G_i = \langle S_i \mid R_i \rangle$. The *free product* of G_1 and G_2 is

$$G_1 * G_2 = \langle S_1 \amalg S_2 \mid R_1 \sqcup R_2 \rangle.$$

Definition (Free product with amalgamation over H)**.** The functions

$$S_i \to S_1 \amalg S_2 \to F(S_1 \sqcup S_2) \to G_1 * G_2$$

induce homomorphisms

$$G_1 \overset{j_1}{\to} G_1 * G_2 \overset{j_2}{\leftarrow} G_2$$

The *free product with amalgamation over H* is the quotient

$$G_1 \underset{H}{*} G_2 = G_1 * G_2 / \langle\!\langle j_1 i_1(h)(j_2 i_2(h))^{-1}, h \in H \rangle\!\rangle.$$

So the following diagram commutes:

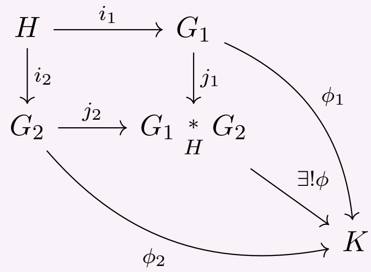
$$\begin{array}{ccc} H & \xrightarrow{i_1} & G_1 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ G_2 & \xrightarrow{j_2} & G_1 \underset{H}{*} G_2 \end{array}$$

Lemma (Universal property of free products with amalgamation)**.** For any group K,

$$\left\{\substack{\text{group homomorphisms}\\ \phi: G_1 \underset{H}{*} G_2 \to K}\right\} \to \left\{\substack{\text{group homomorphisms } \phi_1 : G_1 \to K,\\ \phi_2 : G_2 \to K, \text{ such that}\\ \phi_1 \circ i_1 = \phi_2 \circ i_2}\right\}$$

$$\phi \mapsto [G_i \xrightarrow{j_i} G_1 \underset{H}{*} G_2 \xrightarrow{\phi} K]$$

is a bijection.



Proof. Similar to other universal properties. □

5.1 The Seifert-Van Kampen Theorem

Let X be a space, $A, B \subset X$ be subspaces, $x_0 \in A \cap B$. We get a commutative diagram

$$\begin{array}{ccc} \pi_1(A \cap B, x_0) & \longrightarrow & \pi_1(A, x_0) \\ \downarrow & & \downarrow \\ \pi_1(B, x_0) & \longrightarrow & \pi_1(X, x_0) \end{array}$$

Then the Universal property of free products with amalgamation gives us

$$\pi_1(A, x_0) \underset{\pi_1(A \cap B, x_0)}{*} \pi_1(B, x_0) \to \pi_1(X, x_0).$$

Theorem (Seifert-van Kampen)**.** Let X be a space, $A, B \subset X$ be open subsets, which cover X and such that $A \cap B$ is path-connected. Then for any $x_0 \in A \cap B$, the induced map

$$\phi : \pi_1(A, x_0) \underset{\pi_1(A \cap B, x_0)}{*} \pi_1(B, x_0) \to \pi_1(X, x_0)$$

is a group isomorphism.

Example. Consider S^n for $n \geq 2$. Let

$$A = S^n - \{\text{north pole}\} \cong \mathbb{R}^n \simeq \{*\}$$
$$B = S^n - \{\text{south pole}\} \cong \mathbb{R}^n \simeq \{*\}$$

$A \cap B \cong S^{n-1} \times (-1, 1)$, path-connected for $n \geq 2$, and $\simeq S^{n-1}$. So

$$\pi_1(S^n, \bullet) \cong \{e\} \underset{\pi_1(S^{n-1}, \bullet)}{*} \{e\} = \{e\}.$$

So S^n is simply-connected.

Example. We saw that there is a 2-sheeted covering map $p : S^n \to \mathbb{RP}^n$. For $n \geq 2$, S^n is simply-connected , so this is a universal cover. So

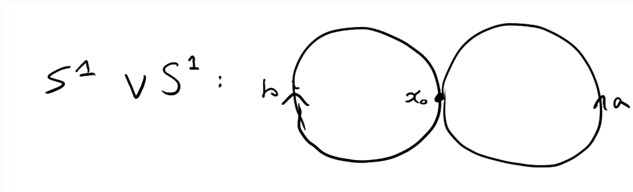
$$\pi_1(\mathbb{RP}^n, \bullet) \overset{\text{bij}}{\to} p^{-1}(\bullet)$$

has 2 elements. So $\pi_1(\mathbb{RP}^n, \bullet) \cong \mathbb{Z}/2\mathbb{Z}$.

Given $(X, x_0), (Y, y_0)$, then

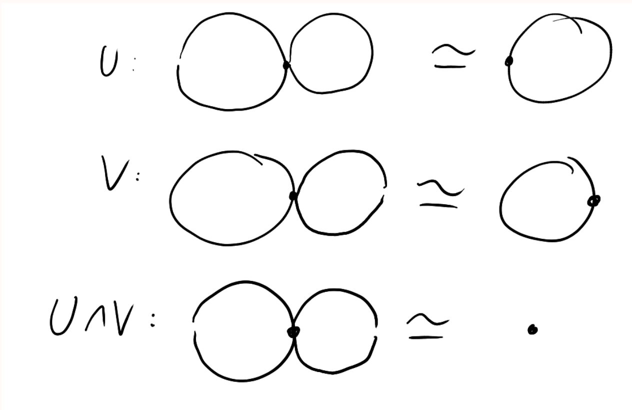
$$X \wedge Y = (X \sqcup Y)/x_0 \sim y_0.$$

Example. Let $S^1 \subset \mathbb{C}$ have basepoint $1 \in S^1 \subset \mathbb{C}$. $S^1 \wedge S^1$ is:



Which is covered by

$$\begin{aligned} (S^1 - \{1\}) \wedge S^1 &= U \\ S^1 \wedge (S^1 - \{1\}) &= V \end{aligned}$$



So Seifert-van Kampen implies

$$\begin{aligned} \pi_1(S^1 \wedge S^1, x_0) &\cong \langle a \mid \rangle \underset{\{e\}}{*} \langle b \mid \rangle \\ &= \langle a, b \mid \rangle \end{aligned}$$

Start of

lecture 11

Example. The function

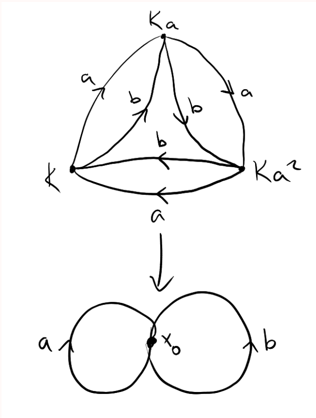
$$\{a, b\} \to \mathbb{Z}/3\mathbb{Z}$$
$$a \mapsto 1$$
$$b \mapsto 1$$

gives a homomorphism $\varphi : \langle a, b \mid \rangle = \pi_1(S^1 \wedge S^2, x_0) \to \mathbb{Z}/3\mathbb{Z}$, surjective. So

$$K = \ker(\varphi) \leq \pi_1(S^1 \wedge S^1, x_0)$$

is a subgroup of index 3. This corresponds to a covering space. What is it? It is a $p : \tilde{X} \to X$ with

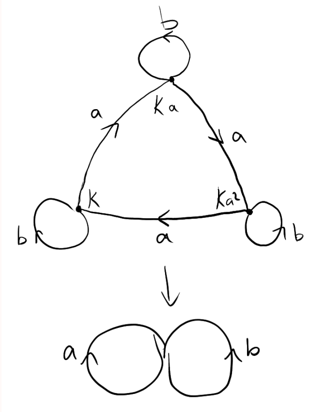
$$p^{-1}(x_0) \cong \frac{\pi_1(S^1 \wedge S^1, x_0)}{K} \cong \mathbb{Z}/3\mathbb{Z}.$$



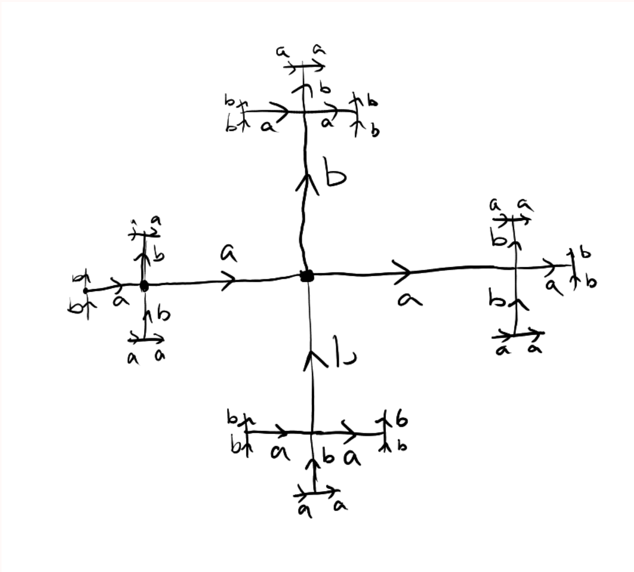
Example. Consider

$$\{a, b\} \to \mathbb{Z}/3\mathbb{Z}$$
$$a \mapsto 1$$
$$b \mapsto 0$$

again giving a $\varphi : \pi_1(S^1 \wedge S^1, x_0) = \langle a, b \mid \rangle \twoheadrightarrow \mathbb{Z}/3\mathbb{Z}$, with kernel K. What is the covering space corresponding to K?

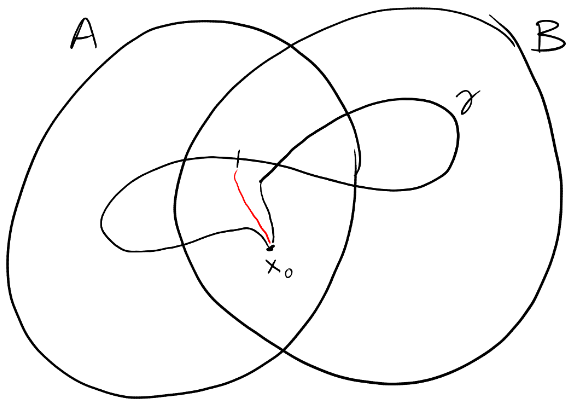


Example. The universal cover of $S^1 \wedge S^1$?



Proof of Seifert-van Kampen (non-examinable). Without loss of generality can assume A, B are path-connected.

(1) ϕ **surjective:** Let $\gamma : I \to X$ be a gloop. Then $\{\gamma^{-1}(A), \gamma^{-1}(B)\}$ is an open cover of I, so by Lebesgue number lemma there is a $n \gg 0$ such that each $\left[\frac{i}{n}, \frac{i+1}{n}\right]$ is sent into A or B (or both). By concatenating intervls which lie in A, or in B, can write $\gamma = \gamma_1 \cdot \gamma_2 \cdots \gamma_k$ with each γ_i being endpoints in $A \wedge B$. Chooce paths u_i from $\gamma_i(1)$ to x_0 in $A \wedge B$.



Then $\gamma \simeq (\gamma_1 \cdot u_i) \cdot (u_i^{-1} \cdot \gamma_2 u_2) \cdot (u_2^{-1} \cdot \gamma_3 \cdot u_3) \cdots (u_{k-1}^{-1} \cdot \gamma_k)$. Each thing in a pair of brackets is a loop based at x_0, lying in A or in B. So $[\gamma] \in \mathrm{Im}(\phi)$ as required.

(2) ϕ **is injective:** The group

$$\pi_1(A, x_0) \underset{\pi_1(A \wedge B, x_0)}{*} \pi_1(B, x_0)$$

by the following description. It is generated by

(i) For $\gamma : I \to A$ a loop in A, $[\gamma]_A$.

(ii) For $\gamma : I \to B$ a loop in B, $[\gamma]_B$.

With relations:

(i) If $\gamma \simeq \gamma'$ in A, then $[\gamma]_A = [\gamma']_A$, and similarly for B.

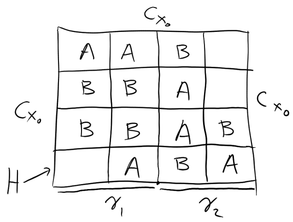
(ii) If $\gamma : I \to A \cap B$ then $[\gamma]_A = [\gamma]_B$.

(iii) $[\gamma]_A \cdot [\gamma']_A = [\gamma \cdot \gamma']_A$, and similarly for B.

Suppose

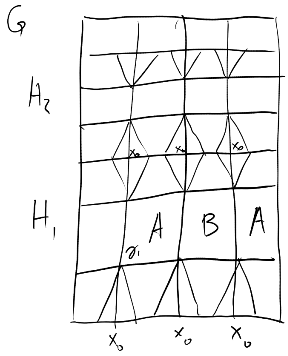
$$\phi([\gamma_1]_{A_{i_1}} \cdot [\gamma_2]_{A_{i_2}} \cdots [\gamma_n]_{A_{i_n}}) = [c_{x_0}]$$

so $\gamma_1 \cdots \gamma_k \simeq_H c_{x_)}$ in X, $H : I \times I \to X$. Can subdivide $I \times I$ into squares of size $\frac{1}{N}$ with $n \mid N$ such that each square is sent into A or B (or both).



Choose paths i_{ij} from $H\left(\frac{i}{N}, \frac{j}{N}\right)$ to x_0 such that:

- If it is a vertex of a box labelled A, the path is in A.
- If it is a vertex of a box labelled B, the path is in B.



G has the property that it decomposes into rectangles with vertices sent to x_0, and each rectangle in A or in B. This shows that $[\gamma_1]_{A_1} \cdots [\gamma_n]_{A_n}$ can be transformed using the 3 kinds of relations described.

□

5.2 Attaching a cell

Let $f : (S^{n-1}, *) \to (X, x_0)$, then

$$Y = X \cup_f D^n = \frac{X \sqcup D^n}{f(x) \sim x \in \partial D^n = S^{n-1}}$$

Then $[x_0] \in Y$ is a basepoint, and the inclusion $(X, x_0) \hookrightarrow (Y, [x_0])$ is a based map.

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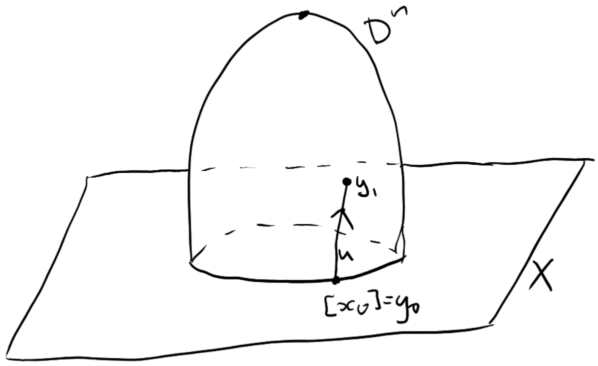
Theorem.

(i) If $n \geq 3$, then $\text{inc}_* : \pi_1(X, x_0) \to \pi_1(Y, [x_0])$ is an isomorphism.

(ii) If $n = 2$, then $\text{inc}_* : \pi_1(X, x_0) \to \pi_1(Y, [x_0])$ is the quotient by the normal subgroup generated by $[f] \in \pi_1(X, x_0)$.

