Methods

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Chapter I

Self-Adjoint ODEs

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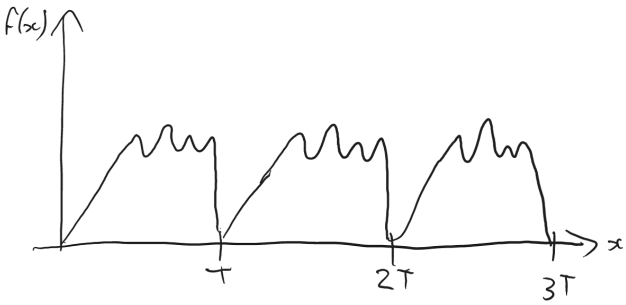
1. Fourier Series

1.1. Periodic Functions

A function $f(x)$ is periodic if

$$f(x+T) = f(x) \qquad \forall x$$

where T is the period.



Example. Simple harmonic motion

$$y = A \sin \omega t$$

where A is amplitude and period $T = \frac{2\pi}{\omega}$ with angular frequency ω (frequency $= \frac{1}{T}$).

Properties of sin and cosine functions

Consider the set of functions

$$g_n(x) = \cos \frac{n\pi x}{L}, \qquad h_n(x) = \sin \frac{n\pi x}{L}$$

which are periodic on the interval $0 \leq n > \infty$, $0 \leq x < 2L$). (Note: Period $T = 2L$). Recall the identities:

$$\begin{aligned}
\cos A \cos B &= \frac{1}{2}(\cos(A-B) + \cos(A+B)) \\
\sin A \sin B &= \frac{1}{2}(\cos(A-B) - \cos(A+B)) \\
\sin A \cos B &= \frac{1}{2}(\sin(A-B) + \sin(A+B))
\end{aligned}$$

Define an *inner product* for two periodic functions f, g on the interval $0 \leq x < 2L$ by:

$$\langle f, g \rangle = \int_0^{2L} f(x) g(x) \mathrm{d}x \qquad (*)$$

STUFF —

For $n \neq m$,

$$\begin{aligned}\langle h_n, h_m \rangle &= \int_0^{2L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} \mathrm{d}x \\ &= \int_0^{2L} \frac{1}{2}\left(\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)\right) \mathrm{d}x \\ &= 0\end{aligned}$$

¡++¿

STUFF —

For $n = m$,

$$\begin{aligned}\langle h_n, h_n \rangle &= \int_0^{2L} \sin^2\frac{n\pi x}{L} \mathrm{d}x \\ &= \frac{1}{2}\int_0^{2L} \left(1 - \cos\frac{2\pi n x}{L}\right) \mathrm{d}x \\ &= L \qquad (n \neq 0)\end{aligned}$$

Hence,

$$\langle h_n, h_m \rangle = \begin{cases} L\delta_{nm} & \forall\ n, m \neq 0 \\ 0 & m = 0 \end{cases} \tag{1.1}$$

Similarly (exercise)

$$\langle g_n, g_m \rangle = \int_0^{2L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L} \mathrm{d}x = \begin{cases} L\delta nm & \forall\ n, m \neq 0 \\ 2L\delta_{0n} & m = 0 \end{cases} \tag{1.2}$$

$$\langle h_n, g_m \rangle = \int_0^{2L} \sin\frac{n\pi x}{L} \cos\frac{m\pi x}{L} \mathrm{d}x = 0 \quad \forall\ n, m \tag{1.3}$$

1.2. Definition of Fourier Series

We can express any 'well-behaved' periodic function $f(x)$ with period $2L$ as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{1.4}$$

where a_n, b_n are constants such that the right hand side is convergent for all x where f is continuous. At a discontinuity x, the Fourier Series approaches the midpoint (replace left hand side)

$$\frac{1}{2}(f(x_+) + f(x_-))$$

Fourier coefficients

Consider

$$\begin{aligned}\langle h_m(x), f(x)\rangle &= \int_0^{2L} \sin\frac{m\pi x}{L} f(x)\mathrm{d}x \\ &= Lb_m\end{aligned}$$

by orthogonal relation (1.1-1.3). Hence we find

$$\boxed{\begin{aligned}b_n &= \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\mathrm{d}x \\ a_n &= \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\mathrm{d}x\end{aligned}} \tag{1.5}$$

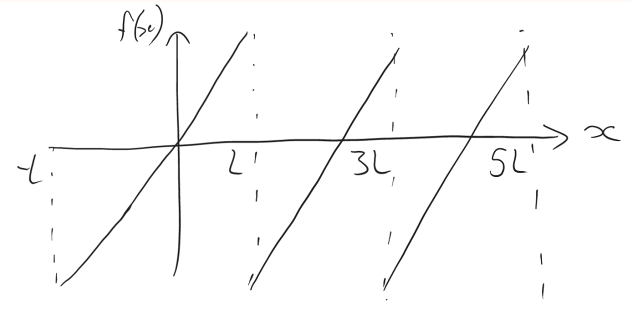
Note. (i) a_n includes $n = 0$, since $\frac{1}{2}a_0$ is the *average* $\langle f(x)\rangle = \frac{1}{2L}\int_0^{2L} f(x)\mathrm{d}x$

(ii) Range of integration is one period, so

$$\int_0^{2L} \mathrm{d}x \cdots = \int_{-L}^{L} \mathrm{d}x \cdots$$

(iii) Think of Fourier series (1.4) as a decomposition into harmonics. Simplest Fourier series are sine and cosine functions: for example pure mode $\sin\frac{3\pi x}{L}$, has $b_3 = 1$, $b_n = 0$ for all $n \neq 3$.

Example (Sawtooth)**.** Consider $f(x) = x$ for $-L \leq x < L$ and periodic elsewhere:



Here, we have

$$a_n = \frac{1}{2}\int_{-L}^{L} x \cos\frac{n\pi x}{L}\mathrm{d}x = 0$$

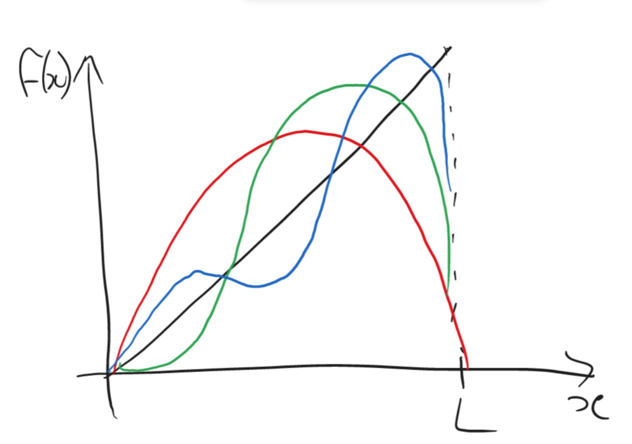
for all n, since the resulting function is odd. However:

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L}\mathrm{d}x \\ &= -\frac{2}{n\pi}\left[x\cos\frac{n\pi x}{2}\right]_0^L + \frac{2}{n\pi}\int_0^L \cos\frac{n\pi x}{L}\mathrm{d}x \\ &= -\frac{2L}{n\pi}\cos n\pi + \frac{2L}{(n\pi)^2}\sin n\pi \\ &= \frac{2L}{n\pi}(-1)^{n+1} \end{aligned}$$

So the sawtooth Fourier series is

$$\begin{aligned} f(x) &= \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L} \\ &= \frac{2L}{\pi}\left(\sin\frac{\pi x}{L} - \frac{1}{2}\sin\frac{2\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} - \cdots\right) \end{aligned}$$

which is slowly convergent.



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1.3. The Dirichlet Conditions (Fourier's theorem)

Sufficiency conditions for a "well-behaved" function to have a unique Fourier Series (1.4):

If $f(x)$ is a bounded periodic function (period $2L$) with a finite number of minima, maxima and discontinuities in $0 \leq x < 2L$, then the Fourier Series (1.4-5) converges to $f(x)$ at all points where f is continuous; at discontinuities the series converges to the midpoint $\frac{1}{2}(f(x_+) + f(x_-))$.

Note.

- Weak conditions (in contrast to Taylor series) but pathological functions are excluded, such as $\frac{1}{x}$, $\sin\frac{1}{x}$,

$$f(x) = \begin{cases} 0 & \text{rational} \\ 1 & \text{irrational} \end{cases}$$

- Converse is not true (consider $\sin\frac{1}{x}$ which has a Fourier series)
- Proof is difficult (see Jeffrey's & Jeffrey's)

Theorem (Convergence of Fourier series)**.** If $f(x)$ has continuous derivatives up to the p-th derivative which is discontinuous, then the Fourier series coefficients converge as $\theta(n^{-(p+1)})$, as $n \to \infty$.

Example ($p = 0$)**.** "Square wave" (Example sheet 1, Q5)

$$f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ -1 & -1 \leq x < 0 \end{cases}$$

then Fourier series

$$f(x) = 4\sum_{m=1}^{\infty} \frac{\sin(2m-1)\pi x}{(2m-1)\pi} \tag{1.7}$$

Example ($p = 1$)**.** General "see-saw" wave. If

$$f(x) = \begin{cases} x(1-\xi) & 0 \le x < \xi \\ \xi(1-x) & \xi \le x < 1 \\ x(1-\xi) & -\xi \le x < 0 \\ \xi(-1-x) & -1 \le x < \xi \end{cases}$$

Show that the Fourier series is

$$f(x) = 2\sum_{n=1}^{\infty} \frac{\sin n\pi\xi \sin n\pi x}{(n\pi)^2} \tag{1.8}$$

For $\xi = \frac{1}{2}$, show

$$f(x) = 2\sum_{m=1}^{\infty} (-1)^{m+1} \frac{\sin(2m-1)\pi x}{((2m-1)\pi)^2}$$

Example ($p = 2$)**.** Take

$$f(x) = \begin{cases} \frac{1}{2}x(1-x) & 0 \le x < 1 \\ \frac{1}{2}x(1+x) & -1 \le x < 0 \end{cases}$$

Show Fourier series is

$$f(x) = 4\sum_{m=1}^{\infty} \frac{\sin(2m-1)\pi x}{((2m-1)\pi)^3} \tag{1.9}$$

Example ($p = 3$)**.** $f(x) = (1-x^2)^2$ with Fourier series $a_n = \theta\left(\frac{1}{n^4}\right)$.

Integration of Fourier Series

It is always valid to integrate the Fourier series (1.4) of $f(x)$ term-by-term to obtain

$$F(x) = \int_{-L}^{x} f(x)\mathrm{d}x$$

because $F(x)$ satisfies Dirichlet conditions if $f(x)$ does. (for example discontinuities in f become continuous in $F(x)$).

Differentiation of Fourier Series

Take care with term-by-term differentiation.

Example (Counter example)**.** Take "square wave" Fourier series (1.7) and find

$$f'(x) \stackrel{?}{=} 4\sum_{m=1}^{\infty} \cos(2m-1)\pi x$$

which is unbounded!

Theorem. If $f(x)$ is continuous and satisfies Dirichlet conditions and $f'(x)$ satisfies Dirichlet conditions, then $f'(x)$ can be found by term-by-term differentiation of Fourier series (1.4) of $f(x)$.

Exercise: Differentiate "see-saw" (1.8) with $\xi = \frac{1}{2}$, to get offset "square-wave" (1.7) (i.e. $x \to x + \frac{1}{2}$).

1.4. Parseval's Theorem

Relation between integral of the square of a function and the sum of the squares of the Fourier coefficients:

$$\begin{aligned}\int_0^{2L} [f(x)]^2 \mathrm{d}x &= \int_0^{2L} \mathrm{d}x \left[\frac{1}{2}a_0 + \sum_n a_n \cos\frac{n\pi x}{L} + \sum_n b_n \sin\frac{n\pi x}{L}\right]^2 \\ &= \int_0^{2L} \mathrm{d}x \left[\frac{1}{4}a_0^2 + \sum_n a_n^2 \cos^2\frac{n\pi x}{L} + \sum_n b_n^2 \sin^2\frac{n\pi x}{L}\right]\end{aligned}$$

by orthogonal relations (1.1-3).

$$\boxed{\int_0^{2L} [f(x)]^2 \mathrm{d}x = L\left[\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]} \tag{1.10}$$

Also called the *completeness relation* because $LHS \geq RHS$ if any basis coefficients are missing.

Example. "Sawtooth" wave $f(x) = x$ on $-L \leq x < L$ with Fourier series (1.6)

$$LHS = \int_{-L}^{L} x^2 \mathrm{d}x = \frac{2}{3}L^3$$

$$RHS = L\sum_{n=1}^{\infty}\frac{4L^2}{n^2\pi^2} = \frac{4L^3}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

(note that we can combine these to notice that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}!$). See Example sheet 1, Q3

Remark. Parseval's theorem for functions $\langle f, f\rangle \equiv \|f\|^2$ is the same as Pythagoras for vectors $\langle v, v\rangle = |v|^2 = x^2 + y^2 + z^2$ (the norm).

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1.5. Alternative Fourier Series

Half-range series

Consider $f(x)$ defined only on $0 \le x < L$. Then we can extend its range over $-L \le x < L$ in two simple ways:

(i) Require it to be odd ($f(-x) = -f(x)$), with period $2L$, Then $a_n = 0$ because cos is even, and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\mathrm{d}x \tag{1.11}$$

This is a Fourier sine series, for example the saw tooth (1.6).

(ii) Require it to be even $f(-x) = f(x)$. Then $b_n = 0$ and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\mathrm{d}x \tag{1.12}$$

for example $f(x) = (1-x^2)^2$ (Example sheet 1 question 1).

Complex Representation

Recall:

$$\cos\frac{n\pi x}{L} = \frac{1}{2}(e^{in\pi x/L} + e^{-in\pi x/L})$$
$$\sin\frac{n\pi x}{L} = \frac{1}{2i}(e^{in\pi x/L} - e^{-in\pi x/L})$$

So Fourier series (1.4) becomes:

$$\begin{aligned} f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L} \\ &= \frac{1}{2}a_0 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n - ib_n)e^{in\pi x/L} + \frac{1}{2}\sum n = 1^{\infty}(a_n + b_n)e^{-in\pi x/L} \\ &= \sum_{m=-\infty}^{\infty} c_m e^{im\pi x/L} \end{aligned} \tag{1.13}$$

For $m > 0$, $m = n$, $c_m = \frac{1}{2}(a_n - ib_n)$. For $m = 0$, $c_0 = \frac{1}{2}a_0$. For $m < 0$, $m = -n$, $c_m = \frac{1}{2}(a_{-m} + ib_{-m})$. Equivalently

$$c_m = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-im\pi x/L}\mathrm{d}x \tag{1.14}$$

Our inner product is upgraded to

$$\langle f, g \rangle = \int f^* g$$

using complex conjugate f^*. Orthogonal:

$$\int_{-L}^{L} e^{-im\pi x/L} e^{in\pi x/L} = 2L\delta_{mn} \tag{1.15}$$

Parseval's:

$$\int_{-L}^{L} |f(x)|^2 \mathrm{d}x = 2L \sum_{m=-\infty}^{\infty} |c_m|^2$$

1.6. Some Fourier Series Motivations

Self-adjoint matrices

Suppose $\mathbf{u}, \mathbf{v}$ are complex N-vectors, with inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\dagger \mathbf{v} \tag{1.16}$$

($\mathbf{u}^\dagger$ means complex conjugate and transpose, i.e. $\mathbf{u}^\dagger = (\mathbf{u}^*)^\top$). Let A be an $N \times N$ matrix which is self adjoint (or Hermitian). Note that by simple algebra, this property means that $\langle A\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, A\mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v}$.

The eigenvalues of A are λ_n and satisfy

$$A\mathbf{v}_n = \lambda_n \mathbf{v}_n \tag{1.17}$$

(where $\mathbf{v}_n$ are eigenvectors). The eigenvalues have the following properties:

(i) The eigenvalues are real ($\lambda_n^* = \lambda_n$).

(ii) If $\lambda_n \neq \lambda_m$ then the eigenvectors are orthogonal

$$\langle \mathbf{v}_n, \mathbf{v}_m \rangle = 0$$

(iii) If we rescale our eigenvectors to be unit length then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N\}$ are an orthonormal basis.

Given $\mathbf{b}$ we can solve for $\mathbf{x}$ given

$$A\mathbf{x} = \mathbf{b} \tag{1.18}$$

Express

$$\mathbf{b} = \sum_{n=1}^{N} b_n \mathbf{v}_n$$

where b_n are knowns. Seek a solution

$$\mathbf{x} = \sum_{n=1}^{N} c_n \mathbf{v}_n$$

where c_n are unknowns. Substitute into (1.18):

$$A\mathbf{x} = \sum_{n=1}^{N} A c_n \mathbf{v}_n = \sum_{n=1}^{N} c_n \lambda_n \mathbf{v}_n$$

$$\mathbf{b} = \sum_{n=1}^{N} b_n \mathbf{v}_n$$

equate and use orthogonality

$$c_n \lambda_n = b_n \implies c_n = \frac{b_n}{\lambda_n}$$

So the solution is

$$\mathbf{x} = \sum_{n=1}^{N} \frac{b_n}{\lambda_n} \mathbf{v}_n \tag{1.19}$$

Solving inhomogeneous ODE with Fourier series

We wish to find $y(x)$ given $f(x)$ for

$$\mathcal{L}y \equiv -\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f(x) \tag{1.20}$$

(the minus sign is by convention, and $f(x)$ is the driving force / source). Boundary conditions:

$$y(0) = y(L) = 0$$

The related eigenvalue problem is

$$\mathcal{L}y_n = \lambda_n y_n$$

with the same boundary conditions. Has eigenfunctions and eigenvalues

$$y_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \tag{1.21}$$

(verify this, also self adjoins ODE with orthogonal eigenfunctions).

Seek solution as half range sine series. Try

$$y(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}$$

where c_n are unknowns. Expand

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

where b_n are knowns. Using (1.11)

$$b_n \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \mathrm{d}x$$

Substitute into (1.20):

$$\mathcal{L}y = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\sum_n c_n \sin \frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} c_n \left(\frac{n\pi}{L}\right)^2 \sin \frac{n\pi x}{L} \overset{\text{want}}{=} \sum_n b_n \sin \frac{n\pi x}{L}$$

By orthogonality (1.1) we have

$$c_n \left(\frac{n\pi}{L}\right)^2 = b_n \implies c_n = \frac{b_n}{\left(\frac{n\pi}{L}\right)^2}$$

and solution is

$$y(x) = \sum_n \frac{b_n}{\left(\frac{n\pi}{L}\right)^2} \sin \frac{n\pi x}{L} = \sum_n \frac{b_n}{\lambda_n} y_n \tag{1.22}$$

Example ("square wave" source)**.** $L = 1$. Define $f(x) = 1$, $0 \le x < 1$, odd function. This has Fourier series (1.7)

$$f(x) = 4 \sum_m \frac{\sin(2m-1)\pi x}{(2m-1)\pi}$$

So the solution (1.22) should be

$$y(x) = \sum_n \frac{b_n}{\lambda_n} y_n = 4 \sum_m \frac{\sin(2m-1)\pi x}{((2m-1)\pi)^3}$$

$(n = 2m-1)$. But this is the Fourier series (1.9) for

$$y(x) = \frac{1}{2} x(1-x) \tag{1.23}$$

Start of lecture 4

2. Sturm-Liouville Theory

2.1. Review of second-order linear ODEs

We wish to solve general inhomogeneous ODE

$$\mathcal{L}y \equiv \alpha(x)y'' + \beta(x)y' + \gamma(y)y = f(x) \tag{2.1}$$

- The *homogeneous equation*
$$\boxed{\mathcal{L}y \equiv 0} \tag{2.2}$$
has two independent solutions $y_1(x)$, $y_2(x)$ (besides trivial $y \equiv 0$), with the *complementary function* $y_c(x)$ the general solution of (2.2):
$$y_c(x) = Ay_1(x) + By_2(x) \tag{2.3}$$
where A, B are arbitrary constants.

- The inhomogeneous equation
$$\mathcal{L}y = f(x) \tag{2.4}$$
(i.e. the driving force or source term $f(x)$) has a special solution called the particular integral y_p. The general solution of (2.4) is then
$$y(x) = y_p(x) + Ay_1(x) + By_2(x) \tag{2.5}$$

- Two *boundary* or initial data are required to determine A, B.
  (a) *Boundary conditions* (BC) Solve (2.4) on $a < x < b$ given y at $x = a, b$ (Dirichlet) *or* specify y' at $x = a, b$ (Neumann) *or* mixed $y + ky'$ etc. Homogeneous boundary conditions are often assumed, $y(a) = y(b) = 0$ to admit the trivial solution $y \equiv 0$. Can be achieved by adding complementary function (2.3).
  $$\tilde{y} = y + Ay_1 + By_2$$
  such that $\tilde{y}(a) = \tilde{y}(b) = 0$.
  (b) Alternatively we may be given *initial conditions* Solve (2.4) for $x \geq a$, given y, y' at $x = a$.

General eigenvalue problem

To solve (2.1) employing eigenfunction expansions (like Fourier series (1.22)) we must solve the *related* eigenvalue problem

$$\alpha(x)y'' + \beta(x)y' + \gamma(x)y = -\lambda\rho(x)y \tag{2.6}$$

with specified boundary conditions. This form often occurs in higher dimensions after seperation of variables.

2.2. Self-adjoint operators

Definition (Inner product)**.** For two (complex-valued) functions f, g on $a \le x \le b$ define

$$\langle f, g \rangle = \int_a^b f^*(x) g(x) \mathrm{d}x$$

(later f, g assumed to be real so we will drop the complex conjugate part). The norm of f is $\|f\| = \sqrt{\langle f, f \rangle}$.

The Sturm-Liouville equation

The eigenvalue problem (2.6) greatly simplifies if $\mathcal{L}$ is self-adjoint, i.e. it can be expressed in *Sturm-Liouville form*

$$\boxed{\mathcal{L}y \equiv -(py')' + qy = \lambda \omega y} \tag{2.7}$$

where $\omega(x)$ is non-negative.where the *weight function* $\omega(x)$ is non-negative $\omega(x) \ge 0$ for all x.

Converting to Sturm-Liouville form: Multiply (2.6) by an integrating factor $F(x)$ to find

$$F\alpha y'' + F\beta y' + F\gamma y = -\lambda F \rho y$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(F\alpha y') - F'\alpha y' - F\alpha' y' + F\beta y' + F\gamma y = -\lambda F \rho y$$

We want to eliminate the y' term, so we want

$$F'\alpha = F(\beta - \alpha') \implies \frac{F'}{F} = \frac{\beta - \alpha'}{\alpha}$$

so

$$\boxed{F(x) = \exp\left(\int^x \frac{\beta - \alpha'}{\alpha} \mathrm{d}x\right)} \tag{2.8}$$

and $(F\alpha y')' + F\gamma y = -\lambda F \rho y$, so $p(x) = F(x)\alpha(x)$, $q(x) = -F(x)\gamma(x)$ and $\omega(x) = F(x)\rho(x)$ (note $F(x) > 0$).

Example. The Hermite equation for simple harmonic motion:

$$y'' - 2xy' + 2ny = 0$$

We want to put into Sturm-Liouville form (2.7). Comparing to (2.6) we have $\alpha = 1$, $\beta = -2x$, $\gamma = 0$, $\lambda\rho = 2n$. By (2.8),

$$F = \exp\left(\int^x \frac{-2x - 0}{1} \mathrm{d}x\right) = e^{-x^2}$$

Hence

$$\mathcal{L}y \equiv -(e^{-x^2} y')' = 2ne^{-x^2} y \tag{2.9}$$

Definition (Self-adjoint differential operator)**.** $\mathcal{L}$ is self-adjoint on $a \le x \le b$ for all pairs of functions y_1, y_2 satisfying appropriate boundary conditions if

$$\langle y_1, \mathcal{L}y_2 \rangle = \langle \mathcal{L}y_1, y_2 \rangle$$

or

$$\boxed{\int_a^b y_1^*(x)\mathcal{L}y_2(x)\mathrm{d}x = \int_a^b (\mathcal{L}y_1(x))^* y_2(x)\mathrm{d}x} \tag{2.10}$$

Boundary conditions: substitute Sturm-Liouville form (2.7) in (2.10) to find

$$\begin{aligned}\langle y_1, \mathcal{L}y_2 \rangle - \langle \mathcal{L}y_1, y_2 \rangle &= \int_a^b [-y_1(py_2')' + y_1 q y_2 + y_2(py_1')' - y_2 q y_1]\mathrm{d}x \\ &= \int_a^b [-(py_1y_2')' + (py_1'y_2)']\mathrm{d}x \\ &= [-py_1y_2' + py_1'y_2]_a^b\end{aligned}$$

and we want this to be 0 for given boundary conditions at $x = a = b$.

Self-adjoint compatible boundary conditions include:

- Homogeneous $y(a) = y(b) = 0$ or $y'(a) = y'(b) = 0$ or mixed $y + ky' = 0$ (note regular Sturm-Liouville $\equiv$ homogeneous boundary conditions)
- Periodic $y(a) = y(b)$
- Singular points of ODE $\rho(a) = \rho(b) = 0$
- Combinations of the above.

2.3. Properties of self-adjoint operators

(1) Eigenvalues λ_n are *real*.

(2) Eigenfunctions y_n are *orthogonal*.

(3) Eigenfunctions y_n form a *complete set*.

Start of lecture 5

Real eigenvalues

Given

$$\mathcal{L}y_n = \lambda_n \omega y_n \tag{2.12}$$

take complex conjugate (note both $\mathcal{L}$ and ω are real):

$$\mathcal{L}y_n^* = \lambda_n^* \omega y_n^*$$

Consider

$$\int_a^b (y_n^* \mathcal{L} y_n - (\mathcal{L} y_n^*) y_n \mathrm{d}x = (\lambda_n - \lambda_n^*) \int_a^b \omega y_n y_n^* \mathrm{d}x$$
$$= 0$$

(the equals zero comes from the fact that we know that the original expression was zero, because $\mathcal{L}$ is self-adjoint)
But the right hand side is

$$\int \omega |y_n|^2 \mathrm{d}x > 0$$

so $\lambda_n = \lambda_n^*$ so λ_n is real. We will *assume* that y_n are real.

Orthogonal eigenfunctions

Consider (2.12) with a second eigenvalue $\lambda_m \neq \lambda_n$.

$$\mathcal{L} y_m = \lambda_m \omega y_m$$

then from (2.10)

$$0 = \int_a^b (y_m \mathcal{L} y_n - y_n \mathcal{L} y_m) \mathrm{d}x$$
$$= (\lambda_n - \lambda_m) \int_a^b \omega y_n y_m \mathrm{d}x$$

But since $\lambda_m \neq \lambda_n$,

$$\boxed{\int_a^b \omega y_n y_m \mathrm{d}x = 0 \quad \forall n \neq m} \tag{2.13}$$

so y_n, y_m are orthogonal with respect to $\omega(x)$ on the interval $a \leq x < b$.
Define the inner product with respect to weight function $w(x)$ on $a \leq x \leq b$ as

$$\langle f, g \rangle_\omega = \int_a^b \omega(x) f^*(x) g(x) \mathrm{d}x = \langle \omega f, g \rangle = \langle f, \omega g \rangle \tag{2.14}$$

so orthogonal relation (2.13) becomes

$$\langle y_n, y_m \rangle_\omega = 0 \quad \forall n \neq m \tag{2.15}$$

Eigenfunction expansions

Completeness (not proven here) implies we can approximate any "well-behaved" function $f(x)$ on $a \leq x \leq b$ by the series

$$\boxed{f(x) = \sum_{n=1}^{\infty} a_n y_n(x)} \tag{2.16}$$

To find *expansion coefficients* consider

$$\int_a^b \mathrm{d}x\omega(x)y_m(x)f(x) = \sum_{n=1}^{\infty} a_n \int_a^b \omega y_n y_m \mathrm{d}x$$
$$= a_m \int_a^b \omega y_m^2 \mathrm{d}x$$

by orthogonality. Hence

$$\boxed{a_n = \frac{\int_a^b \omega(x)y_n(x)f(x)\mathrm{d}x}{\int_a^b \omega(x)y_n^2(x)\mathrm{d}x}} \tag{2.17}$$

Eigenfunctions normalized for convenience. *Unit norm* has

$$Y_n(x) \equiv \frac{y_n(x)}{\left(\int_a^b \omega y_n^2 \mathrm{d}x\right)^{\frac{1}{2}}}$$

so $\langle Y_n, Y_m \rangle = \delta_{nm}$ (2.18) are orthogonal with $f(x) = \sum_{n=1}^{\infty} A_n Y_n(x)$ and $A_n = \int_a^b \omega Y_n f \mathrm{d}x$.

Exemplar 1: Recall Fourier series (1.4) in Sturm Liouville form

$$\mathcal{L}y_n \equiv -\frac{\mathrm{d}^2 y_n}{\mathrm{d}x^2} = \lambda_n y_n \tag{1.21}$$

with $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and orthogonality relations (1.1-3).

2.4. Completeness and Parseval's identity

Consider

$$\int_a^b \left[f(x) - \sum_{n=1}^{\infty} a_n y_n\right]^2 \omega \mathrm{d}x = \int_a^b \left[f^2 - 2f\sum_n a_n y_n + \sum_n a_n^2 y_n^2\right] \omega \mathrm{d}x \quad \text{(by orthogonality)}$$
$$= \int_a^b \omega f^2 \mathrm{d}x - \sum_{n=1}^{\infty} a_n^2 \int_a^b \omega y_n^2 \mathrm{d}x$$

because by (2.17) $\int_a^b f y_n \omega \mathrm{d}x = a_n \int_a^b \omega y_n^2 \mathrm{d}x$. If the eigenfunctions are *complete* then series converges

$$\int_a^b \omega f^2 \mathrm{d}x = \sum_{n=1}^{\infty} a_n^2 \int \omega y_n^2 \mathrm{d}x \tag{2.19}$$
$$= \sum_{n=1}^{\infty} A_n^2$$

for unit norm Y_n.

Theorem (Bessel's inequality)**.** If some eigenfunctions are missing, then

$$\int_a^b \omega f^2 \mathrm{d}x \geq \sum_{n=1}^{\infty} A_n^2$$

Definition (Mean square error)**.**

$$\varepsilon_N = \int_a^b \omega[f(x) - S_N(x)]^2 \mathrm{d}x \to 0$$

Define *partial sum*

$$S_N(x) = \sum_{n=1}^{N} a_n y_n \tag{2.20}$$

with $f(x) = \lim_{N\to\infty} S_N(x)$. The *error* in the partial sum (2.20) is minimised by a_n defined in (2.19) for the $N = \infty$ expansion:

$$\begin{aligned}
\frac{\partial \varepsilon_N}{\partial a_n} &= \frac{\partial}{\partial a_n}\left[\int_a^b \omega[f(x) - \sum a_n y_n]^2 \mathrm{d}x\right] \\
&= -2\int_a^b y_n \omega[f - \sum_{n=1}^{N} a_n y_n \mathrm{d}x \\
&= -2\int_a^b (\omega f y_n - a_n \omega y_n^2)\mathrm{d}x \\
&= 0
\end{aligned}$$

if a_n given by (2.17). So a_n is the "best possible choice" (assuming you care about the mean square error).

2.5. Exemplar 2: Legendre polynomials

Consider Legendre's equation arising from spherical polars $x = \cos\theta$

$$(1 - x^2)y'' - 2xy' + \lambda y = 0 \tag{2.21}$$

so $p = 1 - x^2$, $q = 0$, $\omega = 1$.on the interval $-1 \leq x \leq 1$ with y finite at $x = \pm 1$ (regular singular point of ODE). Equation (2.21) is in Sturm Liouville form (2.7) with

$$p = 1 - x^2,\ q = 0,\ \omega = 1$$

How to solve? Seek a power series about $x = 0$:

$$y = \sum_n c_n x^n$$

Substitute

$$(1-x^2)\sum n(n-1)c_n x^{n-2} - 2x\sum_n c_n x^{n-1} + \lambda \sum c_n x^n = 0$$

Equate powers of x^n:

$$(n+2)(n+1)c_{n+2} - n(n-1)c_n - 2nc_n + \lambda c_n = 0$$

$$\implies c_{n+2} = \frac{n(n+1)-\lambda}{(n+1)(n+2)} c_n \tag{2.22}$$

so specifying c_0, c_1 gives 2 independent solutions (near $x = 0$).

$$y_{\text{even}} = c_0 \left[1 + \frac{(-\lambda)}{2!}x^2 + \frac{(6-\lambda)(-\lambda)}{4!}x^4 + \cdots \right]$$

$$y_{\text{odd}} = c_1 \left[x + \frac{(2-\lambda)}{3!}x^3 + \cdots \right]$$

But as $n \to \infty$, $\frac{c_{n+1}c_n}{\rightarrow}1$ so there is a radius of convergence $|x| < 1$ (geometric series), i.e. divergent at $x = \pm 1$. What can be done? *Finiteness...*

Start of lecture 6

Take $\lambda = l(l+1)$ with l integer, then one or other series terminates, i.e. $c_n = 0$ for all $n \ge l+2$. These *Legendre polynomials* $P_l(x)$ are coefficients of (2.21) on $-1 \le x \le 1$ with normalisation convention $P_l(1) = 1$.

Note. $P_l(x)$ has l zeros and $P_l(x)$ is odd if l is odd, and if l is even then $P_l(x)$ is even.

Orthogonality:

$$\int_{-1}^{1} P_n P_m \mathrm{d}x = 0 \quad \forall m \neq n$$

Normalization:

$$\int_{-1}^{1} P_n^2 \mathrm{d}x = \frac{2}{2n+1} \tag{2.24}$$

Prove with Rodriques formula:

$$P_n(x) = \frac{1}{2^n x!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (x^2-1)^n$$

(Example sheet 2, Q5)
Generating function (see later):

$$\begin{aligned}\sum_{n=0}^{\infty} P_n(x)t^n &= \frac{1}{\sqrt{1-2xt+t^2}} \qquad (2.23\text{a})\\ &= 1 + \frac{1}{2}(2xt - t^2) + \frac{3}{8}(2xt-t^2)^2 + \cdots \\ &= \underbrace{1}_{P_0} + \underbrace{x}_{P_1} t + \underbrace{\frac{1}{2}(3x^2-1)}_{P_2} t^2 + \cdots\end{aligned}$$

Exercise: Verify P_3 and find P_4. (binomial expansion).
Recursion relations:

$$(l+1)P_{l+1}(x) = (2l+1)xP_l(x) - lP_{l-1}(x)$$

$$(2l+1)P_l(x) = \frac{\mathrm{d}}{\mathrm{d}x}(P_{l+1}(x) - P_{l-1}(x))$$

Eigenfunction expansions: Any function $f(x)$ on $-1 \leq x \leq 1$ can be expressed as

$$f(x) = \sum_{l=0}^{\infty} a_l P_l(x) \tag{2.25}$$

where

$$a_l = \frac{2l+1}{2} \int_{-1}^{1} f(x)P_l(x)\mathrm{d}x \tag{2.26}$$

Exercise: verify $f(x) = \frac{15}{2}x^2 - \frac{3}{2} = P_0(x) + 5P_2(x)$ using (2.26).

2.6. Sturm Liouville theory and inhomogeneous ODEs

Consider the inhomogeneous (with homogeneous boundary conditions) on $a \leq x \leq b$:

$$\mathcal{L}y = f(x) \equiv \omega(x)F(x) \tag{2.27}$$

Given eigenfunctions $y_n(x)$ satisfying

$$\mathcal{L}y_n = \lambda_n \omega y_n$$

$$y(x) = \sum_n c_n y_n(x)$$

$$F(x) = \sum_n a_n y_n(x)$$

where a_n are known and c_n are unknown. We use

$$a_n = \frac{\int_a^b \omega F y_n \mathrm{d}x}{\int_a^b \omega y_n^2 \mathrm{d}x}$$

Substituting into (2.27):

$$\mathcal{L}y = \mathcal{L}\sum_n c_n y_n = \sum_n c_n \lambda_n \omega y_n = \omega \sum_n a_n y_n$$

By orthogonality (2.13), $c_n\lambda_n = a_n$ or $c_n = \frac{a_n}{\lambda_n}$ so solution is

$$\boxed{y(x) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n} y_n(x)} \tag{2.28}$$

(assuming $\lambda_n \neq 0$ for all n). Recall Fourier series (1.22)
Generalisation: driving forces often induce a linear response term $\tilde{\lambda}\omega y$.

$$\mathcal{L}\tilde{\lambda}\omega y = f(x) \tag{2.29}$$

where $\tilde{\lambda}$ is fixed. The solution (2.28) becomes

$$y(x) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n - \tilde{\lambda}} y_n(x) \tag{2.30}$$

(again $\tilde{\lambda} \neq \lambda_n$ for all n).

Integral solution and Green's function

Recall (2.28)

$$\begin{aligned} y(x) &= \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n} y_n(x) \\ &= \sum_{n} \frac{y_n(x)}{\lambda_n \mathcal{N}} \int_a^b \omega(\xi) F(\xi) y_n(\xi) \mathrm{d}\xi \\ &= \int_a^b \underbrace{\sum_{n=1}^{\infty} \frac{y_n(x) y_n(\xi)}{\lambda_n \mathcal{N}_n}}_{G(x,\xi)} \underbrace{\omega(\xi) F(\xi)}_{f(\xi)} \mathrm{d}\xi \\ &\equiv \int_a^b G(x,\xi) f(\xi) \mathrm{d}\xi \end{aligned} \tag{2.31}$$

where

$$G(x,\xi) = \sum_{n=1}^{\infty} \frac{y_n(x) y_n(\xi)}{\lambda_n \mathcal{N}_n}$$

is eigenfunction expansion of the Green's function.
$G(x,\xi)$ depends only on $\mathcal{L}$ and boundary conditions and *not* forcing term $f(x)$ - it acts like an inverse operator

$$\mathcal{L}^{-1} \equiv \int \mathrm{d}\xi G(x,\xi)$$

(recall matrix $A\mathbf{x} = \mathbf{b} \implies \mathbf{x} = A^{-1}\mathbf{b}$)

Chapter II

PDEs on Bounded Domains

Contents

3. The Wave Equation

3.1. Waves on an elastic string

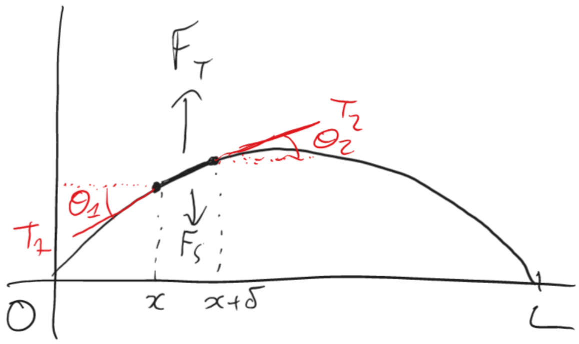
Consider small displacements on a stretched string with fixed ends at $x = 0$ and $x = L$, with boundary conditions

$$\boxed{y(0,t) = y(L,t) = 0} \tag{3.1}$$

and initial conditions

$$\boxed{y(x,0) = p(x) \text{ and } \frac{\partial y}{\partial t}(x,0) = q(x)} \tag{3.2}$$

Derive equation of motion: Balance forces on segment $(x, x+\delta x)$ and take $\delta x \to 0$.



Assume $\left|\frac{\partial y}{\partial x}\right| \ll 1$ for all x, so θ_1, θ_2 are small.