Proof. Let $U = \int(D^n)$, $V = X \cup_f (D^n - \{0\})$. These give an open cover of Y. Choose a path u in $U \cap V$ from y_0 to some $y_1 \in \int(D^n) = U$.



If $n \geq 3$ then:

(i) $U \simeq \{*\}$.

(ii) $U \cap V = S^{n-1} \times (0, 1)$ is simply-connected.

So Seifert-van Kampen gives us:

$$\pi_1(U, y_1) \underset{\underbrace{\pi_1(U \cap V, y_1)}_{=\pi_1(V, y_1)}}{*} \pi_1(V, y_1) \xrightarrow{\sim} \pi_1(Y, y_1).$$

So also, by change of basepoint isomorphism, we get

$$\pi_1(V, y_0) \xrightarrow{\sim} \pi_1(Y, y_0).$$

But V strongly deformation retracts to X, so:

$$\pi_1(X, y_0) \xrightarrow{\sim} \pi_1(V, y_0) \xrightarrow{\sim} \pi_1(Y, y_0).$$

If $n = 2$ then:

(i) $U \simeq \{*\}$.

(ii) $U \cap V \cong S^1 \times (0,1)$.

Now Seifert-van Kampen gives us:

$$\underbrace{\pi_1(U, y_1)}_{\{e\}} \underset{\underbrace{\pi_1(U \cap V, y_1)}_{=\mathbb{Z} \ni 1}}{*} \pi_1(V, y_1) \xrightarrow{\sim} \pi_1(Y, y_1).$$

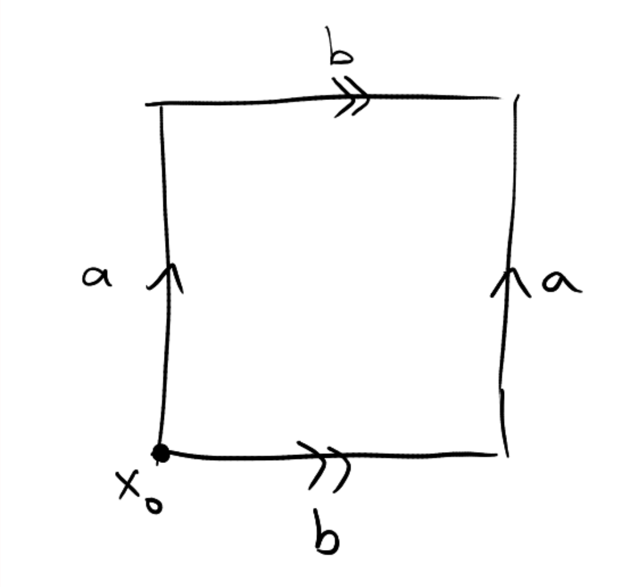
The $1 \in \mathbb{Z}$ corresponds to e in $\pi_1(U, y_1)$, and corresponds to ${u^{-1}}_\#([f])$ in $\pi_1(V, y_1)$. So

$$\frac{\pi_1(V, y_1)}{\langle\langle {u^{-1}}_\#[f] \rangle\rangle} \xrightarrow{\sim} \pi_1(Y, y_0)$$

Then change of basepoint and using the fact that V strongly deformation retracts to X, we get

$$\frac{\pi_1(X, y_0)}{\langle\langle [f] \rangle\rangle} \xrightarrow{\sim} \frac{\pi_1(V, y_0)}{\langle\langle [f] \rangle\rangle} \xrightarrow{\sim} \pi_1(Y, y_0).$$ □

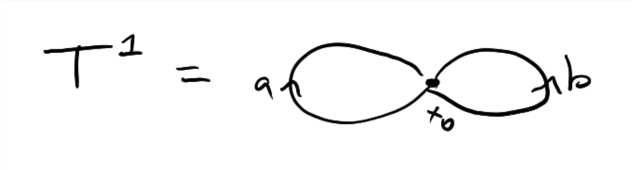
Example. The torus T:



has a cell structure with:

1. A 0-cell x_0,
2. 1-cells a, b,
3. A 2-cell.

The 1-skeleton is:



So

$$\pi_1(T^1, x_0) = \langle a, b \mid \rangle,$$

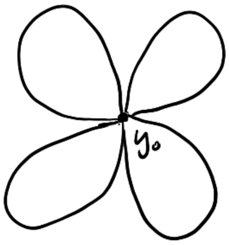
and so

$$\pi_1(T, x_0) = \langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z} \times \mathbb{Z}.$$

Corollary. For $G = \langle S \mid R \rangle$ with S, R finite, there is a 2-dimensional based cell complex (X, x_0) with $\pi_1(X, x_0) \cong G$.

Proof. Let Y be the wedge of $|S|$-many circles. Sending $s \in S$ to the s-th circle x_s gives an isomorphism

$$\langle S \mid \rangle \xrightarrow{\sim} \pi_1(Y, y_0).$$



Each $r \in R$ is an element of $\langle S \mid \rangle$, so gives a $[\gamma_r] \in \pi_1(Y, y_0)$. Attaching 2-cells to Y along $\{\gamma_r\}_{r \in R}$ gives an (X, x_0) with

$$\pi_1(X, x_0) = \frac{\langle S \mid \rangle}{\langle\langle r \in R \rangle\rangle} = \langle S \mid R \rangle. \qquad \square$$

5.3 A refinement of Seifert-van Kampen

Definition (Neighbourhood deformation retract)**.** A subset $A \subset X$ is called a *neighbourhood deformation retract* (NDR) if there is an open neighbourhood $A \subset U \subset X$ and U *strongly* deformation retracts to A.

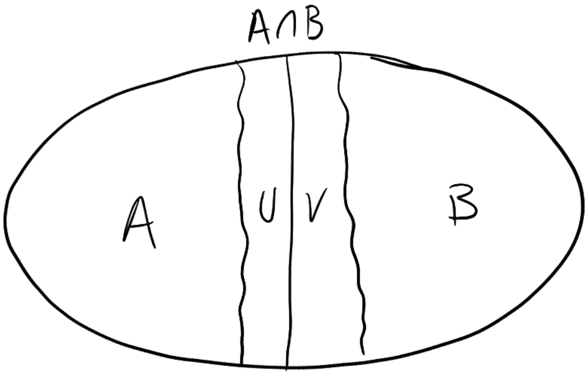
Theorem (Refinement of Seifert-van Kampen)**.** Let X be a space, $A, B \subset X$ closed subsets which cover X and such that $AcapB$ is path-connected and is a neighbourhood deformation retract in both A and B. Then

$$\pi_1(A, x_0) \underset{\pi_1(A \cap B, x_0)}{*} \pi_1(B, x_0) \xrightarrow{\sim} \pi_1(X, x_0).$$

Proof. Let $A \cap B \subset U \subset A$, $A \cap B \subset V \subset B$, U, V open, which strongly deformation retract to $A \cap B$. Observe

$$(A \cup V)^c = B - V \qquad (B \cup U)^c = A - U$$

are closed, so $A \cup V$, $B \cup U$ is an open cover of X.

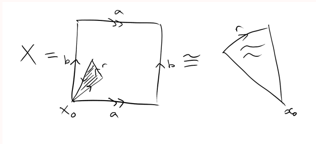


The strong deformation retracts of U and V to $A \cap B$ glue to give a deformation of $(A \cup V) \cap (B \cup U) = U \cap V$ to $A \cap B$. Then we get deformations of $A \cup V$ to A and $B \cup U$ to B. Now we use Seifert-van Kampen for the open cover and we get:

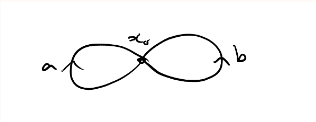
$$\begin{array}{ccccc} \pi_1(B) & \longleftarrow & \pi_1(A \cap B), & \longrightarrow & \pi_1(A) \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ \pi_1(B \cup U) & \longleftarrow & \pi_1((A \cup V) \cap (B \cup U)) & \longrightarrow & \pi_1(A \cup V) \end{array}$$

□

5.4 Surfaces

Example. 

LHS strongly deformation retracts to



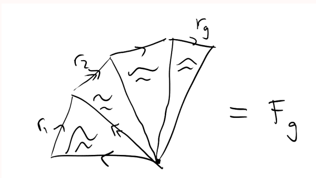
So

$$\pi_1(X, x_0) = \langle a, b \mid \rangle,$$

with $[r] = aba^{-1}b^{-1}$.

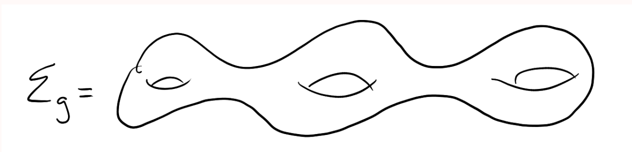
Applying Seifert-van Kampen several times gives:

$$\pi_1(F_g, x_0) \cong \langle a_1, b_1, a_2, b_2, \dots, a_g, b_g \mid \rangle$$

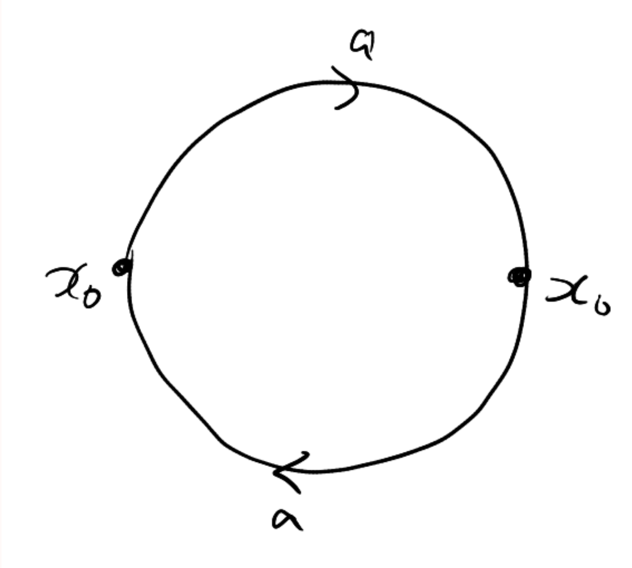


The boundary is $r_1 r_2 \cdots r_g$, so attaching a 2-cell along it to get Σ_g,

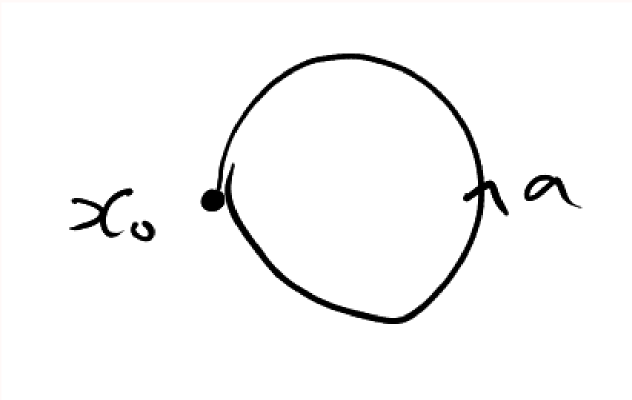
$$\pi_1(\Sigma_g, x_0) = \langle a_1 b_2, \dots, a_g b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle$$



Example. $\mathbb{RP}^2$:



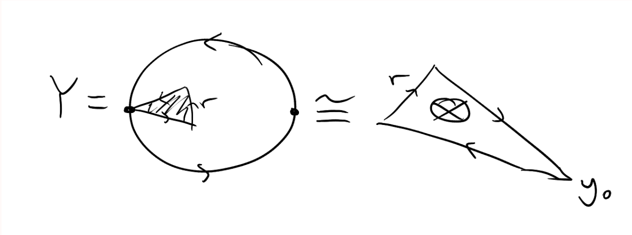
Has 1-skeleton



2-cell is attached along aa. So

$$\pi_1(\mathbb{RP}^2, *) \cong \langle a \mid a^2 \rangle = \mathbb{Z}/2\mathbb{Z}.$$

Example.

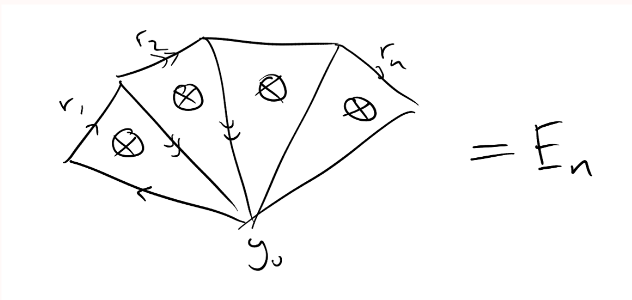


$Y \simeq S^1$, so

$$\pi_1(Y, y_0) = \langle a \mid \rangle,$$

and $[r] = a^2 \in \pi_1(Y, y_0)$. Seifert-van Kampen again:

$$\pi_1(E_n, y_0) \cong \langle a_1, \dots, a_n \mid \rangle$$



The boundary is $r_1 r_2 \cdots r_n$, so attaching a 2-cell along it to get a closed surface S_n, we get

$$\pi_1(S_n, y_0) = \langle a_1, \dots, a_n \mid a_1^2 a_2^2 \cdots a_n^2 \rangle.$$

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lecture 13

6.1 Simplicial complexes

Definition (Affinely independent)**.** A finite set of points $a_0, a_1, \dots, a_r \in \mathbb{R}^m$ is *affinely independent* if

$$\begin{cases} \sum_{i=1}^n t_i a_i = 0 \\ \text{and } \sum_{i=1}^n t_i = 0 \end{cases} \iff (t_1, \dots, t_n) = 0.$$

Lemma. $a_0, \dots, a_n \in \mathbb{R}^m$ is affinely independent if and only if $a_1 - a_0, a_2 - a_0, \dots, a_n - a_0$ is linearly independent.

Proof. Let $a_0, \dots, a_n$ be affinely independent, and suppose

$$\sum_{i=1}^{n} s_i(a_i - a_0) = 0.$$

Then

$$\left(-\sum_{i=1}^{n} s_i\right) a_0 + s_1 a_1 + \dots + s_n a_n = 0$$

and

$$\left(-\sum_{i=1}^{n} s_i\right) + s_1 + \dots + s_n = 0$$

hence $(s_1, \dots, s_n) = 0$. So $a_1 - a_0, \dots, a_n - a_0$ are linearly independent.

Similarly for the converse. □

Definition (n-simplex)**.** If $a_0, \dots, a_n \in \mathbb{R}^m$ are affinely independent, then they define an *n-simplex*

$$\sigma = \langle a_0, \dots, a_n \rangle = \left\{ \sum_{i=0}^{n} t_i a_i \;\middle|\; \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \right\} \subseteq \mathbb{R}^m$$

given by the convex hull of the points $a_0, \dots, a_n$. These are called the *vertices* of σ, and we say that they *span* σ.

If $x \in \langle a_0, \dots, a_n \rangle$, then x can be written *uniquely* as $x = \sum_{i=0}^{n} t_i a_i$ for real numbers $t_0, \dots, t_n$ summing to 1. Call the t_i's the *barycentric coordinates* of x.

Definition (Face)**.** A *face* of a n-simplex $\sigma = \langle a_0, \dots, a_n \rangle$ is a simplex τ spanned by a subset of $\{a_0, \dots, a_n\}$. Write $\tau \leq \sigma$. Write $\tau < \sigma$ if τ is a proper face.

Definition (Boundary)**.** The *boundary* of a simplex σ, written $\partial \sigma$ is the union of all its proper faces.