- Resolve in x direction:

  $$T_1 \cos\theta_1 = T_2 \cos\theta_2$$

  but $\cos\theta = 1 - \frac{1}{2}\theta^2 + \cdots$ so $T_1 \approx T_2 = T$. Hence, tension T is constant independent of x up to $\theta\left(\left|\frac{\partial y}{\partial x}\right|^2\right)$

- Resolve in y direction

$$\begin{aligned}
F_T &= T_2 \sin\theta_2 - T_2 \sin\theta_1 \\
&\approx T\left(\left.\frac{\partial y}{\partial x}\right|_{x+\delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right) \\
&= T\frac{\partial^2 y}{\partial x^2}\delta x
\end{aligned}$$

Thus

$$\begin{aligned} F &= ma \\ &= (\mu\delta x)\frac{\partial^2 y}{\partial t^2} \\ &= F_T + F_g \\ &= T\frac{\partial^2 y}{\partial x^2}\delta x - g\mu\delta x \end{aligned}$$

where μ is the mass per unit length (linear mass density). Define the wave speed $c = \sqrt{\frac{T}{\mu}}$ (constant) and we find

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} - g = c^2\frac{\partial^2 y}{\partial x^2} - g \tag{3.3}$$

Assume gravity is negligible then we have the 1 dimensional wave equation ($\ddot{y} = c^2 y''$):

$$\boxed{\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}} \tag{3.4}$$

3.2. Separation of variables

We wish to solve wave equation (3.4) subject to boundary conditions (3.1) and initial conditions (3.2).
Consider possible solution of *separable form* (ansatz):

$$y(x,t) = X(x)T(t) \tag{3.5}$$

Substitute in (3.4) $\frac{1}{c^2}\ddot{y} = y''$.

$$\begin{aligned} &\frac{1}{c^2}X\ddot{T} = X''T \\ \implies &\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X}. \end{aligned}$$

Start of lecture 7

But $\frac{\ddot{T}}{T}$ depends only on t, and $\frac{X''}{X}$ depends only on x!
So both sides must be equal to a constant, say $-\lambda$, so

$$X'' + \lambda X = 0 \tag{3.6}$$

$$\ddot{T} + \lambda c^2 T = 0 \tag{3.7}$$

3.3. Boundary conditions and normal modes

Three possibilities for λ (+, 0, -) in spatial ODE (3.6) but restricted by (3.1)

(i) $\lambda < 0$. Take $\chi^2 = -\lambda$ then

$$X(x) = Ae^{\chi x} + Be^{-\chi x} = \tilde{A}\cosh \chi x + \tilde{B}\sinh \chi x$$

but boundary conditions imply $X(0) = X(L) = 0 \implies \tilde{A} = \tilde{B} = 0$ (only trivial solution works).

(ii) $\lambda = 0$ then $X(x) = Ax + B$ but then by boundary conditions $A = B = 0$.

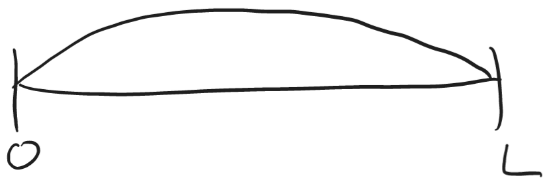
(iii) $\lambda > 0$, then $X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. Here, the boundary conditions (3.1) imply $A = 0$ and $B\sin\sqrt{\lambda}L = 0$, so $\sqrt{\lambda}L = n\pi$. So

$$X_n(x) = B_n \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \tag{3.8}$$

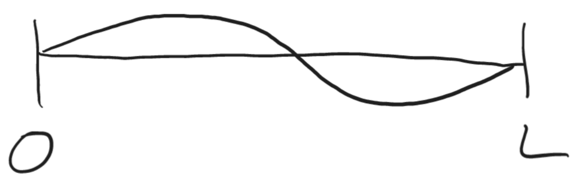
i.e. eigenfunctions and eigenvalues of the system.

These are *normal modes* because spatial shape in x does not change in time (amplitude may vary).

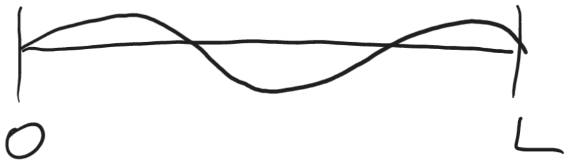
- Fundamental mode ($n = 1$): $\lambda_1 = \frac{\pi^2}{L^2}$. Lowest frequency vibration or first harmonic.



- Second mode ($n = 2$): $\lambda_2 = \frac{4\pi^2}{L^2}$ second harmonic or overtone.



- Third mode $n = 3$ etc



3.4. Initial conditions and temporal solution

Substitute eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ into time ODE (3.7)

$$\ddot{T} + \frac{n^2\pi^2c^2}{L^2}T = 0$$

which has solutions

$$T_n(t) = C_n \cos\frac{n\pi ct}{2} + D_n \sin\frac{n\pi ct}{L} \tag{3.9}$$

Thus a specific solution to (3.4) satisfying boundary conditions (3.1) is

$$\begin{aligned} y_n(x,t) &= T_n(t)X_n(x) \\ &= \left(C_n \cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L} \end{aligned}$$

(absorbing B_n into C_n and D_n). Exercise: verify that this is a solution.

Since the wave equation (3.4) is linear (and boundary conditions (3.1) are homogeneous) we can add the solutions together to find *general string solution*

$$\boxed{y(x,t) = \sum_{n=1}^{\infty}\left(C_n \cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}} \tag{3.10}$$

By construction (3.10) satisfies boundary conditions (3.1), so now impose initial conditions (3.2):
For $t = 0$ we have

$$y(x,0) = p(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}$$

by (3.10) and also by (3.10):

$$\frac{\partial y}{\partial t}(x,0) = q(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} D_n \sin\frac{n\pi x}{L}$$

So the coefficients are those for Fourier sine series given by (1.12):

$$C_n = \frac{2}{L}\int_0^L p(x)\sin\frac{n\pi x}{L}\mathrm{d}x$$

$$D_n = \frac{2}{2\pi c}\int_0^L q(x)\sin\frac{n\pi x}{L}\mathrm{d}x \tag{3.11}$$

Hence (3.10-11) is the solution to (3.4) satisfying (3.1-2).

Example. Pluck string at $x = 3$, drawing it back as

$$y(x,0) = p(x) = \begin{cases} x(1-\xi) & 0 \le x \le \xi \\ \xi(1-x) & \xi < x \le 1 \end{cases}$$

$$\frac{\partial y}{\partial t}(x,0) = q(x) = 0$$

Then with Fourier series (1.8)

$$C_n = \frac{2\sin n\pi\xi}{(n\pi)^2}, \quad D_n = 0$$

so we have solution

$$y(x,t) = \sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}\sin n\pi\xi\sin n\pi x\cos n\pi ct$$

Take $\xi = \frac{1}{2}$ then $C_{2m} = 0$, $C_{2m-1} = \frac{2(-1)^{m+1}}{((2m-1)\pi)^2}$. For a guitar, $\frac{1}{4} \le \xi \le \frac{1}{3}$, for a violin, $\xi \approx \frac{1}{7}$.

Separation of Variables Methodology

(1) Obtain linear PDE for system (with boundary conditions and initial conditions)

(2) Separate variables to tield decoupled ODEs

(3) Impose homogeneous boundary conditions to find eigenvalues and eigenfunctions

(4) Use these eigenvalues (constants of separation) to find eigenfunctions in the other variables.

Aside: Solution in characteristic coordinates

Recall sine / cosine summation identities which means our general solution (3.10) becomes

$$\begin{aligned} y(x,t) = \frac{1}{2}\sum_{n=1}^{\infty}\Big[&C_n\sin\frac{n\pi}{L}(x-ct) + D_n\cos\frac{n\pi}{L}(x-ct) \\ &+ C_n\sin\frac{n\pi}{L}(x+ct) + D_n\cos\frac{n\pi}{L}(x+ct)\Big] \\ \equiv f(x-ct) + g(x+ct) \end{aligned} \tag{3.12}$$

The standing wave solution (3.10) is made up of a right-moving wave (along the characteristic $x - ct = \eta$, constant) and a left-moving wave ($x + ct = \xi$, constant) i.e. a general solution with arbitrary f, g (see later).

Special case: $g(x) = 0$ in (3.1), then $f = g = \frac{1}{2}p$ at $t = 0$.

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3.5. Oscillation energy

A vibrating string has kinetic energy due to its motion (for example particle $\frac{1}{2}mv^2$)

$$KE = \frac{1}{2}\mu \int_0^L \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x$$

and potential energy due to stretching Δx

$$\begin{aligned} PE &= T\Delta x \\ &= T\int_0^L \left(\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} - 1\right) \mathrm{d}x \\ &\approx \frac{1}{2}T\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x \qquad \text{for } \left|\frac{\partial y}{\partial x}\right| \ll 1 \end{aligned}$$

The total summed energy becomes ($c^2 = \frac{T}{\mu}$)

$$E = \frac{1}{2}\mu \int_0^L \left[\left(\frac{\partial y}{\partial t}\right)^2 + c^2\left(\frac{\partial y}{\partial x}\right)^2\right] \mathrm{d}x \tag{3.13}$$

Substitute (3.10) and use orthogonality (1.1)

$$\begin{aligned} E &= \frac{1}{2}\mu \sum_{n=1}^{\infty} \int_0^L \left[\left(-\frac{n\pi c}{L}C_n \sin\frac{n\pi ct}{L} + \frac{n\pi c}{2}D_n \cos\frac{n\pi ct}{L}\right)^2 \sin^2\frac{n\pi x}{L}\right. \\ &\qquad\qquad \left. + c^2\left(C_n \cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L}\right)^2 \frac{n^2\pi^2}{L}\cos^2\frac{n\pi x}{L}\right] \mathrm{d}x \\ &= \frac{1}{4}\mu \sum_{n=1}^{\infty} \frac{n^2\pi^2c^2}{L}\left(C_n^2 + D_n^2\right) \qquad (3.14) \\ &= \sum_{\text{normal modes}} [\text{energy in } n\text{-th mode}] \end{aligned}$$

This is constant, so energy is conserved in time (no dissipation).

3.6. Wave reflection and transmission

Recall travelling wave solution (3.12). A simple *harmonic* travelling wave is

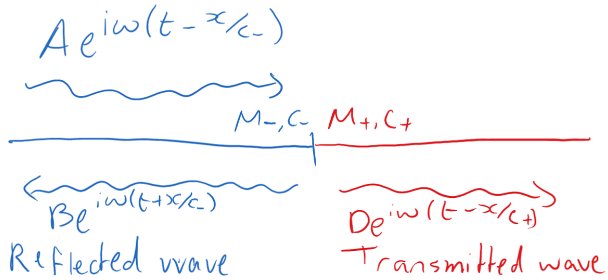
$$y = \mathrm{Re}[Ae^{-\omega(t-x/c)}] = |A|\cos\left(\omega\left(t - \frac{x}{c}\right) + \phi\right)$$

where the *phase* is $\phi = \arg A$ and wavelength is $\frac{2\pi c}{\omega}$.

Consider a density discontinuity on a string at $x = 0$, with

$$\mu = \begin{cases} \mu_- & \text{for } x < 0 \text{ hence } c_- = \sqrt{\frac{t}{\mu_-}} \\ \mu_+ & \text{for } x > 0 \text{ hence } c_+ = \sqrt{\frac{T}{\mu_+}} \end{cases}$$

assuming constant tension. Incident wave on junction



Boundary (or junction) conditions at $x = c$:

- String does not break, i.e. y is continuous for all t.

$$\implies A + B = D \qquad (*)$$

- Forces balance $T\left.\frac{\partial y}{\partial x}\right|_{x=0_-} = T\left.\frac{\partial y}{\partial x}\right|_{x=0_+}$ i.e. $\frac{\partial y}{\partial x}$ is continuous for all t

$$\implies -\frac{i\omega A}{c_-} + \frac{i\omega B}{c_-} = -\frac{i\omega D}{c_+} \qquad (\dagger)$$

$$(*) - \frac{c_-}{i\omega}(\dagger) \implies 2A = D + D\frac{c_-}{c_+} = \frac{D}{c_+}(c_+ + c_-)$$

So given A, we have the solution:

$$D = \frac{2c_+}{c_- + c_+}A \qquad B = \frac{c_+ - c_-}{c_+ + c_-}A \qquad (3.16)$$

where D is the transmitted amplitude and B is the reflected amplitude. In general, different phase shift ϕ is possible.

Limiting cases:

(1) Continuity $c_- = c_+ \implies D = A, B = 0$.

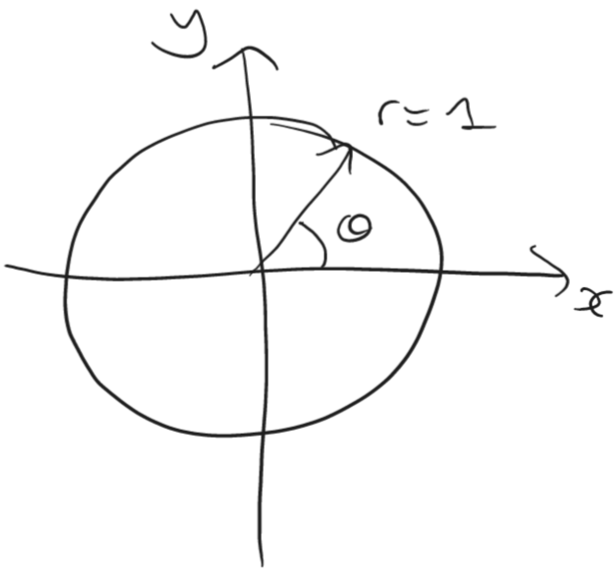
(2) Dirichlet boundary conditions $\frac{\mu_+}{\mu_-} \to \infty$ (fixed end $y = 0$ at $x = 0$) then $\frac{c_+}{c_-} \to 0 \implies D = 0, B = -A$ i.e. total reflection with opposite phase ($\phi = \pi$)

(3) Neumann boundary conditions $\frac{\mu_+}{\mu_-} \to 0$ (free end of string - very light string $x > 0$) then $\frac{c_+}{c_-} \to \infty \implies D = 2A, B = A$ (boundary condition $\left.\frac{\partial y}{\partial x}\right|_{x=0}$). Total reflection with same phase ($\phi = 0$).

3.7. Wave equation in 2D plane polars

The 2D wave equation for $u(r, \theta, t)$ becomes

$$\boxed{\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u} \tag{3.17}$$



with boundary conditions at $r = 1$ on a unit disc (drum)

$$u(1, \theta, t) = 0 \ \forall t \tag{3.18}$$

(fixed rim) and initial conditions for $t = 0$

$$u(r, \theta, 0) = \phi(r, \theta), \quad \frac{\partial u}{\partial t}(r, \theta, 0) = \psi(r, \theta) \tag{3.19}$$

Temporal separation

Substitute

$$u(r, \theta, t) = T(t)V(r, \theta) \tag{3.20}$$

into (3.17) to get

$$\ddot{T} + \lambda c^2 T = 0 \tag{3.21}$$

$$\nabla^2 V + \lambda V = 0 \tag{3.22}$$

which in polars is

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \lambda V = 0$$

Spatial separation

Now try

$$V(r, \theta) = R(r)H(\theta)$$

in (3.22):

$$H'' + \mu H = 0 \tag{3.23}$$

$$r^2 R'' + rR' + (\lambda r^2 - \mu)R = 0 \tag{3.24}$$

where λ, μ are separation constants.

Polar solution: Configuration implies periodic boundary conditions

$$H(0) = H(2\pi)$$

with $\mu > 0$, so the eigenvalue $\mu = m^2$ (m integer) with solution

$$H_m(\theta) = A_m \cos m\theta + B_m \sin m\theta \tag{3.25}$$

Radial equation: divide (3.24) by r to bring it into Sturm Liouville form (2.7) with $\mu = m^2$

$$\frac{\mathrm{d}}{\mathrm{d}r}(rR') - \frac{m^2}{r}R = -\lambda r R \quad (0 \le r \le 1) \tag{3.26}$$

where $p(r) = r$, $q(r) = \frac{m^2}{r}$ and weight $\omega(r) = r$, with self-adjoint boundary conditions with $R(1) = 0$ and bounded at $R(0)$, since $p(0) = 0$ a regular singular point.

Bessel's equation

Substitute $z \equiv \sqrt{\lambda}r$ in (3.26) to find

$$\boxed{z^2\frac{\mathrm{d}^2 R}{\mathrm{d}z^2} + z\frac{\mathrm{d}R}{\mathrm{d}z} + (z^2 - m^2)R = 0} \tag{3.27}$$

which is Bessel's equation

$$(zR')' + \left(z - \frac{m^2}{z}\right)R = 0$$

Frobenius solution: substitute power series

$$R = z^p \sum_{n=0}^{\infty} a_n z^n$$

to obtain

$$\sum_n a_n[(n+p)(n+p-1)z^{n+p} + (n+p)z^{n+p} + z^{n+p+2} - m^2 z^{n+p}] = 0$$

Equate powers of z: considering coefficient of z^p,

$$p^2 - m^2 = 0 \implies p = m, -m$$

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Regular solution $p = m$, has recursion solution

$$(n+m)^2 a_n + a_{n-2} - m^2 a_n = 0$$

$$\implies a_n = \frac{-1}{n(n+2m)} a_{n-2}$$

Put $n \to 2n'$

$$\implies a_{2n'} = \frac{-1}{4n'(n'+m)} a_{2n'-2}$$

so stepping up from a_0 we have (dropping primes)

$$a_{2n} = \frac{(-1)^n}{2^{2n} n!(n+m)(n+m-1)\cdots(m+1)} a_0$$

Take $a_0 = \frac{1}{2^m m!}$ (convention) to find the *Bessel function of the first kind*:

$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} \left(\frac{z}{2}\right)^{2n} \tag{3.2}$$

Exercise: Use $y = \sqrt{z}R$ in Bessel equation (3.27) to find

$$y'' + y\left(1 + \frac{1}{4z} - \frac{m^2}{z"}\right) = 0$$

So as $z \to \infty$, $y'' = -y$ so we have solutions

$$R = \frac{1}{\sqrt{z}}(A\cos z + B\sin z).$$

Also works for $m = \gamma$ (non-integer) if $(n+m)! \to \Gamma(n+m+1)$. Second solution with $p = -m$ (integer) is the Neumann function (Bessel function of the second kind)

$$Y_m(z) = \lim_{\gamma \to m} \frac{J_\gamma(z)\cos(\gamma\pi) - J_{-\gamma}(z)}{\sin\gamma\pi}$$

Exercise*: Use (3.28) to show that

$$\frac{\mathrm{d}}{\mathrm{d}z}(z^m J_m(z)) = z^m J_{m-1}(z)$$

and hence

$$J_m'(z) + \frac{m}{z} J_m(z) = J_{m-1}(z) \tag{3.29}$$

Repeat with z^{-m} to find *recursion relations*.

$$J_{m-1}(z) + J_{m+1}(z) = \frac{2m}{z} J_m(z) \tag{3.30}$$

$$J_{m-1}(z) - J_{m+1}(z) = 2J_m'(z)$$

Asymptotic behaviour $J_m(z), Y_m(z)$:

- Small $z \to 0$, $J_0(z) \to 1$, $J_m(z) \to \frac{1}{m!}\left(\frac{z}{2}\right)^m$, $m > 0$.

  $$Y_0(z) \to \frac{2}{\pi} \ln\left(\frac{z}{2}\right), \quad Y_m(z) \to -\frac{(m-1)!}{\pi}\left(\frac{2}{z}\right)^m \tag{3.31}$$

  (Y_m is divergent as $z \to 0$)

- Large $z \to \infty$: oscillatory solutions:

  $$J_m(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right) \tag{3.32}$$

  $$Y_m(z) \approx \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

Zeros of Bessel function $J_m(z)$

Define j_{mn} to be n-th zero,

$$J_m(jmn) = 0 \quad (z > 0)$$

From (3.32) this occurs when (approximately)

$$\cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right) = 0$$

i.e. $z - \frac{m\pi}{2} - \frac{\pi}{4} = n\pi - \frac{\pi}{2}$ (modal point). So zero at

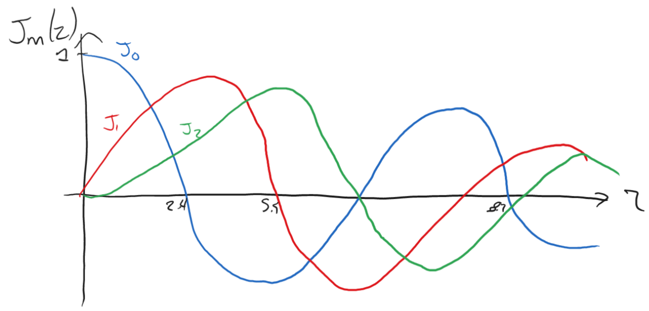
$$z \approx n\pi + \frac{m\pi}{2} - \frac{\pi}{4} \equiv \tilde{j}_{mn} \tag{3.33}$$

(Accuracy: $\frac{j_{mn} - \tilde{j}_{mn}}{j_{mn}} < \frac{0.1}{n}$, for $n > \frac{m^2}{2}$ (non-examinable))
For $J_0(z)$ actual values are

$$j_{01} = 2.405, \quad j_{02} = 5.520, \quad j_{03} = 8.653$$

$$j_{0n} \approx n\pi - \frac{\pi}{4}$$

(precision $\sim \frac{1\%}{n}$).



3.8. 2D Wave equation (continued): Vibrating drum

From section 3.8, radial solutions to (3.26) become

$$R_m(z) = R_m(\sqrt{\lambda}r) = AJ_m(\sqrt{\lambda}r) + BY_m(\sqrt{\lambda}r)$$

Impose boundary conditions:

- Regularity at $r = 0 \implies B = 0$ by (3.31)
- Unit disk $r = 1$ with $R = 0$ implies

  $$J_m(\sqrt{\lambda}) = 0$$

  But these zeros occur at

  $$j_{mn}(\approx \tilde{j}_{mn} = n\pi + \frac{m\pi}{2} - \frac{\pi}{4})$$

  so our eigenvalues must be

  $$\lambda_{mn} = j_{mn}^2 \tag{3.34}$$

With the polar mode (3.26) the spatial solution is

$$V_{mn}(r, \theta) = H_m(\theta)R_{mn}(\sqrt{\lambda_{mn}}r) = (A_{mn}\cos m\theta + B_{mn}\sin m\theta)J_m(j_{mn}r) \tag{3.35}$$

The temporal solution to (3.21) $\ddot{T} = -\lambda c^2 T$ are $T_{mn}(t) = \cos(j_{mn}ct)$ and $\sin(j_{mn}ct)$. For our linear homogeneous PDE (3.17)we can sum together to obtain general solution (noting the special case for $m = 0$):

$$\begin{aligned} u(r, \theta, t) &= \sum_{n=1}^{\infty} J_0(j_{mn}r)(A_{0n}\cos(j_{0n}ct) + C_{0n}\sin(j_{0n}ct)) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(j_{mn}r)(A_{nm}\cos m\theta + B_{nm}\sin m\theta)\cos(j_{mn}ct) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(j_{mn}r)(C_{mn}\cos m\theta + D_{mn}\sin m\theta)\sin(j_{nm}ct) \end{aligned} \tag{3.36}$$

Now impose *initial conditions* (3.19) at $t = 0$

$$u(r,\theta,0) = \phi(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(j_{mn}r) \times (A_{nn}\cos m\theta + B_{nn}\sin m\theta) \tag{3.37}$$

$$\frac{\partial u}{\partial t}(r,\theta,0) = \psi(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} j_{mn} c J_m(j_{mn}r) \times (C_{mn}\cos m\theta + D_{mn}\sin m\theta)$$

Orthogonality: Find coefficients by multiplying by J_m, cos, sin and exploit orthogonality (1.1-3) and Example sheet 1, Q8.

$$\int_0^1 J_m(j_{mn}r)J_m(j_{mk}r)r\mathrm{d}r = \frac{1}{2}[J_m'(j_{mn})]^2\delta_{nk} \tag{3.28}$$

$$= \frac{1}{2}[J_{m+1}(j_{mn})]^2\delta_{nk} \qquad \text{by recursion} \tag{3.29}$$

Now integrate to obtain A_{mn}

$$\int_0^{2\pi}\mathrm{d}\theta\cos p\theta\int_0^1 r\mathrm{d}r J_{pq}(j_{pq}r)\phi(r,\theta) = \frac{\pi}{2}[J_{p+1}(j_{pq})]^2 A_{pq}$$

Exercises: Find B, C, D.

Example. Initial radial profile.

$$u(r,\theta,0) = \phi(r) = 1 - r^2$$

$$\implies m = 0, \quad B_{mn} = 0, A_{mn} = 0, m \neq n$$

$$\frac{\partial u}{\partial t}(r,\theta,0) = 0$$

$$\implies C_{mn} = D_{mn} = 0$$

We need to find:

$$\begin{aligned} A_{0n} &= \frac{2}{J_1(j_{0n})^2}\int_0^1 J_0(j_{0n}r)(1-r^2)r\mathrm{d}r \\ &= \frac{2}{J_1(j_{0n})^2}\frac{J_2(j_{0n})}{j_{0n}^2} \\ &\approx \frac{J_2(j_{0n})}{n} \end{aligned}$$

as $n \to \infty$. (Exercise* using (3.29-30)).

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4. The Diffusion Equation

4.1. Physical origin of heat equation

Applies to processes that "diffuse" due to spatial gradients. An early example was Fick's law with flux $\mathbf{J} = -D\nabla c$ with concentration c and diffusion coefficient D. For *heat flow* we have Fourier's law

$$\mathbf{q} = -k\nabla\theta \tag{4.1}$$

($\mathbf{q}$ is heat flux, k is thermal conductivity, θ is temperature) In a volume V, the overall heat energy Q is

$$Q = \int c_V \rho\theta \mathrm{d}V \tag{4.2}$$

so rate of change due to heat flow

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int c_v \rho \frac{\partial\theta}{\partial t}\mathrm{d}V \tag{$*$}$$

Now integrate (4.1) over surface S enclosing V

$$\begin{aligned}-\frac{\mathrm{d}Q}{\mathrm{d}t} &= \int_S \mathbf{q}\cdot\hat{\mathbf{n}}\mathrm{d}S \\ &= \int_S (-k\nabla\theta)\cdot\hat{\mathbf{n}}\mathrm{d}S \\ &= \int (-k\nabla^2\theta)\mathrm{d}V \end{aligned} \tag{$\dagger$}$$

Equating ($*$) and ($\dagger$) we find

$$\int \left(c_v\rho\frac{\partial\theta}{\partial t} - k\nabla^2\theta\right)\mathrm{d}V = 0$$

True for all V, so integrand must vanish, so

$$\frac{\partial\theta}{\partial t} - \frac{k}{c_v\rho}\nabla^2\theta = 0$$

so if we set $D = \frac{k}{c_v\rho}$ we have

$$\boxed{\frac{\partial\theta}{\partial t} = D\nabla^2\theta} \tag{4.3}$$

Brownian motion (random walk)

Gas particles are diffusing by scattering every Δt with probability PDF $p(\xi)$ of moving distance ξ with

$$\langle\xi\rangle = \int p(\xi)\xi \mathrm{d}\xi = 0$$

Suppose the PDF after $N\Delta t$ steps is $P_{N\Delta t}(x)$, then for the $(N+1)\Delta t$ step:

$$\begin{aligned} P_{(N+1)\Delta t}(x) &= \int_{-\infty}^{\infty} p(\xi)P_{N\Delta t}(x-\xi)\mathrm{d}\xi \\ &\approx \int_{-\infty}^{\infty} p(\xi)\left[P_{N\Delta t}(x) + P'_{N\Delta t}(x)(-\xi) + P''_{N\Delta t}(x)\frac{\xi^2}{2} + \cdots\right]\mathrm{d}\xi \\ &= P_{N\Delta t}(x) - P'_{N\Delta t}(x)\langle\xi\rangle + P''_{N\Delta t}(x)\frac{\langle\xi^2\rangle}{2} + \cdots \end{aligned}$$

Note that $\langle\xi\rangle$ is the mean of ξ which is 0. Denote $P_{N\Delta t}(x) = P(x, N\Delta t)$, then we have

$$P(x,(N+1)\Delta t) - P(x, N\Delta t) = \frac{\partial^2}{\partial x^2}P(x, N\Delta t)\frac{\langle\xi\rangle^2}{2}$$

Assuming $\frac{\langle\xi\rangle^2}{2} = D\Delta t$ then $\Delta t \to 0$. We find

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2} \tag{4.4}$$

4.2. Similarity solution

The characteristic relation between variance and time, suggest seeking solutions with dimensionless parameter

$$\eta \equiv \frac{x}{2\sqrt{Dt}} \tag{4.5}$$

Can we find solutions $\theta(x,t) = \theta(\eta)$? Change variables in (4.3):

$$LHS: \frac{\partial\theta}{\partial t} = \frac{\partial\eta}{\partial t}\frac{\partial\theta}{\partial\eta} = -\frac{1}{2}\frac{x}{\sqrt{D}t^{3/2}}\theta' = -\frac{1}{2}\frac{\eta}{t}\theta'$$

$$RHS: D\frac{\partial^2\theta}{\partial x^2} = D\frac{\partial}{\partial x}\left(\frac{\partial\eta}{\partial x}\frac{\partial\theta}{\partial\eta}\right) = D\frac{\partial}{\partial x}\left(\frac{1}{2\sqrt{D}t}\theta'\right) = \frac{D}{4Dt}\theta'' = \frac{1}{4t}\theta''$$

Equating

$$\theta'' = -2\eta\theta' \tag{4.6}$$

Take $\psi = \theta'$, $\frac{\psi'}{\psi} = -2\eta \implies \ln\psi = -\eta^2 + \text{const}$

$$\implies \psi = \theta' = (\text{const})e^{-\eta^2}$$

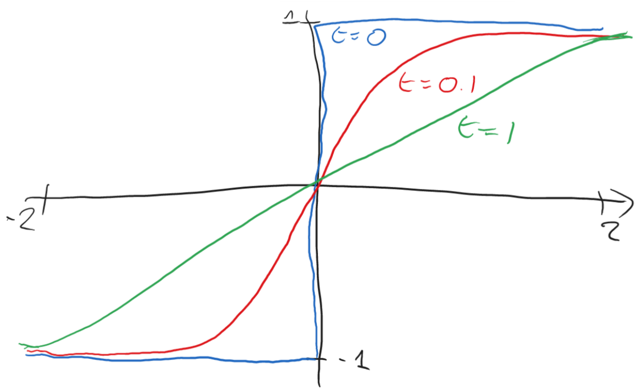
Integrate to find

$$\theta = c\frac{2}{\sqrt{\pi}}\int_0^\eta e^{-u^2}\mathrm{d}u = c\,\mathrm{erf}\left(\frac{x}{2\sqrt{Dt}}\right) \tag{4.7}$$

where the error function is

$$\mathrm{erf}(Z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-u^2}\mathrm{d}u$$

This describes discontinuous initial conditions that spread over time ($D = 1$):



4.3. Heat conduction in a finite bar

Suppose we have a bar of length wL with $-L \leq x \leq L$ and initial temperature:

$$\theta(x, 0) = H(x) = \begin{cases} 1 & 0 \leq x \leq L \\ 0 & -L \leq x < 0 \end{cases} \tag{4.8}$$

with boundary conditions

$$\theta(h, t) = 1, \quad \theta(-h, t) = 0 \tag{4.9}$$

Transforming boundary conditions: The boundary conditions (4.9) are not homogeneous. Can we identify steady state solution (time independent) that reflects late-time behaviour? Try

$$\theta_s(x) = Ax + B$$

satisfies $\frac{\partial^2\theta}{\partial x^2} = 0$. To satisfy (4.9), $A = \frac{1}{2L}$, $B = \frac{1}{2}$.

$$\theta_S = \frac{x + L}{2L} \tag{4.10}$$

Transform and solve for

$$\hat{\theta}(x,t) = \theta(x,t) - \theta_S(x)$$

with homogeneous boundary conditions

$$\hat{\theta}(-L,t) = \hat{\theta}(L,t) = 0$$

and initial conditions

$$\hat{\theta}(x,0) = H(x) - \frac{x+L}{2L}$$

Separation of variables: try

$$\hat{\theta}(x,t) = X(x)T(t)$$
$$\implies X'' = -\lambda X, \dot{T} = -D\lambda T \tag{4.12}$$

Boundary conditions imply $\lambda > 0$ with

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

For $\cos(\sqrt{\lambda}L) = 0$

$$\implies \sqrt{\lambda_m} = \frac{m\pi}{2L} \qquad m = 1,3,5,\ldots$$

$\sin(\sqrt{\lambda}L) = 0$

$$\implies \sqrt{\lambda_n} = \frac{n\pi}{L} \qquad n = 1,2,3,\ldots$$

but initial conditions are *odd* ($A_m = 0$) so take

$$X_n = B_n \sin\frac{n\pi x}{L} \quad \lambda_n = \frac{n^2\pi^2}{L^2}$$

Put λ_n into (4.12) $\dot{T} = -D\lambda T$ to find

$$T_n(t) = C_n \exp\left(-\frac{Dn^2\pi^2}{L^2}t\right)$$

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General solution:

$$\hat{\theta}(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-\frac{Dn^2\pi^2}{L^2}t} \tag{4.13}$$

Now impose initial conditions (4.11) at $t = 0$

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{\hat{\phi}(x,0)}_{H(x)-\frac{x+L}{2L}} \sin\frac{n\pi x}{L}\mathrm{d}x \\
&= \frac{2}{L}\int_0^L (H(x) - \frac{1}{2})\sin\frac{n\pi x}{L}\mathrm{d}x - \frac{2}{L}\int_0^L \frac{x}{2L}\sin\frac{n\pi x}{L}\mathrm{d}x \\
&= \frac{2}{n\pi}(\text{n odd}) - \frac{(-1)^{n+1}}{n\pi} \\
&= \frac{1}{n\pi}
\end{aligned}$$

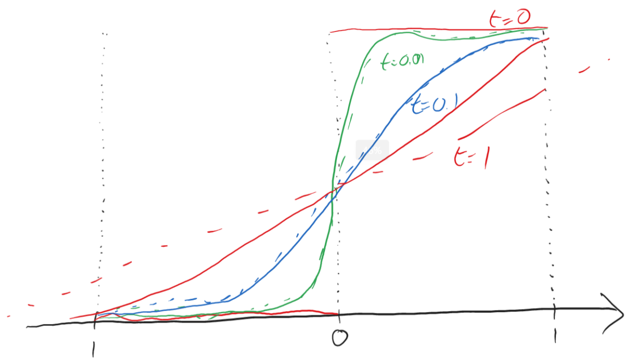
Solution

$$\hat{\theta}(x,t) = \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin \frac{n\pi x}{L} e^{-D\frac{n^2\pi^2}{L^2}t}$$

or with original boundary conditions (4.9)

$$\theta(x,t) = \frac{x+L}{2L} + \hat{\theta}(x,t) \tag{4.14}$$

Plot with $L = 1$ and $D = 1$.



(the dotted lines are the fundamental solutions). Approximate solution (4.7) ($\frac{1}{2}\,\mathrm{erf}\left(\frac{x}{2\sqrt{Dt}}\right)$) are excellent for $t \leq 0.1$.

5. The Laplace Equation

Laplace's equation

$$\boxed{\nabla^2 \phi = 0} \tag{5.1}$$

has wide application in math physics & applications:

- Steady state heat flow
- Potential theory $\mathbf{F} = -\nabla\phi$ (also $\nabla^2\phi = \rho(\mathbf{x})$)
- Incompressible fluid flow $\mathbf{v} = \nabla\phi$ etc.

We solve (5.1) in a domain D subject to boundary conditions either:

- Dirichlet: ϕ given on boundary surface ∂D
- Neumann: $\mathbf{n}\cdot\nabla\phi$ given on boundary surface ∂D.

5.1. 3D Cartesian coordinates

Equation (5.1) becomes

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{5.2}$$

Seek separable solution $\phi(x,y,z) = X(x)Y(y)Z(z)$

$$X''YZ + XY''Z + XYZ'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = -\lambda_l(\text{constant})$$

and

$$\frac{Y''}{Y} = -\lambda_m(\text{constant})$$

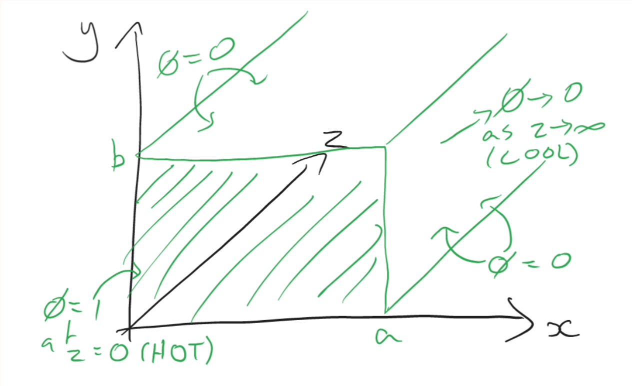
so

$$\frac{Z''}{Z} = -\lambda_n = \lambda_l + \lambda_m$$

General solution from eigenmodes

$$\phi(x,y,z) = \sum_{l,m,n} a_{lmn} X_l(x) Y_m(y) Z_n(z) \tag{5.4}$$

Example (Steady heat conduction)**.** ((4.3) with $\frac{\partial \phi}{\partial t} = 0 \implies$ (5.1)) Consider a semi-infinite rectangular bar



with boundary conditions $\phi = 0$ at $x = 0, a$ and $y = 0, b$. $\phi = 1$ at $z = 0$, $\phi \to 0$ as $z \to \infty$. Solve for eigenmodes successively:

- $X'' = -\lambda_l X$ with $X(0) = X(a) = 0$

$$\lambda_l = \frac{l^2\pi^2}{a^2}, \quad X_l = \sin\frac{l\pi x}{a} \quad l = 1, 2, \dots$$

- $Y'' = -\lambda_m Y$

$$\lambda_m = \frac{m^2\pi^2}{b^2} \quad Y_m = \sin\frac{m\pi y}{b} \quad m > 0$$

- $Z'' = -\lambda_n Z = (\lambda_l + \lambda_m)z = \pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right) Z$ with boundary conditions $Z \to 0$ as $z \to \infty$

$$Z_{lm} = \exp\left[-\left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right)^{\frac{1}{2}} \pi z\right]$$

So our *general solution* (5.4) becomes

$$\phi(x, y, z) = \sum_{l,m} a_{lm} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \exp\left[-\left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right)^{\frac{1}{2}} \pi z\right]$$

Now fix a_{km} using $\phi(x, y, z) = 1$ using Fourier sine b_n (1.12)

$$\begin{aligned} a_{lm} &= \frac{2}{b}\int_0^b \mathrm{d}y \frac{2}{a}\int_0^a \mathrm{d}x \underbrace{1 \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b}}_{\text{square wave FS (1.7)}} \\ &= \frac{4a}{al\pi}\frac{4b}{bm\pi}(l, m, \text{ odd}) \\ &= \frac{16}{\pi^2 lm}(l, m \text{ odd}) \end{aligned}$$

so the heat flow solution is

$$\phi(x, y, z) = \sum_{l,m \text{ odd}} \frac{16}{\pi^2 lm} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \exp\left[-\left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right)^{\frac{1}{2}} \pi z\right]$$

5.2. 2D Plane Polar coordinates

Recall

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \tag{5.6}$$

and try $\phi(r,\theta) = R(r)H(\theta)$ to find

$$H'' + \mu H = 0 \quad \text{and} \quad r(rR')' - \mu R = 0$$

- Polar equation periodic boundary conditions $\implies$ $\mu = m^2$. (as before (3.25), $H_m(\theta) = \cos m\theta$ and $\sin m\theta$)
- Radial equation $r(rR')' - m^2 R = 0$ (5.7). Try $R = \alpha r^\beta \implies \beta^2 - m^2 = 0$, $\beta = \pm m$

  $$R_m = r^m \text{ and } r^{-m}$$

  If $m = 0$, $(rR')' = 0 \implies rR' = \text{const} \implies R = \log r$.

  $$R_0 = \text{const and } \log r.$$

General solution:

$$\theta(r,0) = \frac{a_0}{2} + c_0 \log r + \sum_{m=1}^{\infty} (a_m \cos m\theta + b_m \sin m\theta) r^m$$
$$+ \sum_{m=1}^{\infty} (c_m \cos m\theta + d_m \sin m\theta) r^{-m} \tag{5.8}$$

Example. Soap film on unit disk. Solve (5.6) with a distorted circular disk (wire) radius $r = 1$ with given boundary conditions:

$$\phi(1, \theta) = f(\theta)$$

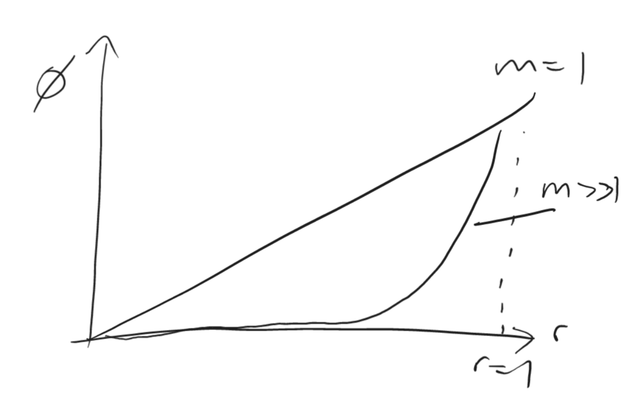
to find $\phi(r, \theta)$ on $r < 1$. Regularity at $r = 0$ implies $c_m = d_m = 0$ for all n, so (5.8) becomes

$$\theta(r, \theta) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty} (a_M \cos m\theta + b_m \sin m\theta) r^m$$

At $r = 1$, $\phi(1, 0) = f(\theta) = \frac{1}{2}a_0 + \sum_m (a_m \cos m\theta + b_m \sin m\theta)$ so the Fourier series coefficients (1.5) are

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos m\theta \mathrm{d}\theta, \quad b_m = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin m\theta \mathrm{d}\theta$$

Note high harmonics are confined near $r = 0$ edge becomes r^m term



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5.3. 3D Cylindrical Polar Coordinates

Here

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{5.9}$$

Substitute $\phi(r, \theta, z) = R(r)H(\theta)Z(z)$ to find

$$H'' = -\mu H, \quad Z'' = \lambda Z$$

$$r(rR')' + (\lambda r^2 - \mu)R = 0$$

- Polar (as before) $\mu_m = m^2$,

$$H_m = \cos m\theta \text{ and } \sin m\theta$$

- Radial (Bessel's equation (3.26))

$$r(rR')' + (\lambda r^2 - m^2)R = 0$$

with solutions $R = J_m(kr)$ and $Y_m = (kr)$. Setting boundary conditions $R = 0$ at $r = 1$ means

$$J_m(ka) \implies k = \frac{j_{mn}}{a}$$

where j_{mn} is the n-th zero (see (3.32)). Radial eigenfunction

$$R_{mn} = J_m\left(\frac{j_{mn}}{a}r\right) \tag{3.10}$$

(eliminate Y_m since $Y_m \to -\infty$ as $r \to 0$)

- Z equation: $Z'' = k^2 Z$ implies $Z = e^{-kz}$ and $z = e^{kz}$ (usually eliminate e^{kz} with $z \to 0$ as $z \to \infty$.)