Definition (Interior)**.** The *interior* of σ, written $\mathring{\sigma}$, is $\sigma - \partial\sigma$.

Lemma. Let σ be a p-simplex in $\mathbb{R}^m$ and τ be a p-simplex in $\mathbb{R}^n$. Then σ and τ are homeomorphic.

Proof. Let $\sigma = \langle a_0, \ldots, a_p \rangle$, $\tau = \langle b_0, \ldots, b_p \rangle$. Define

$$h : \sigma \to \tau$$
$$\sum_{i=0}^{p} t_i a_i \mapsto \sum_{i=0}^{p} t_i b_i$$

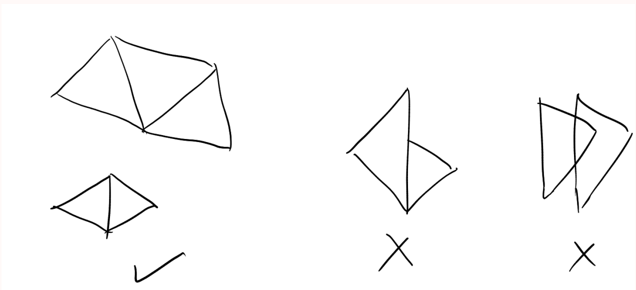
This is well-defined and a bijection, by uniqueness of barycentric coordinates. As the $a_i - a_0$ are linearly independent, h extends to an affine map $\hat{h} : \mathbb{R}^n \to \mathbb{R}^m$, so h is continuous. So is its inverse. □

Definition (Geometric simplicial complex)**.** A *geometric (or Euclidean) simplicial complex* in $\mathbb{R}^m$ is a finite set K of simplexes in $\mathbb{R}^m$ such that:

(i) If $\sigma \in K$ and $\tau \leq \sigma$, then $\tau \in K$.

(ii) If $\sigma, \tau \in K$, then $\sigma \cap \tau = \emptyset$ or $\sigma \cap \tau$ is a face of σ and of τ.

Example.



Definition (Dimension of a simplicial complex)**.** The *dimension* of a simplicial complex K is the largest p such that K contains a p-simplex.

Definition (Polyhedron of a simplicial complex)**.** The *polyhedron* of K is the space

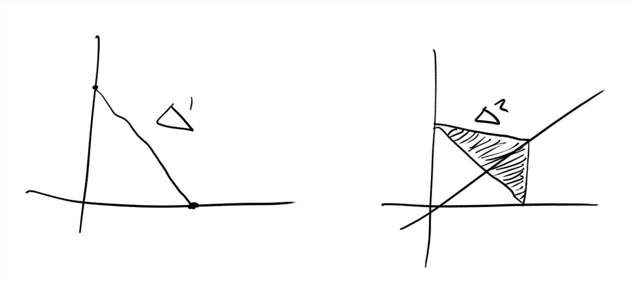
$$|K| = \bigcup_{\sigma \in K} \sigma \subseteq \mathbb{R}^m.$$

Remark. $|K|$ is compact and Hausdorff.

Definition (d-skeleton of a simplicial complex)**.** The *d-skeleton* $K_{(d)}$ of K is the sub-simplicial complex containing all simplexes of K of dimension $\leq d$.

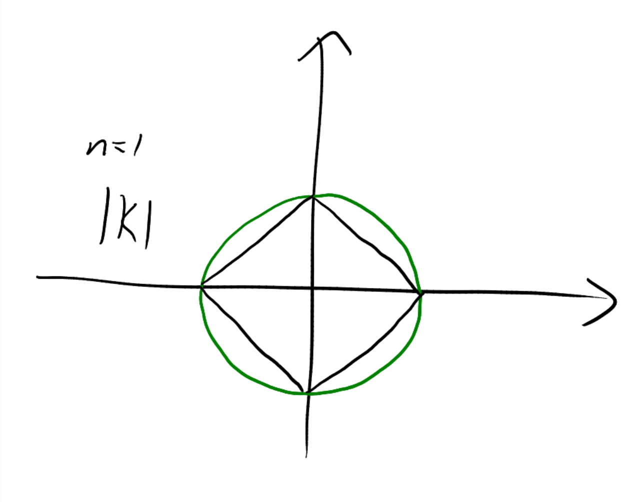
Definition (Triangulation)**.** A *triangulation* of a space X is a geometric simplicial complex K and a homeomorphism $h : |K| \xrightarrow{\cong} X$.

Example. The *standard n-simplex* is $\Delta^n = \langle e_1, \ldots, e_{n+1} \rangle \subseteq \mathbb{R}^{n+1}$. It, along with its faces, defines a simplicial complex.



Example. The *simplicial $(n-1)$-sphere* is the simplicial complex given by the proper faces of Δ^n. Its polyhedron is $\partial\Delta^n \subseteq \mathbb{R}^{n+1}$.

Example. In $\mathbb{R}^{n+1}$ consider the $2^{n+!}$ simplexes given by $\langle \pm e_1, \pm e_2, \dots, \pm e_{n+1}\rangle$ and let K be given by these and all their faces.



Define

$$h : \mathbb{R}n + 1 \setminus \{0\} \supseteq |K| \to S^n$$
$$x \mapsto \frac{x}{|x|}$$

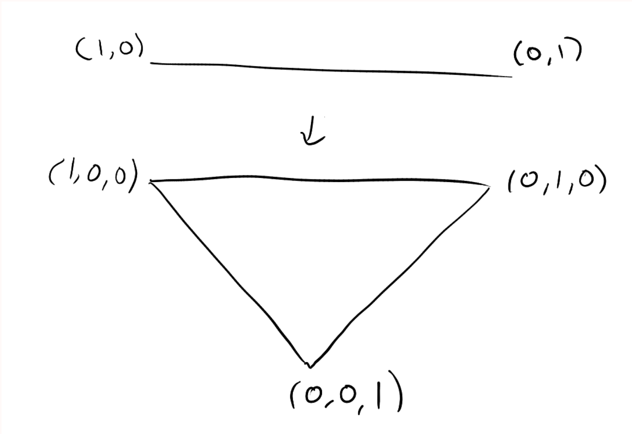
this is continuous, and a bijection. As $|K|$ and S^n are compact Hausdorff, it is a homeomorphism.

Definition (Simplicial map)**.** Write V_K for the set of vertices (i.e. the 0-simplces) of K.

A *simplicial map* f from K to L is a frunction $f : V_K \to V_L$ such that if $\sigma = \langle a_0, \dots, a_n\rangle \in K$ then $\{f(a_0), \dots, f(a_n)\}$ spans a simplex of L, called $f(\sigma)$. Write $f : K \to L$.

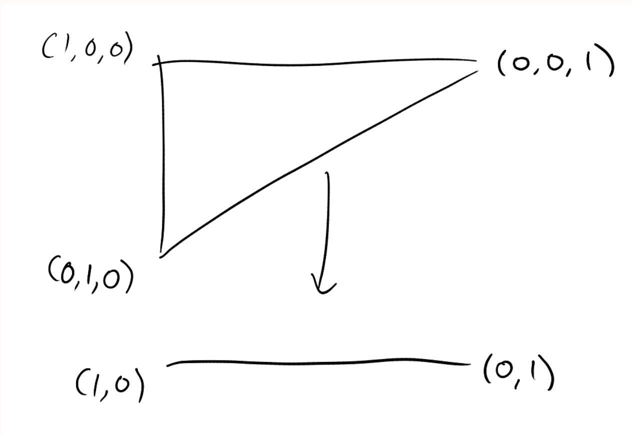
Example. The map

$$
\begin{aligned}
f : \Delta^1 &\to \Delta^2 \\
(1,0) &\mapsto (1,0,0) \\
(0,1) &\mapsto (0,1,0)
\end{aligned}
$$



Example. The map

$$
\begin{aligned}
g : \Delta^2 &\to \Delta^1 \\
(1,0,0) &\mapsto (1,0) \\
(0,1,0) &\mapsto (1,0) \\
(0,0,1) &\mapsto (0,1)
\end{aligned}
$$



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Lemma. A simplicial map $f : K \to L$ of simplicial complexes induces a continuous map $|f| : |K| \to |L|$, and $|f \circ g| = |f| \circ |g|$.

Proof. For a $\sigma \in K$, $\sigma = \langle a_0, a_1, \dots, a_p \rangle$, define

$$\begin{aligned} f_\sigma : \sigma &\to |L| \\ \sum_{i=0}^{p} t_i a_i &\mapsto \sum_{i=0}^{p} t_i f(a_i) \end{aligned}$$

which is linear in the t_i, hence continuous. If $\tau \le \sigma$, then $f_\tau = f_\sigma|_\tau$, so $f_\sigma|_{\sigma\cap\sigma'} = f_{\sigma'}|_{\sigma\cap\sigma'}$, so the f_σ glue to a continuous $|f| : |K| = \bigcup_{\sigma \in K} \sigma \to |L|$. The formula for $|f|$ shows that it behaves as claimed under composition. □

So we can recover f from $|f|$ and the discrete sets $V_K \subset |K|$, $V_L \subset |L|$, i.e. a simplicial map is the same as a continuous map $|K| \to |L|$ which sends vertices to vertices, and is affine on each simplex.

Definition (Star and link)**.** For a $x \in |K|$,

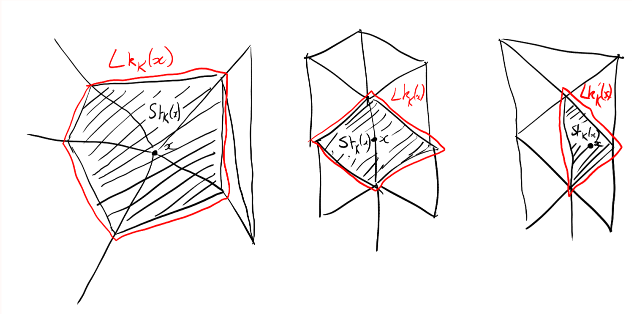
(i) The (open) *star* of x is the union of the interiors of the simplexes which contain x

$$\mathrm{St}_K(x) = \bigcup_{\sigma \in K} \mathring{\sigma} \subset \mathbb{R}^m$$

The complement of $\mathrm{St}_K(x)$ is the union of simplexes which do not contain x, a polyhedron, so closed. Thus $\mathrm{St}_K(x)$ is open.

(ii) The *link* of x, $\mathrm{Lk}_K(x)$ is the union of those simplexes which do not contain x, but are faces of a simplex which does not contain x.

Example.



6.2 Simplicial approximation

Definition (Simplicial approximation)**.** Let $f : |K| \to |L|$ be a continuous map. A *simplicial approximation* to f is a function $g : V_K \to V_L$ such that

$$f(\operatorname{St}_K(v)) \subset \operatorname{St}_L(g(v))$$

for all $v \in V_K$.

Lemma. If g is a simplicial approximation to a continuous map f, then g is a simplicial map, and f is is homotopic to $|g|$. Furthermore, this homotopy may be taken relative to

$$\{x \in |K| \mid f(x) = |g|(x)\}.$$

Proof. To show that g defines a simplicial map, for $\sigma \in K$ we must show that the images under g of the vertices of σ span a simplex of L.

For $x \in \mathring{\sigma}$, we have $x \in \bigcap_{v \in V_\sigma} \operatorname{St}_K(v)$, so

$$f(x) = \bigcap_{v \in V_\sigma} f(\operatorname{St}_K(v)) \subset \bigcap_{v \in V_\sigma} \operatorname{St}_L(g(v)).$$

If τ is the unique simplex of L with $f(x) \in \mathring{\tau}$, then each $g(v)$ is a vertex of σ. So the $\{g(v)\}$ span a face of τ, which is a simplex of L.

Want to show $f \simeq |g|$. If $|L| \subseteq \mathbb{R}^m$, then let

$$\begin{aligned} H : |K| \times I &\to |L| \subset \mathbb{R}^m \\ (x,t) &\mapsto t \cdot f(x) + (1-t)|g|(x) \end{aligned}$$

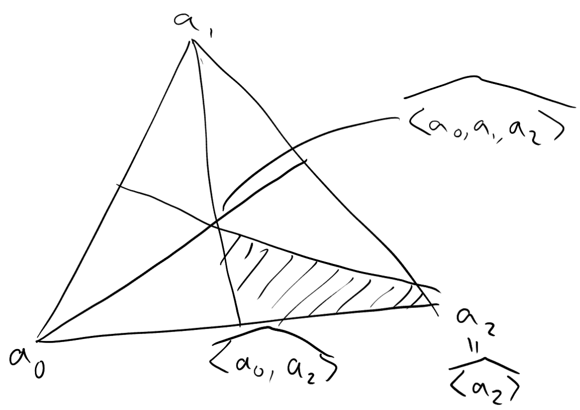
This is continuous, so need to show it indeed lies in $|L|$. Let $x \in \mathring{\sigma} \subset |K|$ and suppose $f(x) \in \mathring{\tau} \subset |L|$. If $\sigma = \langle a_0, \dots, a_p \rangle$ then by the above each $g(a_i)$ is a vertex of τ. Then

$$|g|(x) = \sum_{i=0}^{p} t_i g(a_i) \in \tau$$

as it is a convex linear combination of vertices of τ. As $f(x) \in \tau$, each of $tf(x) + (1 - t)|g|(x)$ lies in τ too. $\square$

Definition (Barycentre)**.** The *barycentre* of a simplex $\sigma = \langle a_0, \dots, a_p \rangle$ is the point

$$\hat{\sigma} = \frac{1}{p+1}(a_0 + a_1 + \cdots + a_p).$$



Definition (Barycentre subdivision)**.** The *barycentre subdivision* of a simplicial complex K is

$$K' = \{\langle \widehat{\sigma_0}, \dots, \widehat{\sigma_p} \rangle \mid \sigma_i \in K, \text{ and } \sigma_0 < \sigma_1 < \cdots < \sigma_p\}$$

We will use the notation $K^{(r)} = (K^{(r-1)})'$ (where $K^{(0)} = K$).

Note. It is not obvious that this is a simplicial complex.

Proposition. K' is a simplicial complex, and $|K'| = |K|$.

Proof. If $\sigma_0 < \sigma_1 < \cdots < \sigma_p$ then the $\widehat{\sigma_i}$ are affinely independent: Suppose $\sum_{i=0}^p t_i\widehat{\sigma_i} = 0$ and $\sum_{i=0}^p t_i = 1$. Let $j = \max\{i \mid t_i \neq 0\}$. Then

$$\widehat{\sigma_j} = -\sum_{i=0}^{p} \frac{t_i}{t_j}\widehat{\sigma_i} \in \sigma_{j-1},$$

so $\widehat{\sigma_j}$ lies in a proper face of σ_j, which is not possible. Thus all t_i must be 0.