So general solution is

$$\phi(r,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(a_{mn}\cos m\theta + b_{mn}\sin m\theta) \times J_m(\frac{j_{mn}}{a}r)e^{-j_{mn}z/a} \tag{5.11}$$

Exercise: Describe steady-state heat flow in a semi-infinite circular wire with boundary conditions $\phi = 0$ at $r = a$, $\phi = T_0$ at $z = 0$ and $\phi \to 0$ as $z \to \infty$ (see section 3.9 and 5.9). Show that the solution is

$$\phi(r,\theta,z) = \sum_{k=1}^{\infty}\frac{2T_c}{j_{0n}J_1(j_{0n})}J_0\left(\frac{j_{0n}}{a}r\right)e^{-j_{0n}z/a}$$

5.4. 3D Spherical Polar Coordinates

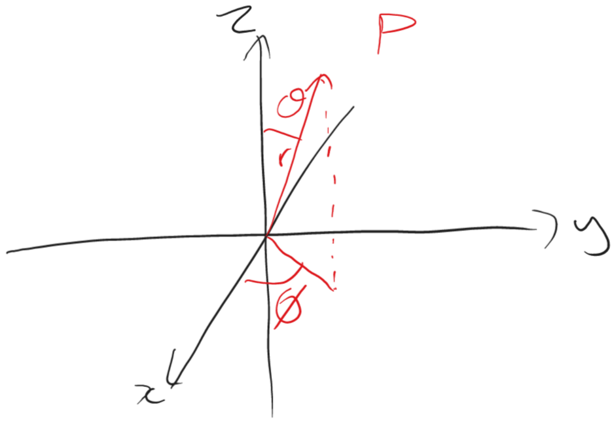
Recall that

$$x = r\sin\theta\cos\theta$$

$$y = r\sin\theta\sin\theta$$

$$z = r\cos\theta$$

and $\mathrm{d}V = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$, $0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$.



Laplace's equation (5.1) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\frac{\partial^2\Phi}{\partial\phi^2}=0 \tag{5.12}$$

Axisymmetric case (no ϕ dependence)

Seek separable $\Phi(r,\theta)=R(r)H(\theta)$.

$$(\sin\theta H')'+\lambda\sin\theta H=0$$
$$(r^2R')'=\lambda R=0 \tag{5.13}$$

Polar (Legendre's) equation: Substitute $x=\cos\theta$ with

$$\frac{\mathrm{d}x}{\mathrm{d}\theta}=-\sin\theta \implies \frac{\mathrm{d}H}{\mathrm{d}\theta}=-\sin\theta\frac{\mathrm{d}H}{\mathrm{d}x}$$
$$-\cancel{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}x}\left[-\sin^2\theta\frac{\mathrm{d}H}{\mathrm{d}x}\right]+\lambda\cancel{\sin\theta}H=0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}H}{\mathrm{d}x}\right)+\lambda H=0$$

which is Legendre's equation (2.21) with eigenvalues $\lambda_l=l(l+1)$ and eigenfunctions (2.23)

$$H_l(\theta)=P_l(x)=P_l(\cos\theta) \tag{5.14}$$

(see section 2.5)

- Radial equation:

  $$(r^2R')'-l(l+1)R=0$$

  Seek solutions $R=\alpha r^\beta$.

  $$\beta(\beta+1)-l(l+1)=0 \implies \left(\beta+\frac{1}{2}\right)^2=\left(l+\frac{1}{2}\right)^2$$

with two solutions $\beta = l$ and $\beta = -l-1$.

$$R_l = r^l \text{ and } r^{-l-1}$$

General axisymmetric solution:

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}(a_l r^l + b_l r^{-l-1})P_l(\cos\theta) \tag{5.15}$$

where a_l, b_l determined by boundary conditions, usually at fixed $r = r_0$. Use orthogonality conditions for P_l's, see (2.24).

Unit sphere solution: Solve $\nabla^2\Phi = 0$ for $r < 1$ given axisymmetric boundary conditions at $r = 1$, $\Phi(1,\theta) = f(\theta)$. Regularity implies that $b_l = 0$, so we have

$$f(\theta) = \sum_{l=0}^{\infty} a_l P_l(\cos\theta)$$

or with $f(\theta) = F(\cos\theta) = F(x)$,

$$F(x) = \sum_{l=0}^{\infty} a_l P_l(x)$$

so by (2.25) so

$$a_l = \frac{2l+1}{2}\int_{-1}^{1} F(x)P_l(x)\mathrm{d}x$$

Exercise: Show $f(\theta) = \sin^2\theta$ yields solution

$$\Phi(r,\theta) = \frac{2}{3}(1 - P_2(\cos\theta)r^2)$$

Generating function for $P_l(x)$ (2.23a)

Consider a charge on z-axis at $z = 1$, $\mathbf{r}_0 = (0,0,1)$ then the potential P becomes

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{|\mathbf{r}-\mathbf{r}_0|}\\ &= \frac{1}{(x^2+y^2+(z-1)^2)^{\frac{1}{2}}}\\ &= \frac{1}{(r^2\sin^2\theta + r^2\cos^2\theta - 2r\cos\theta + 1)^{\frac{1}{2}}}\\ &= \frac{1}{\sqrt{r^2 - 2r\cos\theta + 1}}\\ &= \frac{1}{\sqrt{r^2 - 2r\overline{x} + 1}}\end{aligned}$$

($\overline{x} = \cos\theta$)
Exercise: verify $\Phi = \frac{1}{|\mathbf{r}-\mathbf{r}_0|}$ satisfies $\nabla^2\Phi = 0$ whenever $\mathbf{r} \neq \mathbf{r}_0$.
We can represent any axisymmetric solution (5.12) as a sum (5.15) (with $b_n = 0$) for $r < 1$:

$$\frac{1}{\sqrt{r^2 - 2rx + 1}} = \sum_{l=0}^{\infty} a_l P_l(x) r^l$$

with norm at $x = 1$, $P_l(1) = 1$, we get

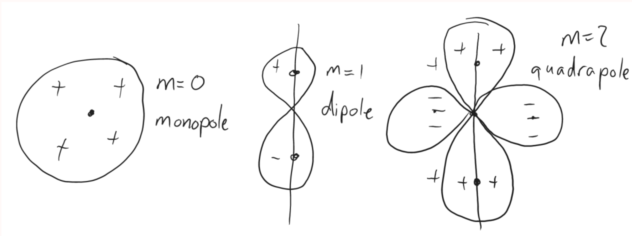
$$\frac{1}{1-r} = \sum_{l=0}^{\infty} a_l r^l$$

so $a_l = 1$ (for a geometric series). Thus generating function for $P_l(x)$ is

$$\frac{1}{\sqrt{1 - 2xr + r^2}} = \sum_{l=0}^{\infty} P_l(x) r^l \tag{5.16}$$

Expand LHS with binomial theorem to find $P_l(x)$ (coefficient of the r^l term) Use to obtain norm condition (2.24). (Example sheet 2, Q5)

Example (Electric multipole)**.**



Start of lecture 13

Chapter III

Inhomogeneous ODEs; Fourier Transforms

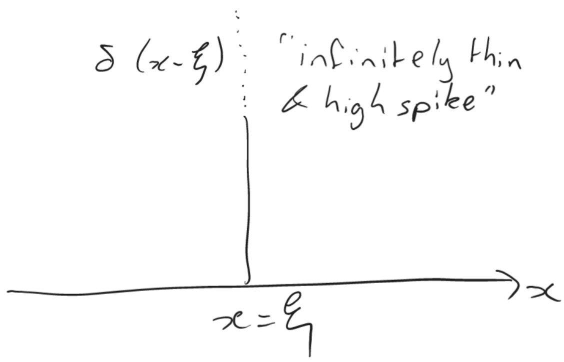
Contents

6. The Dirac Delta Function

6.1. Definition of $\delta(x)$

Define a generalised function $\delta(x-\xi)$ with the following properties:

$$\boxed{\begin{aligned}\delta(x-\xi) &= 0, \quad \forall x \neq \xi \\ \int_{-\infty}^{\infty} \delta(x-\xi)\mathrm{d}x &= 1\end{aligned}} \tag{6.1}$$



This acts as a linear operator $\int \mathrm{d}x\delta(x-\xi)$ on an arbitrary function $f(x)$ to produce a number $f(\xi)$, that is,

$$\boxed{\int_{-\infty}^{\infty} \mathrm{d}x\delta(x-\xi)f(x) = f(\xi)} \tag{6.2}$$

provided $f(x)$ is 'well-behaved' at $x=\xi$ and $\pm\infty$.

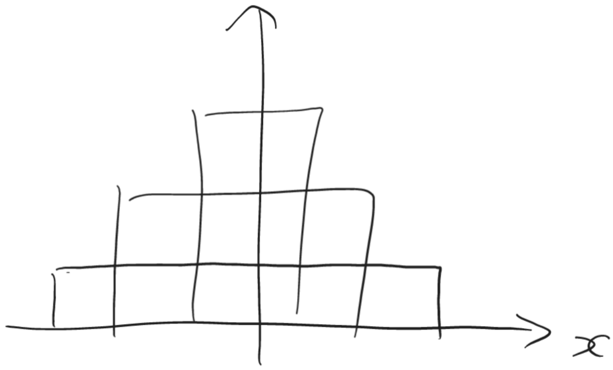
Notes

- The delta function $\delta(x)$ is classified as a distribution (not a function). See lecture notes of Jozsa and Skinner section 6.1 (optional).
- $\delta(x)$ always appears in an integrand as a linear operator where it is well-defined.
- Represents a unit point source (for example mass, charge) or an impulse.

Some limiting approximations

Discrete:

$$\lim_{n\to\infty} \delta_n(x) = \begin{cases} 0 & x > \frac{1}{n} \\ \frac{n}{2} & |x| \leq \frac{1}{n} \\ 0 & x < -\frac{1}{n} \end{cases}$$



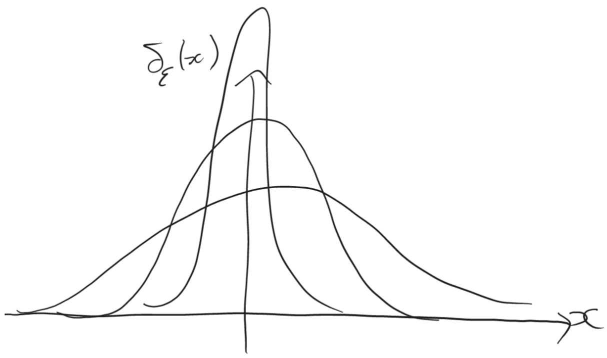
Continuous:

$$\lim_{\varepsilon\to 0}\delta_\varepsilon(x) = \frac{1}{\varepsilon\sqrt{\pi}}e^{-x^2/\varepsilon^2} \tag{6.3}$$

verify (6.2):

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{1}{\varepsilon\sqrt{\pi}}e^{-x^2/\varepsilon^2}f(x)\mathrm{d}x\\ &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-y^2}f(\varepsilon y)\mathrm{d}y\\ &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\mathrm{d}y\frac{1}{\sqrt{\pi}}(f(0)+\varepsilon y f'(0)+\cdots)\\ &= f(0)\end{aligned}$$

$\forall f$. 'well-behaved' at $x = 0$ so that we can take the Taylor expansion, and also need well behaved at $\pm\infty$ so that it doesn't grow faster than $1/e^{-x^2/\varepsilon^2}$



Further examples: ($\lim n\to\infty$)

$$\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi}\int_{-n}^{n} e^{ikx}\mathrm{d}k \tag{6.4}$$

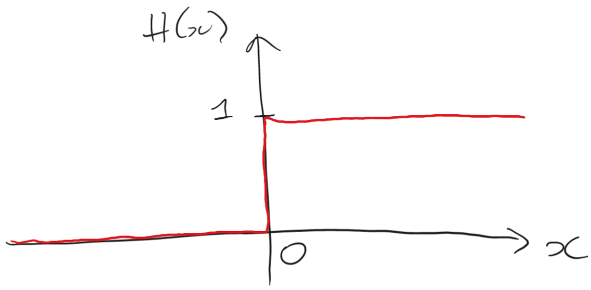
$$\delta_n(x) = \frac{n}{2}\text{sech}^2 nx \tag{6.5}$$

6.2. Properties of $\delta(x)$

Heaviside function $H(x)$

The unit step function,

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6.6}$$



is the *integral of* $\delta(x)$.

$$H(x) = \int_{-\infty}^{x} \delta(x)\mathrm{d}x \tag{6.7}$$

and we can identify $H'(x) = \delta(x)$.

Example. Verify using (6.5) $\delta(x) = \lim_{n\to\infty} \frac{n}{2}\text{sech}^2 nx$. (You will find $\frac{1}{2}(\tanh nx+1)$ is the approximate step function. - also $H(0) = \frac{1}{2}$ (alternate definition).)

Derivative of $\delta(x)$

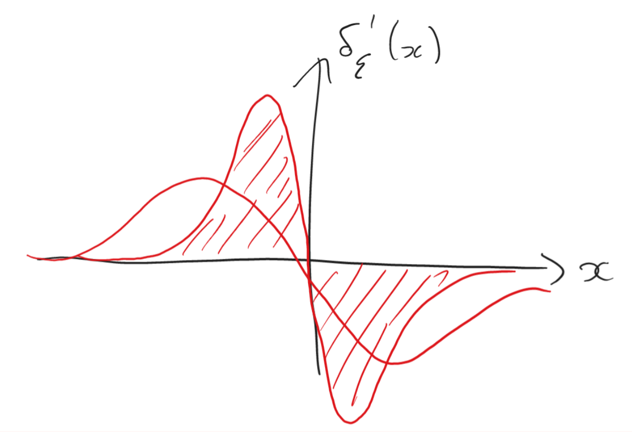
Define $\delta'(x)$ using integration by parts:

$$\begin{aligned} \int_{-\infty}^{\infty} \delta'(x-\xi)f(x)\mathrm{d}x &= [\delta(x-\xi)f(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x-\xi)f'(x)\mathrm{d}x \\ &= -f'(\xi) \end{aligned} \tag{6.8}$$

for all $f(x)$ smooth at $x = \xi$.

Example. Consider Gaussian approximation (6.3)

$$\delta'_\varepsilon(x) = -\frac{2x}{\varepsilon^3\sqrt{\pi}}e^{-x^2/\varepsilon^2}$$



Sampling property

$$\int_a^b f(x)\delta(x-\xi)\mathrm{d}x = \begin{cases} f(\xi) & a<\xi<b \\ 0 & \text{otherwise} \end{cases}$$

Even property

$$\int_{-\infty}^{\infty} f(x)\delta(-(x-\xi))\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\delta(x-\xi)\mathrm{d}x \tag{6.10}$$

$$\begin{aligned} LHS &= \int_{\infty}^{-\infty} f(\xi-u)\delta(u)(-\mathrm{d}u) \\ &= \int_{-\infty}^{\infty} f(\xi-u)\delta(u)\mathrm{d}u \\ &= f(\xi) \\ &= RHS \end{aligned}$$

Scaling property

$$\int_{-\infty}^{\infty} f(x)\delta(a(x-\xi))\mathrm{d}x = \frac{1}{|a|}f(\xi) \tag{6.11}$$

Exercise: Show this using $u = ax$ (noting integral limit order with $a < 0$).

Advanced scaling

Suppose $g(x)$ has n isolated zeros at $x_1, x_2, \dots, x_n$ then (with $g'(x_i) \neq 0$):

$$\delta(g(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|g'(x_i)|} \tag{6.12}$$

(Exercise: Show for g has 1 root at $x = x_i$).

> **Example.**
> $$I = \int_{-\infty}^{\infty} f(x)\delta(x^2 - 1)\mathrm{d}x$$
> $x^2 - 1$ has roots $x = \pm 1$ with $g'(x) = 2x$. So
> $$\begin{aligned} I &= \int_{1-\varepsilon}^{1+\varepsilon} f(x)\frac{\delta(x-1)}{|2x|}\mathrm{d}x + \int_{-1-\varepsilon}^{-1+\varepsilon} f(x)\frac{\delta(x+1)}{|2x|}\mathrm{d}x \\ &= \frac{1}{2}(f(1) + f(-1)) \end{aligned}$$

Isolation property

If $g(x)$ is continuous at $x = 0$ then

$$g(x)\delta(x) = g(0)\delta(x) \tag{6.13}$$

Exercise: evaluate and show

$$\int_0^{\infty} \delta'(x^2 - 1)x^2 \mathrm{d}x = -\frac{1}{4}$$

using $u = x^2 - 1$ and note (6.8) and (6.12).

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6.3. Eigenfunction expansions of $\delta(x)$

Fourier series (complex)

For $-1 \leq x < L$, represent

$$\delta(x) = \sum_{n=-\infty}^{\infty} c_n e^{-n\pi x/L}$$

Fourier series coefficient (1.15):

$$c_n = \frac{1}{2L}\int_{-L}^{L} \delta(x) e^{-n\pi x/L}\mathrm{d}x = \frac{1}{2L}$$

so

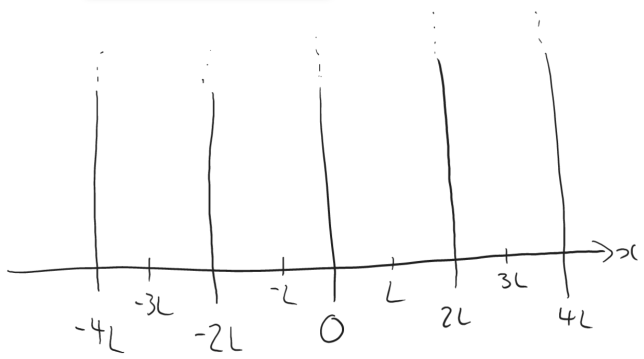
$$\delta(x) = \frac{1}{2L}\sum_{n=-\infty}^{\infty} e^{in\pi x/L} \tag{6.14}$$

Take $f(x) = \sum_{n=-\infty}^{\infty} d_n e^{in\pi x/L}$, then (using section 2.2)

$$\begin{aligned}\int_{-L}^{L} f^*(x)\delta(x)\mathrm{d}x &= \frac{1}{2L}\sum_n d_n \int_{-L}^{L} e^{-in\pi x/L}e^{in\pi x/L}\mathrm{d}x \\ &= \sum_n d_n \\ &= f(0)\end{aligned}$$

The *Diract count* comes from extending periodically to all $\mathbb{R}$:

$$\sum_{m=0\infty}^{\infty} \delta(x - 2mL) = \frac{1}{2L}\sum_{n=-\infty}^{\infty} e^{in\pi x/L}$$



General eigenfunctions

Suppose $\delta(x - \xi) = \sum_{n=1}^{\infty} a_n y_n(x)$, $a \leq x \leq b$ with coefficients (2.17):

$$\begin{aligned}a_n &= \frac{\int_a^b \omega(x) y_n(x)\delta(x-\xi)\mathrm{d}x}{\int_a^b \omega y_n^2 \mathrm{d}x} \\ &= \frac{\omega(\xi) y_n(\xi)}{\int_a^b \omega y_n^2 \mathrm{d}x} \\ &= \omega(\xi) Y_n(\xi)\end{aligned}$$

for unit norm Y_n (2.18). Then

$$\begin{aligned}\delta(x-\xi) &= \omega(\xi)\sum_{n=1}^{\infty} Y_n(\xi)Y_n(x) \\ &= \omega(x)\sum_{n=1}^{\infty} Y_n(\xi)Y_n(x)\end{aligned}$$

since

$$\frac{\omega(x)}{\omega(\xi)}\delta(x-\xi) = \delta(x-\xi)$$

by (6.13). Hence

$$\boxed{\delta(x-\xi) = \omega(x)\sum_{n=1}^{\infty}\frac{y_n(\xi)y_n(x)}{\mathcal{N}_n}} \tag{6.15}$$

where $\mathcal{N}_n = \int_a^b \omega y_n^a \mathrm{d}x$.

Example. Consider Fourier series $y(0) = y(1) = 0$ with $y_n(x) = \sin n\pi x$. Here, from (1.11) we have

$$\delta(x-\xi) = 2\sum_{n=1}^{\infty}\sin n\pi\xi \sin n\pi x \tag{$*$}$$

Exercise:

(i) Integrate both sides to show

$$\sum_{m=1}^{\infty}\frac{(-1)^{m+}}{2m-1} = \frac{1}{4}$$

when $\xi = \frac{1}{2}$.

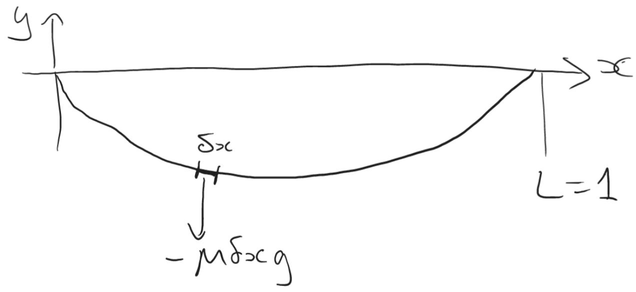
(ii) Integrate twice and compare with $G(x,\xi)$ (1.25) or (2.31).

7. Green's Function

7.1. Physical motivation: Static forces on a string

Consider a *massive* static string (tension T, density μ) with fixed ends

$$y(0) = y(1) = 0 \tag{7.1}$$



By resolving forces, we have (3.3)

$$T\frac{\partial^2 y}{\partial x^2} - \mu g = 0$$

(time independent). So solve inhomogeneous ODE subject to (7.1) with $f(x) = -\frac{\mu g}{T}$.

$$-\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f(x) \tag{7.2}$$

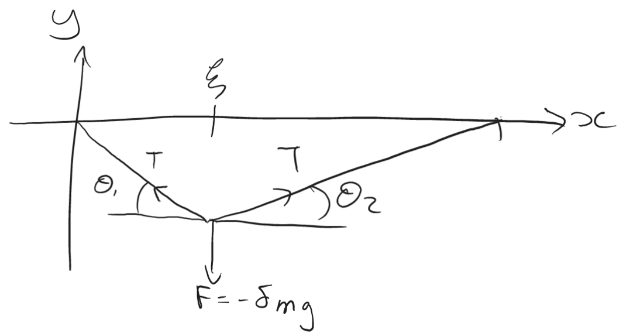
Solution 1: Direct integration for uniform mass density ODE (7.2) implies:

$$-y = -\frac{\mu g}{2T}x^2 + k_1 x + k_2$$

Boundary conditions (7.1) implies

$$y(x) = \left(-\frac{\mu g}{T}\right)\frac{1}{2}x(1-x) \tag{7.3}$$

Solution 2: Superposition of point masses on light string $\tilde{\mu} \to 0$. Consider point mass δm $(= \mu\delta x)$ suspended at $x = \xi$:



Resolve in y dimension to find $y_i(\xi_i)$:

$$\begin{aligned} 0 &= T(\sin\theta_1 + \sin\theta_2) - \delta m g \\ &= T\left(\left(-\frac{y_i}{\xi_i}\right) + \left(\frac{-y_i}{1-\xi_i}\right)\right) - \delta m g \\ \implies -T(y_i(1-\xi_i) + y_i\xi_i) &= \delta m g \xi_i(1-\xi_i) \end{aligned}$$

so

$$y_i(\xi_i) = \left(-\frac{\delta m g}{T}\right)\xi_i(1-\xi_i)$$

Hence solution

$$y_i(x) = \left(-\frac{\delta m g}{T}\right)\begin{cases} x(1-\xi_i) & x < \xi_i \\ \xi_i(1-x) & x < \xi_i \end{cases} = f_i G(x,\xi)$$

where f_i is the source (here, $(-\frac{\delta m g}{T})$ and $G(x,\xi)$ is the solution for unit point mass (Green's function). Now sum N point masses δm at $x = \{\xi_i\}$ by linearity

$$y(x) = \sum_{i=1}^{N} f_i G(x,\xi_i)$$

or in *continuum limit* with

$$f_i = -\frac{\delta m g}{T} = -\frac{\mu \delta x g}{T} \equiv f(x)\mathrm{d}x$$

$f(x) = -\frac{\mu g}{T}$. We have $(x \to \xi)$:

$$\begin{aligned} y(x) &= \int_0^1 f(\xi)G(x,\xi)\mathrm{d}\xi \qquad (7.5) \\ &= \left(-\frac{\mu g}{T}\right)\left[\int_0^x \xi(1-x)\mathrm{d}\xi + \int_x^1 x(1-\xi)\mathrm{d}\xi\right] \\ &= \left(-\frac{\mu g}{T}\right)\left(\left[\frac{\xi^2}{2}(1-x)\right]_0^x + \left[x\left(\xi - \frac{\xi^2}{2}\right)\right]_x^1\right) \\ &= \left(-\frac{\mu g}{T}\right)\left(\frac{x^2}{2}(1-x) - 0 + \frac{x}{2} - x\left(x - \frac{x^2}{2}\right)\right) \\ &= \left(\frac{-\mu g}{T}\right)\frac{1}{2}x(1-x) \end{aligned}$$

so it matches (7.3).

7.2. Definition of Green's function

We wish to solve inhomogeneous ODE (section 2.1) on $a \le x \le b$.

$$\mathcal{L}y \equiv \alpha(x)y'' + \beta(x)y' + \gamma(x)y = f(x) \tag{7.6}$$

$f(x)$ is a source. With $\alpha \neq 0$, β, γ continuous and bounded. Homogeneous boundary conditions $y(a) = y(b) = 0$. The Green's function for the operator $\mathcal{L}$ is the solution for a unit point source (or impulse) at $x = \xi$.

$$\mathcal{L}G(x, \xi) = \delta(x - \xi) \tag{7.7}$$

which satisfies $G(a, \xi) = G(b, \xi) = 0$ (or similar). By linearity, we construct solutions by integrating over source $f(x)$ with G:

$$y(x) = \int_a^b G(x, \xi)f(\xi)\mathrm{d}\xi \tag{7.8}$$

Formally verify this:

$$\mathcal{L}y = \int \mathcal{L}_(x)G(x, \xi)f(\xi)\mathrm{d}\xi = \int \delta(x - \xi)f(\xi)\mathrm{d}\xi = f(x)$$

so the solution (7.8) is given by the inverse operator $\mathcal{L}^{-1} = \int \mathrm{d}x G(x, \xi)$.

Defining properties (summary)

The Green's function splits into two parts:

$$G(x, \xi) = \begin{cases} G_1(x, \xi) & a \le x < \xi \\ G_2(x, \xi) & \xi < x \le b \end{cases} \tag{7.9}$$

such that:

(1) Homogeneous solutions: G solves homogeneous equation for all $x \neq \xi$. So

$$\mathcal{L}G_1 = 0, \quad \mathcal{L}G_2 = 0 \tag{7.10}$$

(2) Homogeneous boundary conditions: G satisfies homogeneous boundary conditions so

$$G_1(a, \xi) = 0, \quad G_2(b, \xi) = 0 \tag{2.11}$$

(3) Continuity condition: G is continuous at $x = \xi$ so

$$G_1(\xi, \xi) = G_2(\xi, \xi)$$

(4) Jump condition: Derivative discontinuous at $x = \xi$ with

$$[G']_{\xi_-}^{\xi_+} = \left.\frac{\mathrm{d}G_2}{\mathrm{d}x}\right|_{x=\xi_+} - \left.\frac{\mathrm{d}G_1}{\mathrm{d}x}\right|_{x=\xi_-} = \frac{1}{\alpha(\xi)} \tag{7.13}$$

where $\alpha(x)$ is defined in (7.6).

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7.3. Constructing $G(x, \xi)$: Boundary Value Problems

Solve

$$\mathcal{L}G(x, \xi) = \delta(x - \xi)$$

on $a \leq x \leq b$ with $G(a, \xi) = G(b, \xi) = 0$.

1 & 2 Solves homogeneous equation with homogeneous boundary conditions

Assume 2 independent homogeneous solutions $y_1(x)$, $y_2(x)$ known.
For $a \leq x < \xi$: $G_1(x, \xi) = Ay_1(x) + By_2(x)$ such that $Ay_1(a) + By_2(a) = 0$ (i.e. choose suitable A, B). This defines a complementary function (2.3) $y_-(x)$ such that $y_-(a) = 0$

$$\boxed{G_1 = Cy_-(x) \text{ with } y_-(a) = 0} \tag{7.14}$$

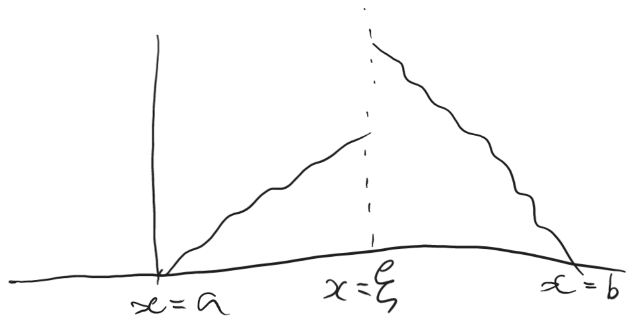
For $\xi < x \leq b$: Similarly find

$$\boxed{G_2 = Dy_+(x) \text{ with } y_+(b) = 0} \tag{7.15}$$

where $y_+(x)$ is a complementary function (2.3).

3. Why is G continuous at $x = \xi$?

Suppose G were discontinuous locally, so $G \propto H(x, \xi) + \cdots$ (6.7)



Then we would have $G' \propto \delta(x, \xi)$ and $G'' \propto \delta'(x - \xi)$. So LHS

$$\mathcal{L}G \propto \alpha(x)\delta'(x - \xi) + \beta(x)\delta(x - \xi) + \gamma(x)H(x, \xi)$$

there is no term $\propto \delta'(x - \xi)$. So G isn't discontinuous. Hence we have $[G]_{\xi_-}^{\xi^+} = 0$, so

$$\boxed{Cy_-(\xi) = Dy_+(\xi)} \tag{7.16}$$

4. Why the jump condition for G' at $x = \xi$?

Integrate $\mathcal{L}G = \delta(x, \xi)$ across $x = \xi$:

$$LHS + \int_{\xi_-}^{\xi^+} \mathcal{L}G \mathrm{d}x = \int_{\xi_-}^{\xi^+} (\alpha G'' + \beta G' + \gamma G)\mathrm{d}x$$

$$= \alpha(\xi)[G']_{\xi_-}^{\xi^+} + \underbrace{(\beta - \alpha')[G]_{\xi_-}^{\xi^+}}_{=0 \text{by continuity (7.16)}} + \underbrace{\int_{\xi_-}^{\xi^+} (\gamma - \beta' + \alpha'')G\mathrm{d}x}_{=0 \text{by continuity}}$$

$$RHS = \int_{\xi_-}^{\xi^+} \delta(x - \xi)\mathrm{d}x$$
$$= 1$$

So $[G']_{\xi_-}^{\xi^+} = \frac{1}{\alpha(\xi)}$ so

$$\boxed{Dy_+'(\xi) - Cy_-'(\xi) = \frac{1}{\alpha(\xi)}} \tag{7.17}$$

Wronskian $W(\xi)$

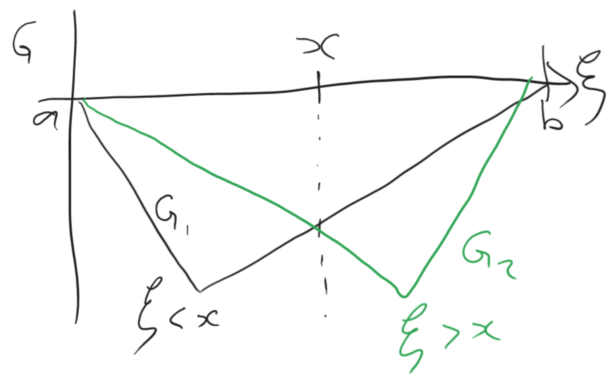
Solving (7.16) and (7.17) we find

$$C(\xi) = \frac{y_+(\xi)}{\alpha(\xi)W(\xi)}, \quad D(\xi) = \frac{y_-(\xi)}{\alpha(\xi)W(\xi)}$$

where $W(\xi) = y_-(\xi)y_+'(\xi) - y_+(\xi)y_-'(\xi) \neq 0$ if y_+, y_- are linearly independent. Hence

$$\boxed{G(x,\xi) = \begin{cases} \frac{y_-(x)y_+(\xi)}{\alpha(\xi)W(\xi)} & a \leq x < \xi \\ \frac{y_+(x)y_-(\xi)}{\alpha(\xi)W(\xi)} & \xi < x \leq b \end{cases}} \tag{7.20}$$

So the solution to (7.6) with $y(a) = y(b) = 0$



$$
\begin{aligned}
y(x) &= \int_a^b G(x,\xi)f(\xi)\mathrm{d}\xi \\
&= \int_a^x G_2(x,\xi)f(\xi)\mathrm{d}\xi + \int_x^b G_1(x,\xi)f(\xi)\mathrm{d}\xi \\
y(x) &= y_+(x)\int_a^x \frac{y_-(\xi)f(\xi)}{\alpha(\xi)W(\xi)}\mathrm{d}\xi + y_-(x)\int_a^b \frac{y_+(\xi)f(\xi)}{\alpha(\xi)W(\xi)}\mathrm{d}\xi \qquad (7.21)
\end{aligned}
$$

Notes:

(1) If $\mathcal{L}$ is in Sturm Liouville form (2.7) i.e. $\beta = \alpha'$ then denominator $\alpha(\xi)W(\xi)$ is a constant and G is *symmetric*, $G(x,\xi) = G(\xi,x)$. Exercise: show $\frac{\mathrm{d}}{\mathrm{d}x}(\alpha(x)W(x)) = 0$ if $\alpha' = \beta$ and using (2.10) (self-adjoint form).

(2) Often take $\alpha = 1$ (but Sturm Liouville form $\alpha < 0$).

(3) Indefinite integrals $\int_x$ in (7.21) are particular integral in general solution (2.5).

Exercise: For $-y'' = f(x)$, $y(0) = y(1) = 0$ directly construct the Green's function (7.4) (i.e. with $y_1 = x$, $y_2 =$ constant).

Example. Solve $y'' - y = f(x)$ with $y(0) = y(1) = 0$. Construct $G(x, \xi)$:

1&2 Homogeneous solutions $y_1 = e^x$ and $y_2 = e^{-x}$ so with homogeneous boundary conditions (by inspection):

$$G = \begin{cases} C \sinh x & 0 \leq x < \xi \\ D \sinh(1-x) & \xi < x \leq 1 \end{cases}$$

3 Continuity at ξ implies $C \sinh \xi = D \sinh(1 - \xi)$

$$C = \frac{D \sinh(1-\xi)}{\sinh \xi}$$

4 $[G'] = 1$ implies

$$-D \cosh(1-\xi) - C \cosh \xi = 1$$

$(\alpha = 1)$ so

$$-D[\cosh(1-\xi)\sinh \xi + \sinh(1-\xi)\cosh \xi] = \sinh \xi \tag{$*$}$$

$$-D[\sinh 1] = \sinh \xi$$

$$D = -\frac{\sinh \xi}{\sinh 1}, \quad C = -\frac{\sinh(1-\xi)}{\sinh 1}$$

so the solution is

$$y = -\frac{\sinh(1-x)}{\sinh 1} \int_0^x \sinh \xi f(\xi) \mathrm{d}\xi - \frac{\sinh x}{\sinh 1} \int_x^1 \sinh(1-\xi) f(\xi) \mathrm{d}\xi \tag{7.22}$$

Inhomogeneous Boundary conditions

Find y_p solution to $\mathcal{L}y = 0$ satisfying boundary conditions ($y(a) \neq 0$, $y(b) \neq 0$). Find Green's function for $\mathcal{L}y_g = f$ with $y_a(a) = y_g(b) = 0$ where $y_g = y - y_p$. For example

$$y'' - y = f(x)$$

with $y(0) = 0$ and $y(1) = 1$.

$$y_p = A \sinh x + B \cosh x$$

$y_p(0) = 0$ implies $B = 0$, $y_p(1) = 1$ implies $A = \frac{1}{\sinh 1}$. Solve for $y_g = y - y_p$ with homogeneous boundary conditions. Solution is

$$y(x) = \frac{\sinh x}{\sinh 1} + y_g(x)$$

(i.e. equation (7.22))

Higher-order ODEs (BVP)

If $\mathcal{L}y = f(x)$ to n-th order (coefficient $\alpha(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$) with homogeneous boundary conditions then we generalize Green's function $\mathcal{L}(x,\xi) = \delta(x-\xi)$ with properties:

1&2 G_1, G_2 homogeneous solutions satisfying homogeneous boundary conditions.