K' is a simplicial complex: Let $\langle\widehat{\sigma_0}, \dots, \widehat{\sigma_p}\rangle \in K'$. A face is given by omitting some $\widehat{\sigma_j}$'s. But omitting some σ_j's from $\sigma_0 < \sigma_1 < \cdots < \sigma_p$ still gives a strictly increasing chain of simplexes of K. Let $\sigma' = \langle\widehat{\sigma_0}, \dots, \widehat{\sigma_p}\rangle$, $\tau' = \langle\widehat{\tau_0}, \dots, \widehat{\tau_1}\rangle$ and consider $\sigma' \cap \tau'$. This is inside $\sigma_p \cap \tau_p$, which is a simplex δ of K. So there are simplexes

$$\sigma'' = \langle\widehat{\sigma_0} \cap \delta, \dots, \widehat{\sigma_p} \cap \delta\rangle, \qquad \tau'' = \langle\widehat{\tau_0} \cap \delta, \dots, \widehat{\tau_p} \cap \delta\rangle$$

of K. Now $\sigma' \cap \tau' = \sigma'' \cap \tau''$ This reduces to the case that σ'' and τ'' are contained in a simplex δ of K. Now we split into cases. If σ'' and τ'' contain $\widehat{\delta}$, then let $\overline{\sigma}$, $\overline{\tau}$ be the faces of σ'', τ'' opposite to $\widehat{\delta}$. Then $\sigma'' \cap \tau''$ is spanned by $\widehat{\delta}$ and $\overline{\sigma}'' \cap \overline{\tau}''$. But $\overline{\sigma}'' \cap \overline{\tau}'' \subset \partial\delta$, which has a smaller dimension than σ, so can suppose it is a simplex of $\partial\delta$ by induction on dimension. If σ'' or τ'' does not contain $\widehat{\delta}$, then $\sigma'' \cap \tau'' \subset \partial\delta$, so again finish by induction on dimension.

$|K'| = |K|$: Note $\langle\widehat{\sigma_0}, \widehat{\sigma_1}, \dots, \widehat{\sigma_p}\rangle \le \sigma_p \le |K|$, so $|K'| \subseteq |K|$. Conversely, if $x \in \sigma = \langle a_0, \dots, a_p\rangle \subset |H|$ is written as

$$x = \sum_{i=0}^{p} t_i a_i,$$

can reorder the a_i so that $t_0 \ge t_1 \ge t_2 \ge \cdots \ge t_p$, so

$$\begin{aligned} x &= \underbrace{(t_0 - 1)}_{\ge 0} a_0 + 2\underbrace{(t_1 - t_2)}_{\ge 0}\left(\frac{a_0 + a_1}{2}\right) + 3\underbrace{(t_2 - t_3)}_{\ge 0}\left(\frac{a_0 + a_1 + a_2}{3}\right) + \cdots \\ &= (t_0 - t_1)\widehat{\langle a_0\rangle} + 2(t_1 - t_1)\widehat{\langle a_0, a_1\rangle} + 3(t_2 - t_3)\widehat{\langle a_0, a_1, a_2\rangle} + \cdots \\ &\in \langle\widehat{\langle a_0\rangle}, \widehat{\langle a_0, a_1\rangle}, \dots, \widehat{\langle a_0, \dots, a_p\rangle}\rangle \\ &\subseteq |K'| \end{aligned}$$

□

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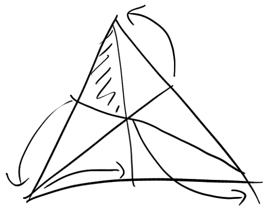
The *vertices* of K' are in bijection with the simplexes of K. Choose a function $K \to V_K$ which assigns to σ *some* vertex of σ. So

$$\begin{aligned} g : V_{K'} \cong K &\to V_K \\ \widehat{\sigma}(\leftrightarrow \sigma) &\mapsto v_\sigma \end{aligned}$$



If $\langle \widehat{\sigma_0}, \dots, \widehat{\sigma_p} \rangle$ is a simplex of K' then $\sigma_0 \le \sigma_1 \le \dots \le \sigma_p$ and $g(\widehat{\sigma_i})$ is some vertex of σ, so a vertex of σ_p. The $\{g(\widehat{\sigma_i})\}$ thus spans a face of σ, so is a simplex of K if and only if g is a simplicial map.

Also, if $\widehat{\sigma} \in \tau' = \langle \widehat{\tau}_0, \dots, \widehat{\tau}_p \rangle \in K'$, then $\widehat{\sigma} \in \tau_p$, so σ is a face of τ_p. So $v_\sigma \in \sigma \subseteq \tau_p$. So $\mathring{\tau}' \subset \mathring{\tau}_p \subseteq \mathrm{St}_K(v_\sigma)$. Thus

$$\mathrm{St}_{K'}(\widehat{\sigma}) \subset \mathrm{St}_K(v_\sigma = g(\sigma))$$

so g is a simplicial approximation to $\mathrm{id} : |K'| \to |K|$. So $|g| \simeq \mathrm{id}$.

Definition (Mesh)**.** The *mesh* of K is

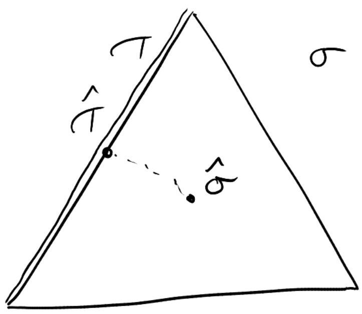
$$\mu(K) = \max\{|v_0 - v_1| \mid \langle v_0, v_1 \rangle \in K\}.$$

Lemma. Suppose K has dimension $\le n$, then $\mu(K^{(r)}) \le \left(\frac{n}{n+1}\right)^r \mu(K)$, so $\mu(K^{(r)}) \to 0$ as $r \to \infty$.

Proof. Enough to treat the case $r = 1$.

Let $\langle \widehat{\tau}, \widehat{\sigma} \rangle \in K'$, so $\tau \le \sigma$ in K.

$$|\widehat{\tau} - \widehat{\sigma}| \le \max\{|v - \widehat{\sigma}| \mid v \text{ is a vertex of } \sigma\}.$$



Let $\sigma = \langle v_0, v_1, \dots, v_m \rangle$ with $m \le n$, and reorder so that the maximum is attained at $v = v_0$.

$$\begin{aligned} |v_0 - \widehat{\sigma}| &= \left| v_0 - \frac{1}{m+1} \sum_{i=0}^m v_i \right| \\ &= \left| \frac{m+1}{m+1} v_0 - \frac{1}{m+1} \sum_{i=0}^m v_i \right| \\ &= \frac{1}{m+1} \left| \sum_{i=0}^m v_0 - v_i \right| \\ &\le \frac{1}{m+1} \sum_{i=1}^m |v_0 - v_i| \\ &\le \frac{m}{m+1} \mu(K) \\ &\le \frac{n}{n+1} \mu(K) \end{aligned}$$ □

Theorem (Simplicial Approximation Theorem)**.** Let $f : |K| \to |L|$ be a continuous map. Then there is a $r \gg 0$ and a simplicial map $g : K^{(r)} \to L$ such that g is a simplicial approximation to f.

If f is simplicial on some $|N| \subset |K|$, can take $g|_{V_N} = f|_{V_N}$.

Proof. The $\mathrm{St}\,_L(\omega)$, $\omega \in V_L$ is an open cover of $|L|$, so

$$\{f^{-1} \mathrm{St}\,_K(\omega)\}_{\omega \in V_L}$$

is an open cover of $|K|$; pick $\delta > 0$ using Lebesgue number lemma for this cover. Choose $r \gg 0$ such that $\mu(K^{(r)}) < \delta$. For each $v \in V_{K^{(r)}}$ have

$$\mathrm{St}\,_{K^{(r)}}(v) \subseteq B_{\mu(K^{(r)})}(v) \subseteq f^{-1}(\mathrm{St}\,_L(\omega))$$

for some $w \in V_L$. Define $g : V_{K^{(r)}} \to V_L$ by $g(v) = w$. Then

$$f(\mathrm{St}\,_{K^{(r)}}(v)) \subseteq \mathrm{St}\,_L(g(v)).$$

So g is a simplicial approximation to f. So g is a simplicial map.

The final step is by choosing w carefully when $v \in V_N$. □

Corollary. If $n < m$, then any map $f : S^n \to S^m$ is homotopic to a constant map.

Proof. Spheres are polyhedra: $S^n = |K|$, $S^n = |L|$, then f is homotopic to $|g|$, for some $g : K^{(r)} \to L$. This can not hit any m-simplex of L, as K has dim $\leq n$. So $|g|$ must miss every point on the interior of some m-simplex: it is not onto. Thus it factors through $(S^n - \{*\}) \simeq *$, so is homotopic to a constant map. □

Start of

lecture 16

7.1 Simplicial homology

Definition. Let K be a simplicial complex. We define $\mathcal{O}_n(K)$ to be the free abelian group ($\mathbb{Z}$-module) with basis

$$\{[v_0, v_1, \dots, v_n] \mid \text{the } v_i \text{ are vertices of } K \text{ which span a simplex}\}.$$

The v_i are considered to be *ordered*, and could span a simplex of dim $< n$, i.e. could have repeats.

Definition. Define $T_n(K) \leq \mathcal{O}_n(K)$ to be the subgroup spanned by:

(i) $[v_0, v_1, \dots, v_n]$ containing a repeat.

(ii) $[v_0, v_1, \dots, v_n] - \text{sgn}(\sigma)[v_{\sigma(0)}, \dots, v_{\sigma(n)}]$ for a permutation σ of $\{0, 1, \dots\}$.

Define $C_n(K) = \mathcal{O}_n(K)/T_n(K)$, the quotient group.

Lemma. There is a non-canonical isomorphism $C_n(K) \cong \mathbb{Z}\{n\text{-simplices of } K\}$.

Proof. Choose a total order $\prec$ of V_K. Then each n-simplex σ of K determines a canonical ordered simplex $[\sigma] \in \mathcal{O}_n(K)$ by ordering its vertices such that $a_0 \prec a_1 \prec \cdots \prec a_n$. This gives:

$$\begin{aligned} \phi : \mathbb{Z}\{n\text{-simplices of } K\} &\to \mathcal{O}_n(K) \\ \sigma &\mapsto [\sigma] \end{aligned}$$

ans so gives

$$\phi' : \mathbb{Z}\{n\text{-simplices of } K\} \to C_n(K)$$

For each $[a_0, a_1, \dots, a_n] \in \mathcal{O}_n(K)$, there is a *unique* permutation τ of $\{0, 1, \dots, n\}$ such that $a_{\tau(0)} \prec a_{\tau(1)} \prec \dots \prec a_{\tau(n)}$. Let

$$\operatorname{sgn}[a_0, \dots, a_n] := \operatorname{sgn}(\tau) \in \{\pm 1\}.$$

Define

$$\begin{aligned} \rho : \mathcal{O}_n(K) &\to \mathbb{Z}\{n\text{-simplices of } K\} \\ [a_0, \dots, a_n] &\mapsto \begin{cases} (\operatorname{sgn}[a_0, \dots, a_n])\langle a_0, \dots, a_n\rangle & \text{no repeats} \\ 0 & \text{repeats} \end{cases} \end{aligned}$$

For this to descend to $C_n(K)$, need $T_n(K)$ to be in $\ker(\rho)$. Certainly $[a_0, \dots, a_n]$'s with repeats are in $\ker(\rho)$.

$$\begin{aligned} \rho([v_0, \dots, v_n] - \operatorname{sgn}(\sigma)[v_{\sigma(0)}, \dots, v_{\sigma(n)}]) &= \operatorname{sgn}[v_0, \dots, v_n]\langle v_0, \dots, v_n\rangle \\ - \operatorname{sgn}(\sigma) \operatorname{sgn}[v_{\sigma(0)}, \dots, v_{\sigma(n)}]\langle v_{\sigma(0)}, \dots, v_{\sigma(n)}\rangle \\ &= 0 \end{aligned}$$

as required. So we get $\rho' : C_n(K) \to \mathbb{Z}\{n\text{-simplices of } K\}$. Not $\rho' \circ \phi'(\sigma) = \sigma$. If $[a_0, \dots, a_n]$ has no repeats, then

$$\begin{aligned} \phi' \circ \rho'([a_0, \dots, a_n]) &= \phi'(\operatorname{sgn}[a_0, \dots, a_n]\langle a_0, \dots, a_n\rangle) \\ &= \operatorname{sgn}[a_), \dots, a_n][a_{\tau(0)}, \dots, a_{\tau(n)}] \\ &= [a_0, \dots, a_n] \mod T_n(K) \end{aligned}$$

So ϕ' and ρ' are inverse. □

Definition (d_n)**.** Define a homomorphism

$$\begin{aligned} d_n : \mathcal{O}_n(K) &\to \mathcal{O}_{n-1}(K) \\ [v_0, \dots, v_n] &\mapsto \sum_{i=0}^n (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_n] \end{aligned}$$

where the $\widehat{v_i}$ means "skipel this element".

Lemma. d_n sends $T_n(K)$ into $T_{n-1}(K)$.

Proof. Note

$$\begin{aligned} &d_n([v_0, \dots, v_n] - \operatorname{sgn}(\sigma)[v_{\sigma(0)}, \dots, v_{\sigma(n)}]) \\ &= \sum_{i=0}^n (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_n] - \sum_{i=0}^n (-1)^i \operatorname{sgn}(\sigma)[v_{\sigma(0)}, \dots, \widehat{v_{\sigma(i)}}, \dots, v_n] \end{aligned}$$

Need to show that this is trivial in $\mathcal{O}_{n-1}(K)/T_{n-1}(K)$. Suppose first $\sigma = (j, j+1)$, a transposition, so $\mathrm{sgn}(\sigma) = -1$. Then

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^i \operatorname{sgn}(\sigma)[v_{\sigma(0)}, \dots, \widehat{v_{\sigma(i)}}, \dots, v_{\sigma(n)}] \\
&\quad= \sum_{i=0}^{j-1}(-1)^{i+1}[v_0, \dots, \widehat{v_i}, \dots, v_{j+1}, v_j, v_{j+2}, \dots, v_n] \\
&\qquad + (-1)^{j+1}[v_0, \dots, v_{j-1}, v_j, v_{j+2}, \dots, v_n] \\
&\qquad + (-1)^{j+2}[v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_n] \\
&\qquad + \sum_{i=j+2}^{n}(-1)^{i+1}[v_0, \dots, v_{j-1}, v_{j+1}, v_j, \dots, v_{j+2}, \dots, \widehat{v_i}, \dots, v_n]
\end{aligned}$$

In the first sum

$$[v_0, \dots, \widehat{v_i}, \dots, v_{j-1}, v_{j+1}, v_{j+2}, \dots, v_n] \equiv -[v_0, \dots, \widehat{v_i}, \dots, v_n] \mod T_{n-1}(K).$$

In the second sum,

$$[v_0, \dots, v_{j-1}, v_{j+1}, v_j, v_{j+2}, \dots, \widehat{v_i}, \dots, v_n] \equiv -[v_0, \dots, \widehat{v_i}, \dots, v_n] \mod T_{n-1}(K).$$

Then

$$\mathrm{RHS} \equiv \sum_{i=0}^{n}(-1)^i[v_0, \dots, \widehat{v_i}, \dots, v_n] \mod T_{n-1}(K)$$

as required. Any σ is a product of $(j, j+10$, so we get the same for any σ.