3 Continuity: $G_1 = G_2$, $G_1' = G_2'$, ..., $G_1^{(n-2)} = G_2^{(n-2)}$ at $x = \xi$.

4 Jump in $(n-1)$ derivative:

$$[G^{(n-1)}]_{\xi_-}^{\xi+} = G_2^{(n-1)}|_{\xi+} - G_1^{(n-1}|_{\xi_-} = \frac{1}{\alpha(\xi)}$$

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Eigenfunction expansion of $G(x,\xi)$

Suppose $\mathcal{L}$ is in Sturm Liouville form (2.7) with eigenfunctions $y_n(x)$ and eigenvalues λ_n, then seek

$$G(x,\xi) = \sum_{n=1}^{\infty} A_n y_n(x)$$

satisfying $\mathcal{L}G = \delta(x-\xi)$.

$$\begin{aligned}
\mathcal{L}G &= \sum_n A_n \mathcal{L} y_n(x) \\
&= \sum_n A_n \lambda_n \omega(x) y_n(x) \\
&= \delta(x-\xi) \\
&= \omega(x) \sum_n \frac{y_n(\xi) y_n(x)}{\mathcal{N}_n}
\end{aligned}$$

with $\mathcal{N}_n = \int \omega y_n^2 \mathrm{d}x$. So $A_n(\xi) = \frac{y_n(\xi)}{\lambda_n \mathcal{N}_n}$ by orthogonality (2.13). Thus

$$\boxed{G(x,\xi) = \sum_{n=1}^{\infty} \frac{y_n(\xi) y_n(x)}{\lambda_n \mathcal{N}_n} = \sum_{n=1}^{\infty} \frac{Y_n(\xi) Y_n(x)}{\lambda_n}} \tag{7.23}$$

which obtained without $\delta(x-\xi)$ in (2.31): refer to section 2.6 Sturm Liouville theory.

7.4. Constructing $G(t,\tau)$: Initial Value Problem

Solve $\mathcal{L}y = f(t)$ for $t \geq a$ with $y(a) = y'(a) = 0$ using $G(t,c)$ satisfying $\mathcal{L}G = \delta(t-\tau)$ with same boundary conditions.

- For $t < \tau$, $G_1 = Ay_1(t) + By_2(t)$ with $W(a) \neq a$, y_1, y_2 independent.

  $$Ay_1(a) + By_2(a) = 0$$

  and

  $$Ay_1'(a) + By_2'(a) = 0$$

  implies

  $$y_1 y_2' - y_2 y_1' = 0$$

  unless $A = B = 0$. So $G_1(t, \tau) \equiv 0$, $a \leq t < \tau$, i.e. no change until impulse at $t = \tau$.

- For $t > \tau$, by G continuity (7.12), $G_2(\tau, \tau) = 0$ so choose $G_2 = Dy_+(t)$, $y_+(t) = Ay_1(t) + By_2(t)$ such that $y_+(\tau) = 0$.

But by discontinuity in G' (7.13):

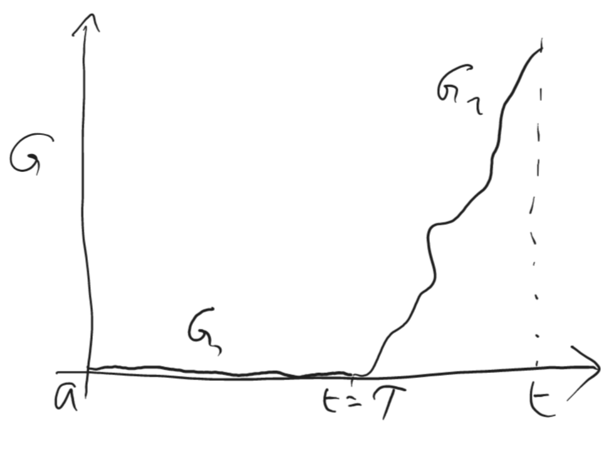
$$[G']_{\tau-}^{\tau+} = G_2'(\tau, \tau) - G_1'(\tau, \tau) = Dy_+'(\tau) = \frac{1}{\alpha(\tau)}$$

i.e. $Ay_1'(\tau) + By_2'(\tau) = \frac{1}{\alpha(\tau)}$ so

$$D(\tau) = \frac{1}{\alpha(\tau) y_+'(\tau)}$$

or solve for A, B. Hence, we have

$$\boxed{G(t, \tau) = \begin{cases} 0 & t < c \\ \frac{y_+(t)}{\alpha(\tau) y_+'(\tau)} & t > \tau \end{cases}} \tag{7.25}$$



The initial value problem is

$$
\begin{aligned}
y(t) &= \int_a^t G(t,\tau) f(\tau) \mathrm{d}\tau \\
&= \int_a^t \frac{y_+(t) f(\tau)}{\alpha(\tau) y'_+(\tau)} \mathrm{d}\tau
\end{aligned}
$$

Causality is "built in", as only forces acting prior to t affect the solution at t.

Example. Solve

$$y'' - y = f(t)$$

with $y(0) = y'(0) = 0$.

1&2 Homogeneous solutions and initial conditions

- $t < \tau$, $G_1 = 0$,
- $t > \tau$, $G_2 = Ae^t + Be^{-t}$

3 Continuity implies $G_2(\tau, \tau) = 0 \implies G_2 = D \sinh(t - \tau)$

1. $[G'] = \frac{1}{\alpha} = 1 \implies G_2'(\tau, \tau) = D \cosh(0) = D = 1.$

Hence, solution (7.26) is

$$y(t) = \int_0^t f(\tau) \sinh(t - \tau) \mathrm{d}t$$

8. Fourier Transforms

8.1. Introduction

Definition. The *Fourier transform* (FT) of a function $f(x)$ is

$$\begin{aligned}\tilde{f}(k) &= \mathcal{F}(f)(k) \\ &= \int_{-\infty}^{\infty} f(x)e^{-ikx}\mathrm{d}x \end{aligned} \tag{8.1}$$

and the *inverse Fourier transform* is

$$\begin{aligned}f(x) &= \mathcal{F}^{-1}(\tilde{f})(x) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}\mathrm{d}k \end{aligned} \tag{8.2}$$

Beware there are several conventions.

The *Fourier inversion theorem* states that

$$\mathcal{F}^{-1}(\mathcal{F}(f))(x) = f(x) \tag{8.3}$$

with a sufficient condition that f and $\tilde{f}$ are absolutely integrable. That is,

$$\int_{-\infty}^{\infty} |f(x)|\mathrm{d}x = M < \infty$$

so $f \to 0$ as $x \to \pm\infty$.

Gaussian example

Find the Fourier transform of

$$f(x) = \frac{1}{\sigma\sqrt{\pi}}e^{-x^2/\sigma^2} \tag{8.4}$$

$$\begin{aligned}\tilde{f}(k) &= \frac{1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2/\sigma^2}e^{-ikx}\mathrm{d}x \\ &= \frac{1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2/\sigma^2}\cos kx\mathrm{d}x\end{aligned}$$

Consider $\frac{\mathrm{d}\tilde{f}}{\mathrm{d}k}$:

$$\begin{aligned}\tilde{f}'(k) &= -\frac{1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} xe^{-x^2/\sigma^2}\sin kx\mathrm{d}x \\ &= \frac{1}{\sigma\sqrt{\pi}}\left[\frac{\sigma^2}{2}e^{-x^2/\sigma^2}\sin kx\right]_{-\infty}^{\infty} - \frac{1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty}\left(\frac{k\sigma^2}{2}\right)e^{-x^2/\sigma^2}\cos kx\mathrm{d}x \\ &= -\frac{k\sigma^2}{2}\tilde{f}(k)\end{aligned}$$

Integrate $\frac{\tilde{f}'}{\tilde{f}} = -\frac{k\sigma^2}{2}$ to find

$$\tilde{f}(k) = Ce^{-k^2\sigma^2/4}$$

But put $k = 0$ into (8.4), $\tilde{f}(0) = 1 \implies C = 1$

$$\boxed{\tilde{f}(k) = e^{-k^2\sigma^2/4}} \tag{8.5}$$

Exercise: Show that $\mathcal{F}^{-1}(e^{-k^2\sigma^2/4}) = f(x)$.

Exponential exercise:

Show that $f(x) = e^{-a|x|}$, $a > 0$ has Fourier Transform

$$\tilde{f}(k) = \frac{2a}{a^2 + k^2} \tag{8.6}$$

in two ways:

(i) Integrate $2\int_0^\infty e^{-ax}\cos kx\mathrm{d}x$ by parts twice.

(ii) Integrate $\int_0^\infty e^{-(a-ik)x}\mathrm{d}x + \int_{-\infty}^0 e^{(a+ik)x}\mathrm{d}x$ directly.

Note if

$$f(x) = \begin{cases} e^{-ax} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$(a > 0)$ then

$$\boxed{\tilde{f}(k) = \frac{1}{ik + a}} \tag{8.6a}$$

8.2. Fourier Transform relation to Fourier series

We can write Fourier series (1.13) as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} \tag{$*$}$$

where $k_n = \frac{n\pi}{L}$, so write $k_n = n\Delta k$ with $\Delta k = \frac{\pi}{L}$, then

$$\begin{aligned} c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-ik_n x}\mathrm{d}x \\ &= \frac{\Delta k}{2\pi}\int_{-L}^{L} f(x)e^{-ik_n x}\mathrm{d}x \end{aligned}$$

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Then the Fourier series $(*)$ becomes

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} e^{ik_n x} \int_{-L}^{L} f(x') e^{-ik_n x'} \mathrm{d}x'$$

But $\sum_{n=-\infty}^{\infty} \Delta k g(k_n) \to \int_{-\infty}^{\infty} g(k) \mathrm{d}k$ with $g(k_n) = \frac{e^{ik_n x}}{2\pi} \int_{-L}^{L} f(x') e^{-ikx'} \mathrm{d}x'$. So take limit as $L \to \infty$ and we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k e^{ikx} \left[\int_{-\infty}^{\infty} f(x') e^{-ikx'} \mathrm{d}x' \right] = \mathcal{F}^{-1}(\mathcal{F}(f))(x)$$

(i.e. equation (8.3))

resolution $\mathcal{F}$ (decomposition) $\to$ synthesis (reconstruction)

Note when $f(x)$ is discontinuous at x (like Fourier series) the Fourier Transform gives

$$\mathcal{F}^{-1}(\mathcal{F}(f))(x) = \frac{1}{2}(f(x_-) + f(x_+)) \tag{8.7}$$

8.3. Fourier Transform Properties

$$\tilde{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-ikx} \mathrm{d}x$$

(1) Linearity:

$$h(x) = \lambda f(x) + \mu g(x) \iff \tilde{h}(k) = \lambda \tilde{f}(k) + \mu \tilde{g}(k) \tag{8.8}$$

(2) Translation:

$$h(x) = f(x - \lambda) \iff \tilde{h}(k) = e^{-i\lambda k} \tilde{f}(k) \tag{8.9}$$

$$\tilde{h}(k) = \int f(x - \lambda) e^{-ikx} \mathrm{d}x = \int f(y) e^{-ik(y+\lambda)} \mathrm{d}y = e^{-i\lambda k} \tilde{f}(k)$$

(3) Frequency:

$$h(x) = e^{i\lambda x} f(x) \iff \tilde{h}(k) = \tilde{f}(k - \lambda) \tag{8.10}$$

(4) Scaling:

$$h(x) = f(\lambda x) \iff \tilde{h}(k) = \frac{1}{|\lambda|} \tilde{f}\left(\frac{k}{\lambda}\right) \tag{8.11}$$

($|\lambda|$ because $x \to -x$ changes limits)

(5) Multiplication by x:

$$h(x) = x f(x) \iff \tilde{h}(k) = i \tilde{f}'(k) \tag{8.12}$$

because

$$\int_{-\infty}^{\infty} x f(x) e^{-ikx} \mathrm{d}x = -\frac{1}{i} \frac{\mathrm{d}}{\mathrm{d}k} \left(\int_{-\infty}^{\infty} f(x) e^{-ikx} \mathrm{d}x \right)$$

(6) Derivative:

$$\boxed{h(x) = f'(x) \iff \tilde{h}(k) = ik\tilde{f}(k)} \tag{8.13}$$

because

$$\begin{aligned}\tilde{h}(k) &= \int_{-\infty}^{\infty} f'(x)e^{-ikx}\mathrm{d}x \\ &= [f(x)e^{-ikx}]_{-\infty}^{\infty} + ik\int_{-\infty}^{\infty} f(x)e^{-ikx}\mathrm{d}x \\ &= ik\tilde{f}(k)\end{aligned}$$

(7) General duality: Consider (8.2) with $x \to -x$

$$f(-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{-ikx}\mathrm{d}k$$

so $k \leftrightarrow x$

$$\implies f(-k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(x)e^{ikx}\mathrm{d}x$$

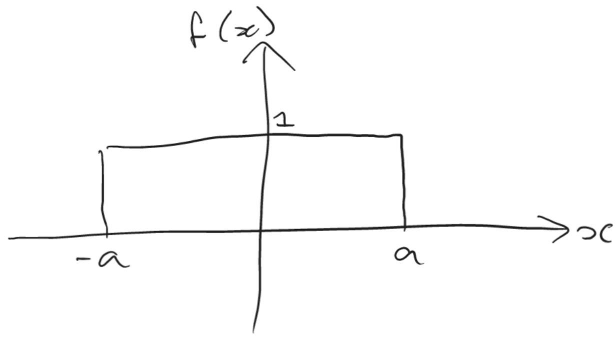
Thus

$$\boxed{g(x) = \tilde{f}(x) \iff \tilde{g}(x) = 2\pi f(-k)} \tag{8.14}$$

We have $f(-x) = \frac{1}{2\pi}\mathcal{F}(\tilde{f})(x) = \frac{1}{2\pi}\mathcal{F}^2(f)(x)$, so repeating, $\mathcal{F}^4(f)(x) = 4\pi^2 f(x)$.

Exercise: Verify 1-7.

"Top hat" example:



Find Fourier Transform for

$$f(x) = \begin{cases} 1 & |x| \leq a \\ 0 & |x| > a \end{cases}$$

$(a > 0)$

$$\begin{aligned}
\tilde{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{-ikx} \mathrm{d}x \\
&= \int_{-a}^{a} \cos kx \mathrm{d}x \\
&= 2\frac{\sin ka}{a}
\end{aligned} \tag{8.15}$$

Fourier inversion, then (8.3) implies

$$\frac{1}{\pi} \int_{-\infty}^{\infty} e^{ikx} \frac{\sin ka}{k} \mathrm{d}k = \begin{cases} 1 & |x| \leq a \\ 0 & |x| > a \end{cases}$$

Now set $x = 0$, then take $k \to x$ to obtain Dirchlet discontinuous formula:

$$\int_0^{\infty} \frac{\sin ax}{x} \mathrm{d}x = \begin{cases} \frac{\pi}{2} & a > 0 \\ 0 & a = 0 \\ -\frac{\pi}{2} & a < 0 \end{cases} = \frac{\pi}{2} \operatorname{sgn}(a) \tag{8.16}$$

Here, we allow $a < 0$, so $\sin(-ax) = -\sin ax$. (See RJ notes for direct inverse Fourier transform of (8.15))

8.4. Convolution and Parseval's Theorem

We want to multiply Fourier Transforms in frequency domain $\tilde{h}(k) = \tilde{f}(k)\tilde{g}(k)$ so consider the inverse:

$$\begin{aligned}
h(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) \tilde{g}(k) e^{ikx} \mathrm{d}k \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) e^{-iky} \mathrm{d}y \right) \tilde{g}(k) e^{ikx} \mathrm{d}k \\
&= \int_{-\infty}^{\infty} f(y) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(k) e^{ik(x-y)} \mathrm{d}k \right) \mathrm{d}y \qquad \text{see (8.9)} \\
&= \int_{-\infty}^{\infty} f(y) g(x-y) \mathrm{d}y \\
&\equiv f * g(x)
\end{aligned} \tag{8.17}$$

(convlution definition). By duality (8.14) we also have

$$h(x) = f(x)g(x) \iff \tilde{h}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(p) \tilde{g}(k-p) \mathrm{d}p \tag{8.18}$$

Parseval's Theorem

Consider $h(x) = g^*(-x)$, then

$$\begin{aligned}\tilde{h}(k) &= \int_{-\infty}^{\infty} g^*(-x)e^{-ikx}\mathrm{d}x \\ &= \left[\int_{-\infty}^{\infty} g(-x)e^{ikx}\mathrm{d}x\right]^* \\ &= \left[\int_{-\infty}^{\infty} g(y)e^{-iky}\mathrm{d}y\right]^* \\ &= \tilde{g}^*(k)\end{aligned}$$

Substitute into (8.17) $g(x) \to g^*(-x)$,

$$\int_{-\infty}^{\infty} f(y)g^*(y-x)\mathrm{d}y = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\tilde{g}^*(k)e^{ikx}\mathrm{d}k$$

Take $x = 0$, then dummy variable $y \to x$ on LHS:

$$\int_{-\infty}^{\infty} f(x)g^*(x)\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\tilde{g}^*(k)\mathrm{d}k \tag{8.19}$$

Or equivalently,

$$\langle g, f\rangle = \frac{1}{2\pi}\langle \tilde{g}, \tilde{f}\rangle_k$$

see section (2.1). Now $g = f^*$:

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{f}(k)|^2\mathrm{d}k} \tag{8.20}$$

which is *Parseval's theorem.*

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8.4a Fourier Transform of Generalised Functions

(See discussion in section 8.3 of R Jorsa notes (or D Skinner))

Dirac delta function $\delta(x)$

Consider the inversion then (8.3):

$$\begin{aligned}f(x) = \mathcal{F}^{-1}(\mathcal{F}(f))(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)e^{-iku}\mathrm{d}u\right]e^{ikx}\mathrm{d}k \\ &= \int_{-\infty}^{\infty} f(u)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-u)}\mathrm{d}k\right]\mathrm{d}u\end{aligned}$$

so identify

$$\delta(x-u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik}(x-u)\mathrm{d}k = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iku}e^{ikx}\mathrm{d}k$$

- If $f(x) = \delta(x-a)$, then $\tilde{f}(k) = e^{-ika}$ (8.21).
- If $f(x) = \delta(x)$ then
$$\tilde{f}(k) = \int_{-\infty}^{\infty} \delta(x) e^{ikx} \mathrm{d}x = 1 \tag{8.22}$$
- If $f(x) = 1$, then
$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} \mathrm{d}x = 2\pi\delta(k) \tag{8.23}$$
by duality (8.14).

Trig Functions

$$\begin{aligned} f(x) = \cos \omega x &\iff \tilde{f}(k) = \pi(\delta(k+w) + \delta(k-w)) \\ f(x) = \sin \omega x &\iff \tilde{f}(k) = i\pi(\delta(k+w) - \delta(k-w)) \end{aligned} \tag{8.24}$$

Exercise: Find $\mathcal{F}^{-1}$ for $\sin \omega k$, $\cos \omega k$ using (8.14).

Heaviside function

Subtle derivation requiring central value $H(0) = \frac{1}{2}$; then $H(x) + H(-x) = 1$ for all x and continuous at $x = 0$. By (8.8) and (8.23)

$$\tilde{H}(k) + \tilde{H}(-k) = 2\pi\delta(k) \tag{$*$}$$

Recall (6.7) $H'(x) = \delta(x)$ which implies

$$ik\tilde{H}(k) = 1 \tag{$\dagger$}$$

by (8.13) and (8.22). But $k\delta(k) = 0$, so ($*$) and ($\dagger$) are consistent if

$$\tilde{H}(k) = \pi\delta(k) + \frac{1}{ik} \tag{8.25}$$

Dirichlet discontinuous formula (8.16): Rewrite as

$$\frac{1}{2}\operatorname{sgn}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{ik} \mathrm{d}x$$

so

$$f(x) = \frac{1}{2}\operatorname{sgn}(x) \iff \tilde{f}(k) = \frac{1}{ik} \tag{8.26}$$

8.5. Applications of Fourier Transforms

Motivation I: ODE for BVP

Consider $y'' - y = f(x)$ with homogeneous boundary conditions $y \to 0$ as $x \to \pm\infty$. Take the Fourier Transform:

$$(ik)^2\tilde{y} - \tilde{y} = (-k^2 - 1)\tilde{y} = \tilde{f}$$

by (8.13). SO the solution is

$$\tilde{y}(k) = -\frac{\tilde{f}(k)}{1+k^2} \equiv \tilde{f}(k)\tilde{g}(k)$$

where $\tilde{g}(k) = -\frac{1}{1+k^2}$ but this is the Fourier Transform of $g(x) = -\frac{1}{2}e^{-|x|}$ (see (8.6)). Thus convolution theorem (8.17) implies

$$\begin{aligned} y(x) &= \int_{-\infty}^{\infty} f(x)g(x-u)\mathrm{d}u \\ &= -\frac{1}{2}\int_{-\infty}^{\infty} f(u)e^{-|x-u|}\mathrm{d}u \\ &= -\frac{1}{2}\int_{-\infty}^{x} f(u)e^{u-x}\mathrm{d}u - \frac{1}{2}\int_{x}^{\infty} f(u)e^{x-u}\mathrm{d}u \end{aligned}$$

which is in the form of a BVP Green's function.
Exercise: Verify by constructing Green's function.