Now suppose $[v_0, \dots, v_n]$ with $v_j = v_{j+1}$ (if it has repeats, we can assume this without loss of generality by using permutations since we showed this doesn't affect values mod $T_{n-1}(K)$). Then:

$$\begin{aligned}
d_n[v_0, \dots, v_n] &= \sum_{i=0}^{j-1}(-1)^i[v_0, \dots, \widehat{v_i}, \dots, v_j, v_{j+1}, \dots, v_n] \\
&\quad + (-1)^j[v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_n] \\
&\quad + (-1)^{j+1}[v_0, \dots, v_j, v_{j+2}, \dots, v_n] \\
&\quad + \sum_{i=j+2}^{n}(-1)^j[v_0, \dots, v_j v_{j+1}, \dots, \widehat{v_i}, \dots, v_n] \\
&\in T_{n-1}(K)
\end{aligned}$$

This is because the sums are both in $T_{n-1}(K)$ since each term has a repeat, and the middle alone terms cancel each other since $v_j = v_{j+1}$. □

So d_n induces a homomorphism $d_n : C_n(K) \to C_{n-1}(K)$ given by the same formula.

Lemma. The composition $d_{n-1} \circ d_n : C_n(K) \to C_{n-1}(K)$ is zero.

Proof. At the level of $\mathcal{O}_n(K)$, compute

$$\begin{aligned}
&d_{n-1} \circ d_n[v_0, \dots, v_n] \\
&\quad = d_{n-1} \left(\sum_{i=0}^{n} (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_n] \right) \\
&\quad = \sum_{i=0}^{n} (-1)^i \left[\sum_{k=0}^{i-1} (-1)^k [v_0, \dots, \widehat{v_i}, \dots, v_n] + \sum_{k=i}^{n-1} (-1)^k [v_0, \dots, \widehat{v_i}, \dots, v_n] \right]
\end{aligned}$$

Coefficient of $[v_0, \dots, \widehat{v_a}, \dots, \widehat{v_b}, \dots, v_n]$ is $(-1)^a(-1)^b + (-1)^a(-1)^{b-1} = 0$. As the $[v_0, \dots, v_n]$ generate, we get $d_{n-1} \circ d_n = 0$. □

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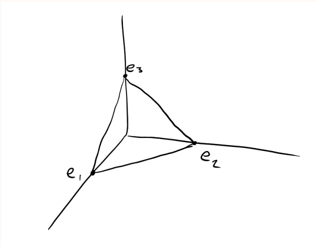
Definition (n-th simplicial homology group)**.** The *n-th simplicial homology group of K* is

$$H_n(K) := \frac{\ker(d_n : C_n(K) \to C_{n-1}(K))}{\operatorname{Im}(d_{n-1} : C_{n+1}(K) \to C_n(K))}.$$

Example. Let K be the union of all the proper faces of the standard 2-simplex $\Delta^2 \subset \mathbb{R}^3$, i.e.

$$K = \{\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1, e_2\rangle, \langle e_2, e_3\rangle, \langle e_3, e_1\rangle\}.$$



Order the vertices as $e_1 \prec e_2 \prec e_3$. Then

$$\begin{aligned} C_0(K) &= \mathbb{Z}\{[e_1], [e_2], [e_3]\} \\ C_1(K) &= \mathbb{Z}\{[e_1, e_2], [e_2, e_3], [e_1, e_3]\} \\ C_n(K) &= 0 \qquad\qquad (n \geq 2) \end{aligned}$$

and

$$d_1 : C_1(K) \to C_0(K), \qquad d_1[e_i, e_j] = [e_j] - [e_i]$$

$$\begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Note $\mathrm{Im}(d_1) = \langle [e_i] - [e_j]\rangle_{\mathbb{Z}}$, so

$$H_0(K) = \frac{\mathbb{Z}\{[e_1], [e_2], [e_3]\}}{\langle [e_i] - [e_j]\rangle} \cong \mathbb{Z}.$$

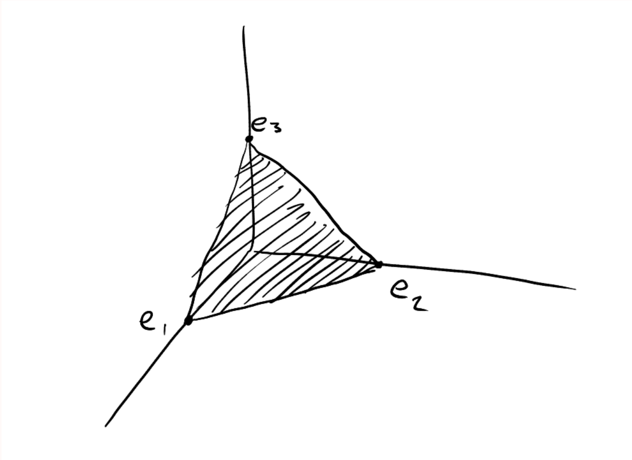
For H_1, note

$$\begin{aligned} \ker(d_1) &= \mathbb{Z}\{[e_1, e_2] \underbrace{-[e_1, e_3]}_{=[e_3, e_1]} + [e_2, e_3]\} \cong \mathbb{Z} \\ \mathrm{Im}(d_2) &= 0 \end{aligned}$$

So $H_1(K) \cong \mathbb{Z}$.

Example. Now let L be the standard 2-simplex $\Delta^2 \subset \mathbb{R}^3$, i.e.

$$L = \{\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1, e_2\rangle, \langle e_1, e_3\rangle, \langle e_2, e_3\rangle, \langle e_1, e_2, e_3\rangle\}.$$



Then

$$\begin{aligned} C_0(L) &= \mathbb{Z}\{[e_1], [e_2], [e_3]\} \\ C_1(L) &= \mathbb{Z}\{[e_1, e_2], [e_1, e_3], [e_2, e_3]\} \\ C_2(L) &= \mathbb{Z}\{[e_1, e_2, e_3]\} \end{aligned}$$

So

$$0 \xrightarrow{d_4} 0 \xrightarrow{d_3} \underbrace{C_2(L)}_{=\mathbb{Z}} \xrightarrow{d_2} \underbrace{C_1(L)}_{=\mathbb{Z}^3} \xrightarrow{d_1} \underbrace{C_0(L)}_{=\mathbb{Z}^3} \to 0.$$

Note

$$d_2[e_1, e_2, e_3] = [e_2, e_3] - [e_1, e_3] + [e_1, e_2] \neq 0,$$

so d_2 is injective, so $H_2(L) = 0$. But also

$$\begin{aligned} H_1(L) &= \frac{\ker(d_1)}{\mathrm{Im}(d_2)} = \frac{\mathbb{Z}\{[e_1, e_2] - [e_1, e_3] + [e_2, e_3]\}}{\mathbb{Z}\{[e_1, e_2] - [e_1, e_3] + [e_2, e_3]\}} = 0 \\ H_0(L) &= \mathbb{Z} \end{aligned}$$

(where $H_0(L) = \mathbb{Z}$ because the relevant groups C_1, C_0 haven't changed since the previous example).

7.2 Some homological algebra

Definition (Chain complex)A *chain complex* is a sequence $C_0, C_1, C_2, \dots$ of abelian groups and homomorphisms $d_n : C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n = 0$ for all n. Write this data as $C_\bullet$, and call the d_n's the *differentials* of $C_\bullet$. Then define

$$H_n(C_\bullet) := \frac{\ker(d_n : C_n \to C_{n-1})}{\operatorname{Im}(d_{n+1} : C_{n+1} \to C_n)}.$$

Notation. Write

$$Z_n(C_\bullet) := \ker(d_n : C_n \to C_{n-1}) \qquad \text{"the } n\text{-cycles of } C_\bullet\text{"}$$
$$B_n(C_\bullet) := \operatorname{Im}(d_{n+1} : C_{n+1} \to C_n) \qquad \text{"the } n\text{-boundaries of } C_\bullet\text{"}$$

Definition (Chain map). A *chain map* $f_\bullet : C_\bullet \to D_\bullet$ is a sequence of homomorphisms $f_n : C_n \to D_n$, such that $f_n \circ d_{n+1} = d_{n+1} \circ f_{n+1}$, i.e. the diagram

$$\begin{array}{ccc} C_{n+1} & \xrightarrow{f_{n+1}} & D_{n+1} \\ \downarrow{\scriptstyle d_{n+1}} & & \downarrow{\scriptstyle d_{n+1}} \\ C_n & \xrightarrow{f_n} & D_n \end{array}$$

commutes.

Definition (Chain homotopy). A *chain homotopy* between $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ is a sequence of homomorphisms $h_n : C_n \to D_{n+1}$ such that

$$g_n - f_n = d_{n+1} \circ f_n + h_{n-1} \circ d_n.$$

The above definition is hard to motivate at the moment, but one should just accept it as it is for now.

Lemma. A chain map $f_\bullet : C_\bullet \to D_\bullet$ induces a homomorphism

$$\begin{aligned} f_* : H_n(C_\bullet) &\to H_n(D_\bullet) \\ [x] &\mapsto [f_n(x)] \end{aligned}$$

Furthermore, if $g_\bullet$ is chain homotopic to $f_\bullet$, then $g_* = f_*$.

Proof. Need to show that f_* is well-defined.

(i) Let $[x] \in H_n(C_\bullet)$, i.e. $x \in C_n$ and $d_n(x) = 0$. Then

$$d_n f_n(x) = f_{n-1} \underbrace{d_n(x)}_{=0} = 0.$$

so $f_n(x) \in Z_n(D_\bullet)$.

(ii) If $[x] = [y] \in H_n(C_0)$, then $x - y \in B_n(C_\bullet)$, $x - y = d_{n+1}(z)$. So

$$f_n(x) - f_n(y) = f_n d_{n+1}(z) = d_{n+1} f_{n+1}(z) \in B_n(D_\bullet),$$

so $[f_n(x)] = [f_n(y)]$. So f_* is a well-defined function. It is a homomorphism.

Now let $g_\bullet$ be chain homotopic to $f_\bullet$, i.e. $g_n - f_n = d_{n+1} \circ h_n + h_{n-1} \circ d_n$. Let $x \in Z_n(C_\bullet)$, so

$$g_n(x) - f_n(x) = d_{n+1} \circ h_n(x) + h_{n-1} \circ d_n(x) \in B_n(D_\bullet).$$

So $g_*([x]) = [g_n(x)] = [f_n(x)] = f_*([x])$. □

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Just as we did for homotopy of maps between spaces, one checks:

(1) being chain homotopic defines an equivalence relation on the set of chain maps from $C_\bullet$ to $D_\bullet$. Write $f_\bullet \simeq g_\bullet$.

(2) if $a_\bullet : A_\bullet \to C_\bullet$ is a chain map, and $f_\bullet \simeq g_\bullet : C_\bullet : D_\bullet$, then $f_\bullet \circ a_\bullet \simeq g_\bullet \circ a_\bullet$ and similarly with post composition.

Definition (Chain homotopy equivalence). A chain map $f_\bullet : C_\bullet \to D_\bullet$ is a *chain homotopy equivalence* if there is a $g_\bullet : D_\bullet \to C_\bullet$, $g_\bullet \circ f_\bullet \simeq \mathrm{id}_{C_\bullet}$, $f_\bullet \circ g_\bullet \simeq \mathrm{id}_{D_\bullet}$.

Lemma. If $f_\bullet : C_\bullet \to D_\bullet$ is a chain homotopy equivalence, then $f_* : H_n(C_\bullet) \to H_n(D_\bullet)$ is an isomorphism.

Proof. Using a homotopy inverse $g_\bullet$, have $f_* \circ f_* = (f_\bullet \circ g_\bullet)_* = (\mathrm{id}_{D_\bullet})_* = \mathrm{id}_{H_n(D_\bullet)}$ and simlarly for $g_* \circ f_* = \mathrm{id}_{H_n(C_\bullet)}$. □

Exercise: Let

$$\mathbb{Z}[n] = (\to 0 \stackrel{d_{n+1}}{\to} \mathbb{Z} \stackrel{d_n}{\to} 0 \to \cdots).$$

Describe

$$\{\mathbb{Z}[n] \to C_\bullet\}/\text{chain homotopy}.$$

7.3 Elementary calculations

Return to the chain complexes $C_\bullet(K)$ associated to a simplicial complex K.

Lemma. Let $f : K \to L$ be a simplicial map. Then the formula

$$\begin{aligned} f_\bullet : C_n(K) &\to C_n(L) \\ [a_0, \dots, a_n] &\mapsto [f(a_0), \dots, f(a_n)] \end{aligned}$$

is a well-defined homomorphism, and defines a chain map $f_\bullet : C_\bullet(K) \to C_\bullet(L)$, and hence gives a $f_* : H_n(K) \to H_n(L)$.

Proof. To be well-defined, need the given formula to send $T_n(K)$ into $T_n(L)$. It does. To be a chain map, need:

$$\begin{aligned} f_{n-1}d_n[a_0, \dots, a_n] &= f_{n-1}\left(\sum_{i=0}^{n}(-1)^i[a_0, \dots, \widehat{a_i}, \dots, a_n]\right) \\ &= \sum_{i=0}^{n}(-1)^i[f(a_0), \dots, \widehat{f(a_i)}, \dots, f(a_n)] \\ &= d_n f_n[a_0, \dots, a_n] \end{aligned}$$

□

Definition (Cone)**.** Say a simplicial complex K is a *cone* with *cone point* $v_0 \in V_K$ if every simplex of K is a face of a simplex which has v_0 as a vertex.

Proposition. If K is a cone with cone point v_0, then the inclusion $i : \{v_0\} \hookrightarrow K$ induces a chain homotopy equivalence $i_\bullet : C_\bullet(\{v_0\}) \to C_\bullet(K)$ and so

$$H_n(K) \cong \begin{cases} \mathbb{Z}\{[v_0]\} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. The only map $r : V_K \to \{v_0\}$ is a simplicial map $r : K \to \{v_0\}$, and $r \circ i \simeq \mathrm{id}_{\{v_0\}}$. I claim that $i \circ r \simeq \mathrm{id}_{C_\bullet(K)}$. Define

$$\begin{aligned} h_n : \mathcal{O}_n(K) &\to \mathcal{O}_{n+1}(K) \\ [a_0, \dots, a_n] &\mapsto [v_0, a_0, \dots, a_n] \end{aligned}$$

Note that this sends $T_n(K)$ into $T_{n+1}(K)$, so descends to $h_n : C_n(K) \to C_{n+1}(K)$. For $n > 0$, then

$$\begin{aligned}
(h_{n-1} \circ d_n + d_{n+1} \circ h_n)[a_0, \dots, a_n] &= \left(\sum_{i=0}^{n} (-1)^i [v_0, a_0, \dots, \widehat{a_i}, \dots, a_n] \right) \\
&\quad + \left([a_0, \dots, a_n] + \sum_{i=0}^{n} (-1)^{i+1} [v_0, a_0, \dots, \widehat{a_i}, \dots, a_n] \right) \\
&= [a_0, \dots, a_n] \\
&= (\mathrm{id} - i_n \circ r_n)[a_0, \dots, a_n]
\end{aligned}$$

since $i_n \circ r_n[a_0, \dots, a_n] = [v_0, \dots, v_0] = 0$ as $n > 0$.

For $n = 0$,

$$\begin{aligned}
(\underbrace{h_{-1}}_{=0} \circ d_0 + d_1 \circ h_0)[a_0] &= d_1[v_0, a_0] - [a_0] - [v_0] \\
&= (\mathrm{id} - i_0 r_0)[a_0]
\end{aligned}$$

So h provides a chain homotopy from $\mathrm{id}_{C_\bullet(K)}$ to $i_\bullet \circ r_\bullet$. □

Corollary. The standard n-simplex $\Delta^n \subset \mathbb{R}^{n+1}$ and all its faces, L, is a cone with any vertex as cone point. So

$$H_i(L) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Corollary. Let K be the union of all the proper faces of $\Delta^n \subset \mathbb{R}^{n+1}$ (i.e. the simplicial $(n-1)$-sphere). Then for $n \ge 2$, we have

$$H_i(K) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} & i = n-1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Note that K is the $(n-1)$-skeleton of L, so $i : K \to L$ gives an isomorphism $C_i(K) \xrightarrow{\cong} C_i(L)$ for $i \le n-1$. These chain complexes are

$$\begin{array}{ccccccccccc}
C_0(L) & \xleftarrow{d_1^L} & C_1(L) & \xleftarrow{d_2^L} & \cdots & \xleftarrow{d_{n-1}^L} & C_{n-1}(L) & \xleftarrow{d_n^L} & C_n(L) & \xleftarrow{d_{n+1}^L} & 0 \\
\| & & \| & & \| & & \| & & \uparrow & & \\
C_0(K) & \xleftarrow{d_1^K} & C_1(K) & \xleftarrow{d_2^K} & \cdots & \xleftarrow{d_{n-1}^K} & C_{n-1}(K) & \xleftarrow{d_n^K} & 0 & \longleftarrow & 0
\end{array}$$

For $i \leq n-2$,

$$H_i(K) = H_i(L) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & 1 \leq i \leq n-2 \end{cases}$$

For $i > n-1$, $H_i(K) = 0$ as there are no i-simplexes.

$$H_{n-1}(K) = \frac{\ker(d_{n-1}^K)}{\mathrm{Im}(d_n^K)} = \ker(d_{n-1}^K) = \ker(d_{n-1}^L).$$

As $H_{n-1}(L) = 0$, $\frac{\ker(d_{n-1}^L)}{\mathrm{Im}(d_n^L)} = 0$, so $\ker(d_{n-1}^L) = \mathrm{Im}(d_n^L)$. As $H_n(L) = 0$, we see that $d_n^L : \mathbb{Z} \cong C_n(L) \to C_{n-1}(L)$ is injective. So $H_{n-1}(K) = \ker(d_{n-1}^L) = \mathrm{Im}(d_n^L) \cong \mathbb{Z}$. It is generated by $d_n^L[e_1, \dots, e_n] \in C_{n-1}(K)$. □

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Lemma. There is an isomorphism

$$H_0(K) \cong \mathbb{Z}\{\pi_0(|K|)\}.$$

Proof. Note we have a homomorphism

$$\begin{aligned} \phi : C_0(K) &\to \mathbb{Z}\{\pi_0(|K|)\} \\ [v] &\mapsto \text{the path component of } v \in |K| \end{aligned}$$

This is onto: any path-connected component of $|K|$ contains a vertex. If $[v, w]$ is an ordered 1-simplex, then $d_1[v, w] = [w] - [v]$. But $[v]$ and $[w]$ lie in the same path-connected component as the 1-simplex $\langle v, w \rangle$ goes between them. So $\mathrm{Im}(d_1) \subset \ker(\phi)$, so we get an induced surjective $\phi : H_0(K) \to \mathbb{Z}\{\pi_0(|K|)\}$. If $\phi([v]) = \phi([w])$, choose a path $\gamma : I \to |K|$ from v to w. By simplicial approximation, $I = \Delta^1$ can be subdivided so that there is a $g : (\Delta^1)^{(r)} \to K$ with $|g| \simeq \gamma$, i.e. there are 1-simplices $[v, v_1], [v_1, v_2], \dots, [v_k, w]$. Then

$$[w] - [v] = d_1([v, v_1] + [v_1, v_2] + \dots + [v_k, w])$$

so $[v] = [w] \in H_0(K)$. □

7.4 Mayer-Vietoris Theorem

Definition (Exact homomorphisms)**.** Say that a pair of homomorphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact* at B if $\mathrm{Im}(f) = \ker(g)$. More generally, a collection of homomorphisms

$$\cdots \to A_i \to A_{i+1} \to A_{i+2} \to A_{i+3} \to \cdots$$

is *exact* if it is exact at each A_j, where A_j has homomorphisms in and out.

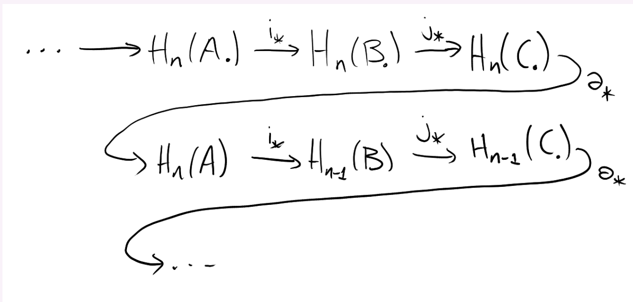
A *short exact sequence* is an exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

Note that in this case, f is injective, g is surjective and $\mathrm{Im}(f) = \ker(g)$.

Chain maps $i_\bullet : A_\bullet \to B_\bullet$ and $j_\bullet : B_\bullet \to C_\bullet$ form a *short exact sequence* of chain complexes if each $0 \to A_n \xrightarrow{i_n} B_n \xrightarrow{j_n} C_n \to 0$ is a short exact sequence.

Theorem. If $0 \to A_\bullet \xrightarrow{i_\bullet} B_\bullet \xrightarrow{j_\bullet} C_\bullet \to 0$ is a short exact sequence of chain complexes, then there are natural homomorphisms $\partial_* : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ such that



is an exact sequence.

Proof. **Constructing ∂_* (snake lemma)**:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_n & \xrightarrow{i_n} & B_n & \xrightarrow{j_n} & C_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{i_{n-1}} & B_{n-1} & \xrightarrow{j_{n-1}} & C_{n-1} & \longrightarrow & 0
\end{array}$$

Let $[x] \in H_n(C_\bullet)$. Then $d_n x = 0$. By surjectivity, there exists $y \in B_n$ such that $j_n y = x$. Then since the diagram commutes, and $d_n j_n y = 0$, we have that $j_{n-1} d_n y = 0$. Then $d_n(y)$ must be in the image of i_{n-1} (since the sequence is exact). Pick the unique z such that $i_{n-1} z = d_n(y)$ (uniqueness follows since i_{n-1} is injective). Then

$$\begin{aligned} i_{n-2} d_{n-1}(z) &= d_{n-1} i_{n-1}(z) \\ &= d_{n-1} d_n(y) \\ &= 0 \end{aligned}$$

Since i_{n-2} is injective, this implies that $d_{n-1}(z) = 0$. So z is a cycle. We now want to define $\partial_*[x] = [z]$, but we must check that this is well-defined: we made a choice when picking y, and also made a choice by picking the representative x.

∂_* **is well-defined**: Suppose $[x] = [x'] \in H_n(C_\bullet)$. Then $x - x' = d_{n+1}(a)$, $a \in C_{n+1}$. The same process for x' gives a $y' \in B_n$. As j_{n+1} is surjective, can write $a = j_{n+1}(b)$. Then

$$j_n(y - y') = x - x' = j_n d_{n+1}(b),$$

so by exactness at B_n,

$$y - y' = d_{n+1}(b) + i_n(c)$$

for some $c \in A_n$. Now z' is such that $i_{n-1}(z') = d_n(y')$, so

$$\begin{aligned} i_{n-1}(z - z') &= d_n(y) - d_n(y') \\ &= d_n(y - y') \\ &= d_n(d_{n+1}() + i_n(c)) \\ &= d_n i_n(c) \\ &= i_{n-1} d_n(c) \end{aligned}$$

injectivity of i_{n-1} again, shows $z - z' = d_n(c)$. So $[z] = [z'] \in H_{n-1}(A_\bullet)$.

∂_* **is a homomorphism**: given $[x_1], [x_2] \in H_n(C_\bullet)$ with corresponding y_1, y_2, z_1, z_2, choose $y_1 + y_2$ to be the lift of $x_1 + x_2$. This gives $z_1 + z_2$ as the result, so

$$\partial_*[x_1 + x_2] = [z_1 + z_2] = [z_1] + [z_2].$$

Exactness at $H_n(C_\bullet)$: Let $[x] \in \text{Im}(j_*)$, so there exists $y \in B_n$ such that $j_n(y) = x$ and y is a cycle. Can use this y to calculate $\partial_*[x]$. As y is a cycle, $d_n(y) = 0$, so z is 0, so $\partial_*[x] = 0$.

Suppose now that $\partial_*[x] = 0$. Calculate this by choosing $y \in B_n$ and taking the corresponding z. Then $z = d_n(t)$ (as $[z] = 0$). Then $j_n(y - i_n(t)) = x$ and $d_n(y - i_n(t)) = d_n y - d_n i_n(t) = i_n(z - z) = 0$. So $j_*[y - i_n(t)] = [x]$, so $[x] \in \text{Im}(j_*)$.

Exactness at $H_n(B_\bullet)$: As $j_n \circ i_n = 0$, $\mathrm{Im}(i_*) \subseteq \ker(j_*)$. Suppose $j_*[y] = 0$. Then $j_n(y) = d_{n+1}(a)$ for some $a \in C_{n+1}$. Let $b \in B_{n+1}$ be such that $j_n(b) = a$. Then

$$j_n(y - d_{n+1}b) = d_{n+1}(a) - j_n d_{n+1}(b) = d_{n+1}(a) - d_{n+1}j_{n+1}(b) = 0.$$

So let $y - d_{n+1}(b) = i_n(t)$. Then $d_n(t - d_{n+1}(b)) = 0$, so $[y] = [y - d_{n+1}(b)] = [i_n(t)] = i_*[t]$.

Exactness at $H_n(A_\bullet)$: Let $[z] \in \partial_*[x]$. Then $i_n(z) = d_{n+1}(y)$, so $i_*[z] = 0$. Let $[z]$ be such that $i_*[z] = 0$. Then $i_n(z) = d_{n+1}(y)$ for some y. Thus $[z] = \partial_*[j_{n+1}(y)]$, so $\ker(i_*) \subseteq \mathrm{Im}(\partial_*)$. □

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Theorem (Mayer-Vietoris Theorem)**.** Let K be a simplicial complex, M, N be subcomplexes, and $L = M \cap N$. Suppose also that M and N cover K, i.e. every simplex of K is in M or N. Write

$$\begin{array}{ccc} L & \overset{j}{\hookrightarrow} & N \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle l} \\ M & \overset{k}{\hookrightarrow} & K \end{array}$$

for the inclusion maps. There are natural homomorphisms $\partial_* : H_n(K) \to H_{n-1}(L)$ such that

$$\cdots \xrightarrow{\partial_*} H_n(L) \xrightarrow{(i_*, j_*)} H_n(M) \oplus H_n(N) \xrightarrow{k_* - l_*} H_n(K) \to H_{n-1}(L) \xrightarrow{(i_*, j_*)} H_{n-1}(M) \oplus H_{n-1}(N) \xrightarrow{k_* - l_*} \cdots$$

$$\cdots \to H_0(L) \xrightarrow{(i_*, j_*)} H_0(M) \oplus H_0(N) \xrightarrow{k_* - l_*} H_0(K) \to 0$$

is a long exact sequence.

Proof. Have a short exact sequence of chain complexes

$$0 \to C_\bullet(L) \overset{(i_*, j_*)}{\longrightarrow} C_\bullet(M) \oplus C_\bullet(N) \overset{k_* - l_*}{\longrightarrow} C_\bullet(K) \to 0$$

because:

(i) $i_\bullet$ and $j_\bullet$ are both injective, so exact at left-hand term.

(ii) $k_\bullet$ and $l_\bullet$ are jointly surjective, as M and N cover K.

(iii) Suppose $(x, y) \in C_n(M) \oplus C_n(N)$ is in $\ker(k_n - l_n)$, i.e. $k_n(x) = l_n(y)$. So both sides are a linear combination of simplices in $M \cap N = L$, i.e. $\exists z \in C_n(L)$ such that $x = i_n(z)$, $y = j_n(z)$. So $\ker(k_n - l_n) \subseteq \mathrm{Im}(i_n \oplus j_n)$. Other constraint is clear.

Now apply the above algebraic theorem. □

7.5 Continuous maps and homotopy inverse

Definition. Simplicial maps $f, g : K \to L$ are *contiguous* if for each $\sigma \in K$, $f(\sigma)$ and $g(\sigma)$ are faces of some simplex τ of L.

Lemma. If f and g are both simplicial approximations to $F : |K| \to |L|$, then f and g are contiguous.

Proof. If $x \in \mathring{\sigma} \subset |K|$ and $F(x) \in \mathring{\tau} \subset |L|$ then as in the proof of "simplicial approximations are simplicial maps", $f(\sigma)$ and $g(\sigma)$ are faces of τ. □

Lemma. If $f, g : K \to L$ are contiguous, then $f_\bullet \simeq g_\bullet : C_\bullet(K) \to C_\bullet(L)$, and so $f_* = g_*$.

Proof. Choose an ordering $\prec$ of V_K, and represent a basis of $C_n(K)$ by $[a_0, \ldots, a_n]$ with $a_0 \prec a_1 \prec \cdots \prec a_n$. Define a homomorphism

$$\begin{aligned} h_n : C_{n+1}(K) &\to C_{n+1}(L) \\ [a_0, \ldots, a_n] &\mapsto \sum_{i=0}^{n} (-1)^i [f(a_0), \ldots, f(a_i), g(a_i), \ldots, g(a_k)] \end{aligned}$$

Direct calculation shows that

$$d_{n+1} \circ h_n + h_{n-1} \circ d_n = g_n - f_n.$$

This is the chain homotopy. □

Lemma. Let K' be the barycentric subdivision, choose $a : K \cong V_{K'} \to V_K$ any function sending $\hat{\sigma}$ to some vertex of σ. Such an $a : V_{K'} \to V_K$ is a simplicial approximation to the identity map on $|K| = |K'|$. Furthermore, all simplicial approximation to id are of this form.

Proof. Seen the first part (just before the definition of mesh). If $g : V_{K'} \to V_K$ is a simplicial approximation to id, then

$$\mathring{\sigma} \subseteq \mathrm{id}(\mathrm{St}_{K'}(\widehat{\sigma})) \subseteq \mathrm{St}_K(g(\widehat{\sigma})).$$

So $g(\widehat{\sigma})$ must be a vertex of σ. □

Proposition. If $a : K' \to K$ is a simplicial approximation to id, then $a_* : H_n(K') \to H_n(K)$ is an isomorphism for all n. Furthermore, this isomorphism is independent of the choice of a.

Proof. First suppose K is a simplex $\sigma = \Delta^n \subset \mathbb{R}^{n+1}$. Then K' is a cone with cone point $\widehat{\sigma}$. Also K is a cone with any vertex as cone point. So $a_* : H_n(K') \to H_n(K)$ is an isomorphism for $n > 0$ (both sides are 0). Also, $H_{*0}(K') \to H_0(K)$ is an isomorphism too, with $a_*[\widehat{\sigma}] = [a(\widehat{\sigma})]$. So done if K is a simplex.

General case: double induction on:

(i) $\dim K$

(ii) number of simplexes of K

Let $\sigma \in K$ be a simplex of maximal dimansion. Then $L = K - \{\sigma\}$ is a simplicial complex. Let

$$S = \{\sigma \text{ and all its faces}\}, \qquad T = S \cap L = \{\text{all proper faces of } \sigma\}.$$

Any simplicial approximation $a : K' \to K$ to id sends L' into L, S' into S, T' into T. So we get a map of Mayer-Vietoris sequences:

$$\begin{array}{ccccccccc}
H_n(T') & \longrightarrow & H_n(S') \oplus H_n(L') & \longrightarrow & H_n(K') & \xrightarrow{\partial_*} & H_{n-1}(T') & \longrightarrow & H_{n-1}(S') \oplus H_{n-1}(L') \\
\downarrow \simeq (a|_{T'})_* & & \downarrow \simeq (a_{K'})_* \oplus (a|_{L'})_* & & \downarrow a_* & & \downarrow \simeq (a_{T'})_* & & \downarrow \simeq \\
H_n(T) & \longrightarrow & H_n(S) \oplus H_n(L) & \longrightarrow & H_n(K) & \xrightarrow{\partial_*} & H_{n-1}(T) & \longrightarrow & H_{n-1}(S) \oplus H_{n-1}(L)
\end{array}$$

T' has dimension strictly smaller than K, so by induction hypothesis, we may assume that $(a|_{T'})_*$ and $(a|_{T'})_*$ are isomorphisms. We can also use the induction hypothesis to assume that the second and fifth downward maps are isomorphisms: we have an isomorphism on the left since S is a simplex (and we did the simplex case earlier), and on the right factor we have strictly fewer simplexes, so we use the induction hypothesis.

Now by the "Five lemma" (see Example Sheet 4, Question 3), we deduce that a_* is an isomorphism. □

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Notation. So: if $a : K' \to K$ is a simplicial approximation to id, then $a_* : H_n(K') \to H_n(K)$ is an isomorphism, and does not actually depend on a.

Call it $\nu_K : H_n(K') \xrightarrow{\sim} H_n(K)$. Write

$$\nu_{K,r} := \nu_{K^{(r-1)}} \circ \cdots \circ \nu_{K'} \circ \nu_K : H_n(K^{(r)}) \xrightarrow{\sim} H_n(K)$$

and

$$\nu_{K,r,s} := \nu_{K^{(r-1)}} \circ \cdots \circ \nu_{K^{(s)}} : H_n(K^{(r)}) \xrightarrow{\sim} H_n(K^{(s)}).$$

Proposition. To each continuous $f : |K| \to |L|$ there is an associated homomorphism $f_* : H_n(K) \to H_n(L)$ given by $f_* = s_* \circ \nu_{K,r}^{-1}$, where $s : K^{(r)} \to L$ is a simplicial approximation to f (which exsists for some $r \gg 0$).

(i) This does not depend on the choices of s and r.

(ii) If $g : |M| \to |K|$ is another map, then $(f \circ g)_* = f_* \circ g_*$.

Proof. For (i) let $s : K^{(r)} \to L$ and $t : K^{(q)} \to L$ be simplicial approximation to f, with $r \ge q$. Let $a : K^{(r)} \to K^{(q)}$ be a simplicial approximation to id. Now $t \circ a, s : K^{(K)} \to L$ are both simplicial approximation to f, so they are contiguous, so $s_* = (t \circ a)_* = t_* \circ a_*$. But $a_* = \nu_{K,r,q}$, so

$$s_* \circ \nu_{K,r}^{-1} = t_* \circ \nu_{K,r,q} \circ \nu_{K,r}^{-1} = t_* \circ \nu_{K,q}^{-1}.$$

For (ii), let $s : K^{(r)} \to L$ approximate f, and let $t : M^{(q)} \to K^{(r)}$ approximate g. Then $s \circ t$ approximates $f \circ g$, and

$$(f \circ g)_* = (s \circ t)_* \circ \nu_{m,q}^{-1} = (s_* \circ \nu_{K,r}^{-1}) \circ (\nu_{K,r} \circ t_* \circ \nu_{m,q}^{-1}) = f_* \circ g_*$$ □

Corollary. If $f : |K| \to |L|$ is a homeomorphism, then $f_* : H_n(K) \to H_n(L)$ is an isomorphism.

Proof. $\text{id} = (f \circ f^{-1})_* = f_* \circ (f^{-1})_*$ and $\text{id} = (f^{-1} \circ f)_* = (f^{-1})_* \circ f_*$, so f_* is invertible. □

Lemma. For a simplicial complex L in $\mathbb{R}^m$, there is a $\varepsilon = \varepsilon(L) > 0$ such that if $f, g : |K| \to |L|$ satisfy $|f(x) - g(x)| < \varepsilon \ \forall x \in |K|$ then $f_* = g_* : H_n(K) \to H_n(L)$.

Proof. The $\{\text{St}_L(w)\}_{w\in V_L}$ give an open cover of $|L|$, so by the Lebesgue number lemma, there exists $\varepsilon > 0$ such that each $B_{2\varepsilon}(x)$ lies in some $\text{St}_L(w)$. Let $f, g : |K| \to |L|$ be as in the statement, using this ε. Then $\{f^{-1}(B_\varepsilon(y))\}_{y\in|L|}$ is an open cover of $|K|$, so there is a $\delta > 0$ such that each $f(B_\delta(x)) \subseteq B_\varepsilon(y)$. Then also $g(B_\delta(x)) \subseteq B_{2\varepsilon}(y)$. Let $r \gg 0$ such that $\mu(K^{(r)}) < \frac{1}{2}\delta$. Then for each $v \in V_{K^{(r)}}$, the diameter of $\text{St}_{K^{(r)}}(v)$ is $< \delta$. So $f(\text{St}_{K^{(r)}}(v))$, $g(\text{St}_{K^{(r)}}(v))$ lie in the common $B_{2\varepsilon}(w) \subseteq \text{St}_L(w)$. Let $s(v) = w$. Then s is a simplicial approximation to both f and g. So $f_* = s_* \circ \nu K, r^{-1} = g_*$. □

Theorem. If $f \simeq g : |K| \to |L|$, then $f_* = g_* : H_n(K) \to H_n(L)$.

Proof. Let $H : |K| \times I \to |L|$ be a homotopy between them. As $|K| \times I$ is compact, H is uniformly continuous. For the $\varepsilon = \varepsilon(L) > 0$ from the lemma, there is a $\delta > 0$ such that $|s - t| < \delta \implies |H(x, s) - H(x, t) < \varepsilon$ for all $x \in |K|$. Choose

$$0 = t_0 < t_1 < \cdots < t_k = 1$$

such that $|t_{i+1} - t_i| < \delta$. Then let $f_i(x) := H(x, t_i)$. Note $f_0 = f$, $f_k = g$. Also, f_i and f_{i+1} are ε-close. So $(f_i)_* = (f_{i+1})_*$, and so $f_* = g_*$. □

Definition (h-triangulation)**.** A *h-triangulation* of a space X is a simplicial complex K and a homotopy equivalence $g : |K| \to X$. We define $H_n(X) := H_n(K)$.

Lemma. The homology of a h-triangulated space does not depend on the choice of h-triangulation.

Proof. Let $\overline{g} : |\overline{K}| \to X$ be another h-triangulation. Then consider

$$|\overline{K}| \xrightarrow{\overline{g}} \to |K|,$$

where the second arrow is a homotopy inverse of g. The composition is a homotopy equivalence, so induces an isomorphism $H_n(\overline{K}) \xrightarrow{\sim} H_n(K)$. If $g : |K| \to X$, $\overline{G} : |\overline{K}| \to \overline{X}$ are h-triangulations with homotopy inverses f and $\overline{f}$, and $h : X \to \overline{X}$ is a map, then

$$\begin{array}{ccc} H_n(X) & \longrightarrow & H_n(\overline{X}) \\ \| & & \| \\ H_n(K) & \xrightarrow{(\overline{f} \circ h \circ g)_*} & H_n(\overline{K}) \end{array}$$

defines induced maps on homology for any map of h-triangulable spaces. □

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7.6 Homology of spheres

Lemma. The sphere S^{n-1} is triangulable, and for $n-1 \geq 1$ we have

$$H_i(S^{n-1}) \cong \begin{cases} \mathbb{Z} & i = 0, n-1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. ∂D^n has polyhedron homeomorphic to S^{n-1}, and we calculated its $H_\bullet$. □

Theorem (Brouwer (higher dimensions))**.**

(i) There is no retraction of D^n to $\partial D^n = S^{n-1}$.

(ii) Any $f : D^n \to D^n$ has a fixed point.

Proof. (i) $\implies$ (ii) by the same argument as the case $n = 2$. Let $r = D^n \to S^{n-1}$ be a retraction, and consider $\mathrm{id}_{S^{n-1}} = r \circ i$. Then

$$\mathrm{id} = (r \circ i)_* : \underbrace{H_{n-1}(S^{n-1})}_{\cong \mathbb{Z}} \xrightarrow{i_*} \underbrace{H_{n-1}(D^n)}_{\cong n-1(\{*\})=0} \xrightarrow{r_*} \underbrace{H_{n-1}(S^{n-1})}_{\cong \mathbb{Z}}$$

for $n - 1 > 0$. □

We have seen a different triangulation of S^n, via the simplicial complex K in $\mathbb{R}^{n+1}$ with simplexes $\langle \pm e_1, \ldots, \pm e_{n+1} \rangle$ and all their faces. We must have

$$H_i(K) \cong \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{otherwise} \end{cases}$$

by independence of triangulation.

Lemma. The element

$$x = \sum_{\mathbf{a} \in \{1,-1\}^{n+1}} a_1 a_2 \cdots a_{n+1} [a_1 e_1, a_2 e_2, \ldots, a_{n+1} e_{n+1}]$$

is a cycle, and generates $H_n(K) \cong \mathbb{Z}$.

Proof. When we apply d_n to x, the simplex

$$[a_1e_1, \dots, \widehat{a_ie_i}, \dots, a_{n+1}e_{n+1}]$$

shows up twice, corresponding to $a_i = +1$, $a_i = -1$. So the coefficients cancel out (so x is a cycle).

It generates as it is clearly not divisible in $C_n(K)$ (coefficients are ± 1). □

The reflection $r_i : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ in the i-th coordinate is

$$\begin{aligned}(r_i)_*(x) &= \sum a_1 \cdots a_{n+1}[a_1e_1, \dots, a_{i-1}e_{i-1}, -a_ie_i, a_{i+1}e_{i+1}, \dots, a_{n+1}e_{n+1}] \\ &= -x\end{aligned}$$

So $(r_i)_* : H_n(S^n) \to H_n(S^n)$ is multiplication by -1.

The antipodal map $a : S^n \to S^n$ is $r_1r_2 \cdots r_{n+1}$. So

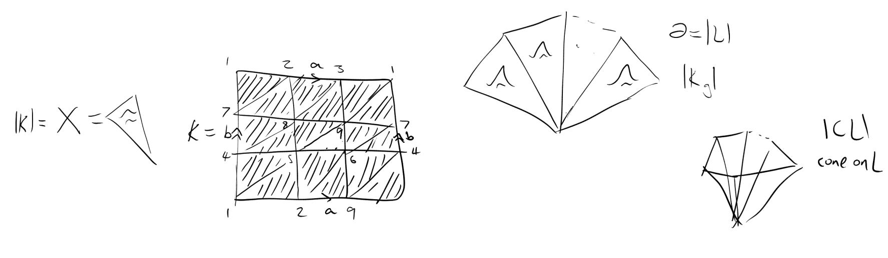
$$a_* : H_n(S^n) \to H_n(S^n)$$

is multiplication by $(-1)^{n+1}$.

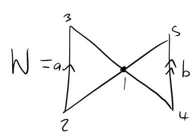
Corollary. If n is even, then $a : S^n \to S^n$ is not homotopic to id.

7.7 Homology of surfaces

Example



Homology of K: consider



Then $|W| \hookrightarrow |K|$ is a homotopy equivalence. Apply Mayer-Vietoris to

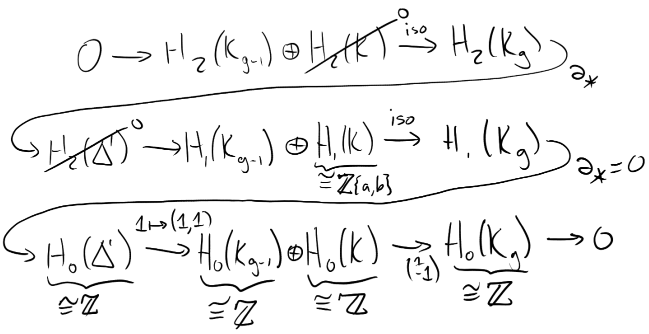
to obtain

$$H_0(K) = \mathbb{Z}\{[1]\}, \qquad H_1(K) = \mathbb{Z}\{a, b\}$$

where $a = [1,2] + [2,3] + [3,1]$ and $b = [1,4] + [4,5] + [5,1]$.

Note: $r = [5,6] + [6,9] + [9,5] \sim a + b - a - b = 0$.

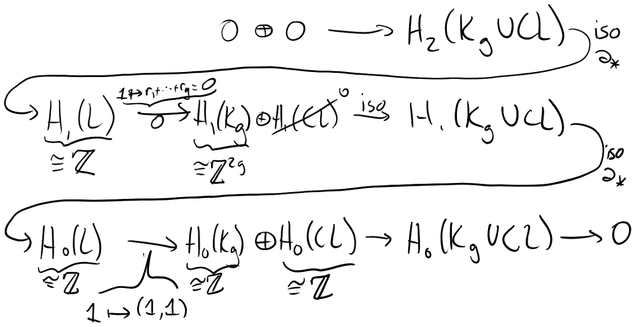
Homology of K_g: can decompose $K_g = K_{g-1} \cup K$, where $K_{g-1} \cap K = \Delta^1$. Mayer-Vietoris gives



so $H_0(K_g) = \mathbb{Z}\{\text{a vertex}\}$, $H_1(K_g) = \mathbb{Z}^{2g}$, $H_2(K_g) = 0$. Note $L = \partial K_g$ is the cycle $r_1 + r_2 + \cdots + r_g = 0$.

Let $\Sigma_g = |K_g| \bigcup_{|L|} |CL|$.

Homology of $K_g \cup CL$: Apply Mayer-Vietoris, using $K_g \cap CL = L$, a a triangulation of S^1.



so $H_0(K_g \cup CL) \cong \mathbb{Z}$, $H_1(K_g \cup CL) \cong \mathbb{Z}^{2g}$, $H_2(K_g \cup CL) \cong \mathbb{Z}$. So this is the homology of Σ_g, so in particular, given different values of g, the Σ_g will be non-homeomorphic, since their homologies will differ.

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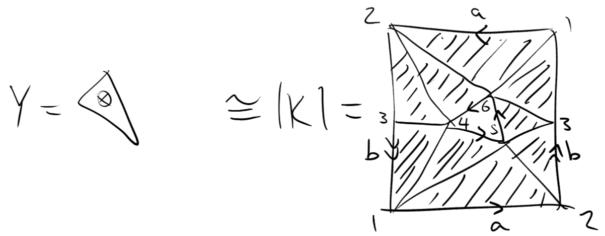
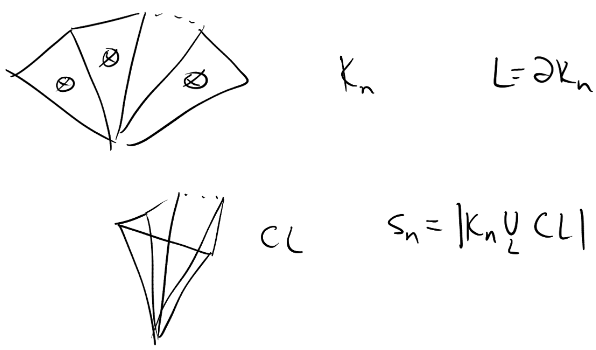
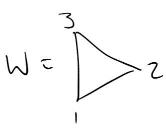
Example (non-orientable surfaces)





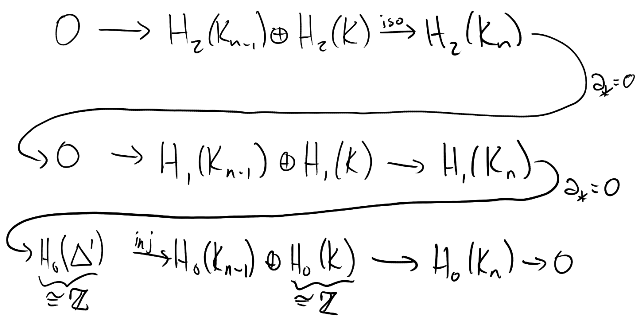
Homology of K: the subcomplex $|W| \subset |K|$ with



is such that $|K|$ deformation retracts to it, so $H_0(K) = \mathbb{Z}\{\text{vertex}\}$, $H_1(K) = \mathbb{Z}\{u\}$, $u = [1,2] + [2,3] + [3,1]$ and $H_i(K) = 0$ for $i > 1$.

Note $r = [4,5] + [5,6] + [6,4]$ is homologous to $2u$.

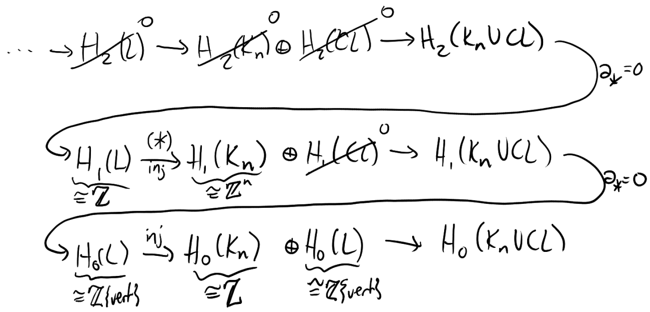
Homology of K_n: $K_n = K_{n-1} \cup K$, $K_{n-1} \cap K = \Delta^1$. So Mayer-Vietoris gives:



So inductively $H_0(K_n) = \mathbb{Z}\{\text{a vertex}\}$, $H_1(K_n) = \mathbb{Z}^n = \mathbb{Z}\{u_1, \ldots, u_n\}$, $H_2(K_n) = 0$ for $i \geq 2$.

The boundary L of K_n is the cycle $r_1 + r_2 + \cdots + r_n = 2(u_1 + \cdots + u_n)$.

Homology of $K_n \cup CL$: Mayer-Vietoris again:



Note $(*)$ sends the generator to $r_1 + r_2 + \cdots + r_n = 2(u_1 + \cdots + u_n)$. So $(*)$ is injective, which tells us that the previous map ∂_* is 0.

So $H_0(K_n \cup CL) = \mathbb{Z}$, $H_1(K_n \cup CL) = \frac{\mathbb{Z}\{u_1,\ldots,u_n\}}{\langle 2(u_1+\cdots+u_n)\rangle} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{n-1}$, and finally $H_i(K_n \cup CL) = 0$ for $i \geq 2$. This is $H_*(S_n)$.

The surfaces:

	S^2	Σ_g	S_n
H_1	0	$\mathbb{Z}^{2g}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{n-1}$
H_2	$\mathbb{Z}$	$\mathbb{Z}$	0

are all not homotopically equivalent to each other.

Theorem. Every triangulable surface is homeomorphic to one of these.

7.8 Rational homology, Euler and Lefschetz numbers

Definition ($O_n(K;\mathbb{Q})$)**.** For a simplicial complex K, let $O_n(K;\mathbb{Q})$ be the $\mathbb{Q}$-vector space with basis the ordered simplexes of K. Define $T_n(K;\mathbb{Q})$ as usual, and so get $C_n(K;\mathbb{Q})$. Define d_n by the same formula, and

$$H_n(K;\mathbb{Q}) = \frac{\ker(d_n : C_n(K;\mathbb{Q}) \to C_{n-1}(K,\mathbb{Q}))}{\operatorname{Im}(d_{n+1} : C_{n+1}(K;\mathbb{Q}) \to C_n(K;\mathbb{Q}))}.$$

This is a $\mathbb{Q}$-vector space.

Lemma. If $H_n(K) \cong \mathbb{Z}^r \oplus$ (finite abelian group), then $H_n(K;\mathbb{Q}) \cong \mathbb{Q}^r$.

Proof. See Example Sheet 4, Question 5. □

Example.

$$H_i(S6n;\mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0, n \\ 0 & \text{otherwise} \end{cases}$$

$$H_i(\Sigma_g;\mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0, 2 \\ \mathbb{Q}^{2g} & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$H_i(S_n;\mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0 \\ \mathbb{Q}^{n-1} & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Definition (Lefschetz number)**.** Let X be a polyhedron ($\cong |K|$) and $f : X \to X$ be a continuous map. The *Lefschetz number* of f is

$$L(f) = \sum_{i=0}^{\infty} (-1)^i \operatorname{Tr}(f_* : H_i(X;\mathbb{Q}) \to H_i(X;\mathbb{Q})).$$

Definition (Euler characteristic)**.** The *Euler characteristic* of X is

$$\chi(X) = L(\mathrm{id}) = \sum_{i=0}^{\infty} (1)^i \dim_{\mathbb{Q}} H_i(X;\mathbb{Q}).$$

Example.

$$\chi(S^n) = 1 + (-1)^n = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$\chi(\Sigma_g) = 2 - 2g, \qquad \chi(S_n) = 2 - n.$$

Example. The antipodal map $a : S^n \to S^n$ induces the identity map on $H_0(S^n;\mathbb{Q})$, and multiplication by $(-1)^{n+1}$ on $H_n(S^n;\mathbb{Q})$. So $L(a) = 1 + (-1)^n(-1)^{n+1} = 0$.

Lemma. Let V be a finite-dimensional vector space, $W \leq V$ a subspace, and $A : V \to V$ a linear map such that $A(W) \leq W$. Let $B = A|_W : W \to W$ and $C : V/W \to V/W$ the map induced by A. Then

$$\mathrm{Tr}(A) = \mathrm{Tr}(B) + \mathrm{Tr}(C).$$

Proof. Let $e_1, \ldots, e_r$ be a basis for W, and extend it to a basis of V. In this basis,

$$A = \begin{pmatrix} B & * \\ 0 & C \end{pmatrix}$$

so $\mathrm{Tr}(A) = \mathrm{Tr}(B) + \mathrm{Tr}(C)$. ☐

Corollary. For a chain map $f_\bullet : C_\bullet(K;\mathbb{Q}) \to C_\bullet(K;\mathbb{Q})$, we have

$$\sum_{i=0}^{\infty} (-1)^i \mathrm{Tr}(f_* : H_i(K;\mathbb{Q}) \to H_i(K;\mathbb{Q})) = \sum_{i=0}^{\infty} (-1)^i \mathrm{Tr}(f_i : C_i(K;\mathbb{Q}) \to C_i(K;\mathbb{Q}))$$

Proof. Consider

$$0 \to B_i(K;\mathbb{Q}) \to Z_i(K;\mathbb{Q}) \to H_i(K;\mathbb{Q}) \to 0$$

$$0 \to Z_i(K;\mathbb{Q}) \to C_i(K;\mathbb{Q}) \xrightarrow{d_i} B_{i-1}(K;\mathbb{Q}) \to 0$$

Let f^C, f^Z, f^B, f^H be the induced maps on chains / cycles / boundary / homology.

$$\begin{aligned}\sum(-1)^i \operatorname{Tr}(f_i^H) &= \sum(-1)^i(\operatorname{Tr}(f_i^Z) - \operatorname{Tr}(f_i^B)) \\ &= \sum(-1)^i(\operatorname{Tr}(f_i^C) - \operatorname{Tr}(f_{i-1}^B) - \operatorname{Tr}(f_i^B)) \\ &= \sum(-1)^i \operatorname{Tr}(f_i^C)\end{aligned}$$

□

Corollary.

$$\chi(|K|) = \sum_{i=0}^{\infty}(-1)^i\#\{i\text{-simplices of } K\}$$

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Lemma. There is a chain map $s_\bullet : C_\bullet(K) \to C_\bullet(K')$ given by sending a simplex σ to an appropriate linear combination of the simplexes of K' which compose σ. On H_* it induces $\nu_K^{-1} : H_n(K) \xrightarrow{\sim} H_n(K')$.

Proof. Start by $s_0([v_0]) = [\widehat{v_0}]$. Supposing $s_0, \dots, s_{n-1}$ have been defined and satisfy $d_i \circ s_i = s_{i-1} \circ d_i$ for all $i < n$. Then define $s_n(\sigma) = [\widehat{\sigma}, s_{n-1}d_n(\sigma)]$, interpreted linearly in the second variable.

Example: $s_1([v_0, v_1]) = [\widehat{\langle v_0, v_1\rangle}, \widehat{\langle v_1\rangle} - \widehat{\langle v_0\rangle}] = [\widehat{\langle v_0, v_1\rangle}, \widehat{\langle v_1\rangle}] - [\widehat{\langle v_0, v_1\rangle}, \widehat{\langle v_0\rangle}]$.

We calculate

$$\begin{aligned}d_n s_n([v_0, \dots, v_n]) &= [\widehat{\langle v_0, \dots, v_n\rangle}, s_{n-1}d_n[v_0, \dots, v_n]] \\ &= s_{n-1}d_n[v_0, \dots, v_n] - [\widehat{\langle v_0, \dots, v_n\rangle}, \underbrace{d_{n-1}s_{n-1}}_{s_{n-2}d_{n-1}} d_n[v_0, \dots, v_n]] \\ &= s_{n-1}d_n[v_0, \dots, v_n]\end{aligned}$$

so $d_n s_n = s_{n-1}d_n$. So this $s_\bullet$ is a chain map. Let $\prec$ be an ordering of V_K and $a : K \cong V_{K'} \to V_K$ send $\widehat{\sigma}$ to the smallest vertex of σ with respect to $\prec$. This is some simplicial approximation to id. Now $a_0 \circ s_0 = \text{id}$. Suppose $a_i \circ s_i = \text{id}$ for $i < n$. If $[v_0, \dots, v_n]$,

$v_0 \prec \cdots \prec v_n$ then

$$\begin{aligned} a_n s_n([v_0, \dots, v_n]) &= a_n([\widehat{\langle v_0, \dots, v_n \rangle}, s_{n-1} d_n [v_0, \dots, v_n]]) \\ &= [v_0, \underbrace{a_{n-1} s_{n-1}}_{\text{id}} d_n [v_0, \dots, v_n]] \\ &= [v_0, d_n[v_0, \dots, v_n]] \\ &= [v_0, \langle v_1, \dots, v_n \rangle - \sum \pm \langle v_0, \dots, \widehat{v_i}, \dots, v_n \rangle] \\ &= [v_0, \dots, v_n] \end{aligned}$$

So $a_* \circ s_* = \text{id}$, so $\nu \circ s_* = \text{id}$, so $s_* = \nu_K^{-1}$. □

Theorem (Lefschetz Fixed Point Theorem)**.** Let $f : X \to X$ be a map of polyhedra. If $L(f) \neq 0$ then f has a fixed point.

Proof. Let $X \cong |K| \subset \mathbb{R}^N$. Suppose f does not have a fixed point. Let

$$\delta := \inf\{|x - f(x)|, x \in X\}.$$

As X is compact, $\delta > 0$. Let$K$ be a triangulation of X with $\mu(K) < \frac{\delta}{2}$, and choose a simplicial approximation $g : K^{(r)} \to K$ to f. For $v \in V_{K^{(r)}}$ we have $f(v) \in f(\operatorname{St}_{K^{(r)}}(v)) \subset \operatorname{St}_K(g(v))$, so $|f(v) - g(v)| < \frac{\delta}{2}$. But $|f(v) - v| > \delta$, so $|g(v) - v| > \frac{\delta}{2}$. So if $v \in \sigma \in K$, then $g(v) \notin \sigma$. The map f_* is defined as $g_* \circ \nu_{K,r}^{-1} = g_* \circ s_*^{(r)}$ where $s_*^{(r)}$ is the r-fold iteration of the map in the previous Lemma. So

$$\begin{aligned} L(f) &:= \sum_i (-1)^i \operatorname{Tr}(f_* : H_i(X) \to H_i(X)) \\ &= \sum_i (-1)^i \operatorname{Tr}(g_i \circ s_i^{(r)} : C_i(K) \to C_i(K)) \end{aligned}$$

by last lecture. If $\sigma \in K$ is an i-simplex, then $s_i^{(r)}(\sigma)$ is a sum of simplexes inside σ, so $g_i s_i^{(r)}(\sigma)$ is a sum of simplexes not including σ. So the matrix for $g_i \circ s_i^{(r)}$ has zeroes on the diagonal, so the trace is 0. □

Example. If X is a contractible polyhedron, then

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

so any $f : X \to X$ has $L(f) = 1$, so has a fixed point.

Example. $S_1 = \mathbb{RP}^2$ has

$$H_n(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2\mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

so

$$H_n(\mathbb{RP}^2; \mathbb{Q}) = \begin{cases} \mathbb{Q} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

so any $f : \mathbb{RP}^2 \to \mathbb{RP}^2$ has $L(f) = 1$, so has a fixed point.

Example. Let G be a topological group which is path-connected and non-trivial. Suppose it is a polyhedron. If $g \neq 1 \in G$, then $g \cdot \bullet : G \to G$ has no fixed points, so

$$0 = L(g \cdot \bullet) = L(1 \cdot \bullet) = L(\mathrm{id}) = \chi(G)$$

as $g \cdot \bullet \simeq 1 \cdot \bullet$ as G is path-connected. This can be used to show that certain topological spaces cannot be given a topological group structure.

Example. Consider S_3, which is the connected sum of three copies of $\mathbb{RP}^2$. Let $f : S_3 \to S_3$ be such that $f \circ f = \mathrm{id}$. We have

$$H_n(S_3; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & n = 0 \\ \mathbb{Q}^2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

so $L(f) = 1 - \mathrm{Tr}(f_* : H_1(S_3; \mathbb{Q}) \to H_1(S_3; \mathbb{Q}))$. As $f \circ f = \mathrm{id}$, $f_* \circ f_* = \mathrm{id}$. So $f_* : H_1(S_3; \mathbb{Q}) \to H_1(S_3, \mathbb{Q})$ squares to id. So its minimal polynomial divides $x^2 - 1$, so it is diagonalisable with eigenvalues ± 1. So its trace is one of $\{2, 0, -2\}$. So $L(f) \in \{-1, 1, 3\}$, none of which are 0, so f has a fixed point.

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