Motivation II: Signal processing (IVP)

Suppose (given) input $\mathcal{J}(t)$ acting on by linear operator $\mathcal{L}_{\text{in}}$ to yield output $\theta(t)$.

$$\theta(t) = \mathcal{L}_{\text{in}}\mathcal{J}(t)$$

The Fourier Transform $\tilde{\mathcal{J}}(\omega)$ is denoted *the resolution*

$$\tilde{\mathcal{J}}(\omega) = \int_{-\infty}^{\infty} \mathcal{J}(t)e^{-i\omega t}\mathrm{d}t \tag{8.27}$$

In frequency domain $\mathcal{L}_{\text{in}}\mathcal{J}(t)$ means $\tilde{\mathcal{J}}(\omega)$ is multiplied by a *transfer function* $\tilde{R}(\omega)$ to yield output

$$\theta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{R}(\omega)\tilde{\mathcal{J}}(\omega)e^{i\omega t}\mathrm{d}\omega \tag{8.28}$$

with *response function* given by

$$R(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{R}(\omega)e^{i\omega t}\mathrm{d}\omega \tag{8.29}$$

By the convolution theorem (8.17), output is

$$\theta(t) = \int_{-\infty}^{\infty} \mathcal{J}(u) R(t-u) \mathrm{d}u$$

We assume no input $\mathcal{J}(t) = 0$ for $t < 0$ and by *causality*, zero output for $R(t) = 0$ for $t < 0$ (i.e. $R(t-u)$ has source $\delta(t-u)$), so we require $0 < u < t$:

$$\theta(t) = \int_{0}^{r} \mathcal{J}(u) R(t-u) \mathrm{d}u \tag{8.30}$$

i.e. the same form as IVP Green's function.

General transfer functions for ODEs

Suppose input / output relation given by linear ODE (n-th order)

$$\mathcal{L}\theta(t) \equiv \left(\sum_{i=0}^{n} a_i \frac{\mathrm{d}^i}{\mathrm{d}t^i} \right) \theta(t) = \mathcal{J}(t) \tag{8.31}$$

where a_i are constant and here set $\mathcal{L}_{\text{in}} = 1$. Take the Fourier Transform:

$$(a_0 + a_1(i\omega) + a_2(i\omega)^2 + \cdots + a_n(i\omega)^n)\theta(\omega) = \tilde{\mathcal{J}}(\omega)$$

so the transfer function (8.28) is

$$\tilde{R}(\omega) = \frac{1}{a_0 + a_1(i\omega) + \cdots + a_n(i\omega)^n} \tag{8.32}$$

Factorise n-th degree polynomial into product of n roots $(i\omega - c_j)^{k_j}$ with $j = 1, 2, \ldots, J$ (with repeated roots $k_j > 1$) i.e. $\sum_{j=1}^{J} k_j = n$. Then

$$\begin{aligned} \tilde{R}(\omega) &= \frac{1}{(i\omega - c_1)^{k_1} + \cdots + (i\omega - c_J)^{k_J}} \\ &= \sum_{j=1}^{J} \sum_{m=1}^{k_j} \frac{\Gamma_{jm}}{(i\omega - c_j)^m} \end{aligned} \tag{8.33}$$

since it can be expanded in partial fractions (constant Γ_{jm}). For repeated roots ($1 \leq m \leq k_j$):

$$\frac{1}{(i\omega - c_j)^{k_j}} \rightarrow \frac{\Gamma_{j1}}{(i\omega - c_j)} + \frac{\Gamma_{j2}}{(i\omega - c_j)^2} + \cdots + \frac{\Gamma_{jk}}{(i\omega - c_j)^{k_j}}$$

To solve we must invert $\frac{1}{(i\omega - a)^m}$, $m \geq 1$. We know (8.6a)

$$\mathcal{F}^{-1}\left(\frac{1}{i\omega - a}\right) = \begin{cases} e^{at} & t > 0 \\ 0 & t < 0 \end{cases}$$

for $\mathrm{Re}(a) < 0$, so we assume $\mathrm{Re}(c_j) < 0$, $\forall j$ (eliminate exponential growing modes).

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For $m = 2$ note $i\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{1}{i\omega - a}\right) = \frac{1}{(i\omega-a)^2}$ and recall (8.12) $\mathcal{F}(tf(t)) = i\tilde{f}'(\omega)$, so

$$\mathcal{F}^{-1}\left(\frac{1}{(i\omega - a)^2}\right) = \begin{cases} te^{at} & t > 0 \\ 0 & t < 0 \end{cases}$$

By induction,

$$\mathcal{F}^{-1}\left(\frac{1}{(i\omega - a)^m}\right) = \begin{cases} \frac{t^{m-1}}{(m-1)!}e^{at} & t > 0 \\ 0 & t < 0 \end{cases} \tag{8.34}$$

Thus the response function takes the form

$$R(t) = \sum_j \sum_m \Gamma_{jm} \frac{t^{m-1}}{(m-1)!} e^{c_j t} \quad t > 0 \tag{8.35}$$

We can solve (8.31) in Green's function form (8.30) or directly invert $\tilde{R}(\omega)\tilde{J}(\omega)$ for polynomial $\tilde{J}(\omega)$.

Example (Damped oscillator)**.** Solve

$$\mathcal{L}y \equiv y'' + 2py' + (p^2 + q^2)y = f(t)$$

with damping $p > 0$ and homogeneous initial conditions $y(0) = y'(0) = 0$. Fourier Transform is

$$(i\omega)^2\tilde{y} + 2ip\omega\tilde{y} + (p^2 + q^2)\tilde{y} = \tilde{f}$$

$$\tilde{y} = \frac{\tilde{f}}{-\omega^2 + 2ip\omega + p^2 + q^2} \equiv \tilde{R}\tilde{f}$$

Inverting with convolution theorem (8.17)

$$y(t) = \int_0^t r(t-\tau)f(\tau)\mathrm{d}\tau$$

with response

$$R(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega(t-\tau)}\mathrm{d}\omega}{p^2 + q^2 + 2ip\omega - \omega^2}$$

Exercise: Show $\mathcal{L}R(t-\tau) = \delta(t-\tau)$ using (8.23). That is, the response function for $R(t-\tau)$ is the Green's function (see Example sheet 3, Q4).

8.6. Discrete Fourier Transforms

Discrete sampling & the Nyquist frequency

Sample a signal $h(t)$ at equal times $t_n = n\Delta$ with time-sampling Δ, and values

$$h_n = h(n\Delta), \quad n = \ldots, -2, 1, 0, 1, 2, \ldots \tag{8.36}$$

i.e. with sampling frequency $\frac{1}{\Delta}$ ($\omega_s = 2\pi f_s = \frac{2\pi}{\Delta}$).
The *Nyquist frequency* $f_c = \frac{1}{2\Delta}$ (8.37) is the highest frequency actually sampled at Δ.
Suppose we have a signal with given frequency f.

$$\begin{aligned} g_f(t) &= A\cos(2\pi ft + \phi) \\ &= \mathrm{Re}(Ae^{2\pi ift+\phi}) \\ &= \frac{1}{2}(Ae^{i\phi}e^{2\pi ift} + Ae^{-i\phi}e^{-2\pi ift}) \end{aligned} \tag{8.38}$$

(i.e. for real complex Fourier Series, the sum of positive frequencies f and negative frequency $-f$ modes).
What happens if we sample at Nyquist $f = f_c$?

$$\begin{aligned} g_{f_c}(t_n) &= A\cos(2\pi\left(\frac{1}{2\Delta}n\Delta + \phi\right) \\ &= A\cos\pi n\cos\phi + A\sin n\pi\sin\phi \\ &= A'\cos(2\pi f_c t_n) \end{aligned} \tag{8.39}$$

with $A' = A\cos\phi$. So phase / amplitude information is lost (no distinction) and we can identify $f_c \leftrightarrow -f_c$ i.e. (8.38) and (8.39) are *aliased* together.
What happens if we sample above $f > f_c$? Exercise: Take $f = f_c + \delta f > f_c$ and show that ($\delta f < f_c$)

$$\begin{aligned} g_f(t_n) &= A\cos(2\pi(f_c + \delta f)t_n + \phi) \\ &= A\cos(2\pi(f_c - \delta f)t_n - \phi) \end{aligned} \tag{8.40}$$

So the effect is to *alias* a "ghost signal" to frequency $f_c - \delta f$ (actually - $-(f_c - \delta f)$).

Sampling Theorem

A signal $g(t)$ is *bandwidth limited* if it contains no frequencies above $\omega_{\max} = 2\pi f_{\max}$, i.e. $\tilde{g}(\omega) = 0$ for $|\omega| > \omega_{\max}$. So

$$\begin{aligned} g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{g}(\omega)e^{i\omega t}\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\omega_{\max}}^{\omega_{\max}}\tilde{g}(\omega)e^{i\omega t}\mathrm{d}\omega \end{aligned} \tag{8.41}$$

Set sampling to satisfy Nyquist condition

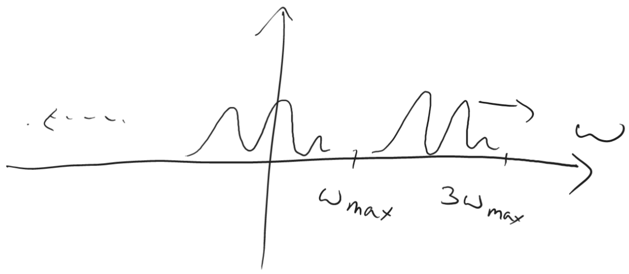
$$\Delta = \frac{1}{2f_{\max}}$$

then

$$g_n \equiv g(t_n) = \frac{1}{2\pi}\int_{-\omega_{\max}}^{\omega_{\max}}\tilde{g}(\omega)e^{i\pi n\omega/\omega_{\max}}\mathrm{d}\omega$$

which is complex Fourier series coefficient (1.13) $c_n \times \frac{\omega_{\max}}{\pi}$ $(x \to \omega)$. The Fourier Series represents a periodic function (period $2\omega_{\max}$)

$$\tilde{g}_{\text{per}}(\omega) = \frac{\pi}{\omega_{\max}} \sum_{n=-\infty}^{\infty} g_n e^{-i\pi n\omega/\omega_{\max}} \tag{8.42}$$



The actual Fourier Transform $\tilde{g}(\omega)$ is found by multiplying by a "top hat"

$$\tilde{h}(\omega) = \begin{cases} 1 & |\omega| \leq \omega_{\max} \\ 0 & \text{otherwise} \end{cases}$$

i.e.

$$\tilde{g}(\omega) = \tilde{g}_{\text{per}}(\omega)\tilde{h}(\omega) \tag{8.43}$$

which is an *exact* relation. Inverting with (8.42):

$$\begin{aligned} g(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}_{\text{per}}(\omega)\tilde{h}(\omega)e^{i\omega t}\mathrm{d}\omega \\ &= \frac{1}{2\omega_{\max}} \sum_{n=-\infty}^{\infty} g_n \int_{-\omega_{\max}}^{\omega_{\max}} \exp\left(i\omega\left(t - \frac{n\pi}{\omega_{\max}}\right)\right)\mathrm{d}\omega \\ &= \sum_{n=-\infty}^{\infty} g_n \frac{\sin(\omega_{\max}t - \pi n)}{\omega_{\max}t - \pi n} \end{aligned} \tag{8.44}$$

So $g(t)$ can be exactly represented after sampling at discrete times t_n (sampling theorem).

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Discrete Fourier Transform

Suppose we have a finite number N of samples

$$h_m = h(t_m), \quad t_m = m\Delta, \quad m = 0, 1, \ldots, N-1 \tag{8.45}$$

We want to approximate the Fourier Transform for N frequencies using equally spaced frequencies $\Delta_f = \frac{1}{N\Delta}$ in the range $-f_c \leq f \leq f_c$. We *could* take $f_n = n\Delta_f = \frac{n}{n\Delta}$ with $n = -\frac{N}{2}, -\frac{N}{2}+1, \ldots, -1, 0, 1, \ldots, \frac{N}{2}$. *But* this has $N+1$ frequencies, with f_c

and $-f_c$ aliased (8.39). Instead, note that $\left(\frac{N}{2}+m\right)\Delta_f = f_c + \delta f$ is aliased back to $\left(\frac{N}{2}-m\right)\Delta_f = -(f_c - \delta f)$ from (8.40) so we choose

$$f_n = \frac{n}{N\Delta} \text{ with } n = 0, 1, 2, \ldots, \frac{N}{2}-1, \frac{N}{2}, \frac{N}{2}+1, \ldots, N-1$$

The *discrete Fourier Transform* at frequency f_n becomes (8.46)

$$\begin{aligned}\tilde{h}(f_n) &= \int_{-\infty}^{\infty} e^{-2\pi i f_n t}\mathrm{d}t \\ &\simeq \Delta \sum_{m=0}^{N-1} h_m e^{-2\pi i f_n t_m} \\ &= \Delta \sum_{m=0}^{N-1} h_m e^{-2\pi i mn/N} \\ &= \Delta \tilde{h}_d(f_n) \end{aligned} \tag{8.47}$$

Recalling section 8.2 Fourier series → Fourier transform Riemann integral. Here $\tilde{h}_d(f_n) \equiv \tilde{h}_n$ is the discrete Fourier Transform. So the matrix $[DFT]_{mn} = e^{-2\pi imn/n}$ defines the discrete Fourier Transform for $\mathbf{h} = \{h_m\}$ (data vector) as $\tilde{\mathbf{h}}_d = [DFT]\mathbf{h}$.
The *inverse* is its adjoint $[DFT]^{-1} = \frac{1}{N}[DFT]^\dagger$ and it's built from roots of unity $\omega = e^{-2\pi i/N}$. For example $N = 4$, $\omega = -i$,

$$DFT = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

The *inverse DFT* is

$$\begin{aligned} h_m &= h(t_m) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{h}(\omega) e^{i\omega t_m}\mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} \tilde{h}(f) e^{2\pi i f t_m}\mathrm{d}f \\ &\simeq \frac{1}{N\Delta}\sum_{m=0}^{N-1} \Delta \tilde{h}_d(f_n) e^{2\pi imn/N} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} \tilde{h}_n e^{2\pi imn/N} \end{aligned} \tag{8.48}$$

or interpolating Fourier series is $h(t) = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{h}_n e^{2\pi int/N}$.

Exercise: Establish Parseval's theorem

$$\sum_{m=1}^{N-1} |h_m|^2 = \frac{1}{N}\sum_{m=0}^{N-1} |\tilde{h}_m|^2 \tag{8.49}$$

The convolution theorem for g_m, h_m is

$$c_k = \sum_{m=0}^{N-1} g_m h_{k-m} \iff \tilde{c}_k = \tilde{g}_k \tilde{h}_k \tag{8.50}$$

Chapter IV

PDEs on Unbounded Domains

Contents

9. Characteristics

9.1. Well-posed Cauchy Problems

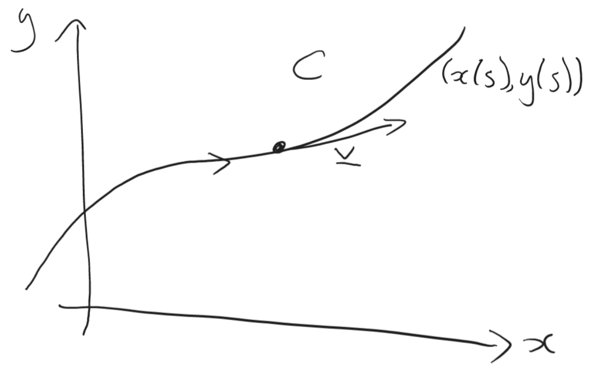
Solving PDEs depends on the nature of the equations in combination with the boundary and / or initial data. A *Cauchy problem* is the PDE for ϕ together with this auxillary data (i.e. ϕ and its derivatives) specified on a surface (or curve in 2D), which is called *Cauchy data*.
A Cauchy problem is *well-posed* if:

(i) a solution exists

(ii) the solution is unique item the solution depends continuously on auxillary data.

9.2. Method of Characteristics

Consider a parametrised curve C given by $(x(S), y(s))$ with tangent vector $\mathbf{v} = \left(\frac{\mathrm{d}x}{\mathrm{d}s}(s), \frac{\mathrm{d}y}{\mathrm{d}s}(s)\right)$.



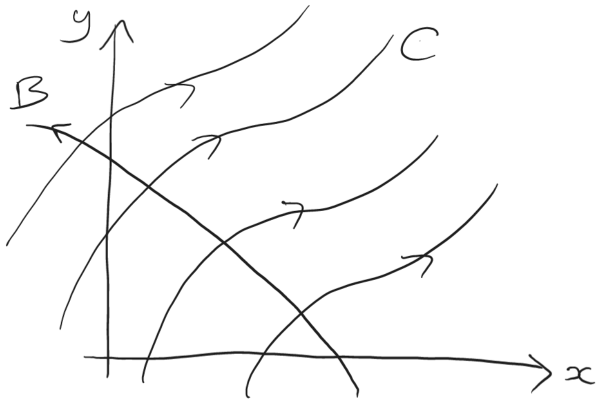
For a function $\phi(x, y)$ we can define a directional derivative along C

$$\left.\frac{\mathrm{d}\phi}{\mathrm{d}s}\right|_C = \frac{\mathrm{d}x(s)}{\mathrm{d}s}\frac{\partial\phi}{\partial x} + \frac{\mathrm{d}y(s)}{\mathrm{d}s}\frac{\partial\phi}{\partial y} = \mathbf{v}\cdot\nabla\phi|_C \tag{9.1}$$

If $\mathbf{v}\cdot\nabla\phi = 0$, then $\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0$ and $\phi =$ constant along C.
Now suppose we have a vector field

$$\mathbf{u} = (\alpha(x, y), \beta(x, y)) \tag{9.2}$$

with its family of integral curves C non-intersecting and filling $\mathbb{R}^2$ (i.e. at a point (x, y) the integral curve has tangent $\mathbf{u}$).



Define a curve B by $((x(t), y(t))$ transverse to $\mathbf{u}$, such that its tangent $\boldsymbol{\omega} = \left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}, \frac{\mathrm{d}y(t)}{\mathrm{d}t}\right)$ is nowhere parallel to $\mathbf{u}$. Label each integral curve C of $\mathbf{u}$ using t at the intersection point with $\mathbf{B}$, then use s to parametrise along the curve (i.e. take $s = 0$ at B).
Our integral curves $(x(s,t), y(s,t))$ satisfy:

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \alpha(x,y), \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = \beta(x,y) \tag{9.3}$$

Solve these to find a family of characteristic curves along which t remains constant (i.e. new coordinates (s,t)).

9.3. Characteristics of a 1st order PDE

Consider 1st order linear PDE

$$\alpha(x,y)\frac{\partial \phi}{\partial x} + \beta(x,y)\frac{\partial \phi}{\partial y} = 0 \tag{9.4}$$

with specified Cauchy data on an initial curve B $(x(t), y(t))$:

$$\phi(x(t), y(t)) = f(t) \tag{9.5}$$

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Note from (9.1) and (9.2) that

$$\alpha\phi_x + \beta\phi_y = \mathbf{u} \cdot \nabla\phi = \left.\frac{\mathrm{d}\phi}{\mathrm{d}s}\right|_C$$

is the directional derivative along integral curves C of $\mathbf{u} = (\alpha, \beta)$, called the *characteristic curves* of the PDE. Since $\frac{\mathrm{d}\phi}{\mathrm{d}s} = \alpha\phi_x + \beta\phi_y = 0$ from (9.4), the function $\phi(x,y)$ will be constant along the curves C, i.e. the Cauchy data $f(t)$ defined on B at $s = 0$ will be propagated constantly along the curve C to give solution

$$\phi(s,t) = \phi(x(s,t), y(s,t)) = f(t) \tag{9.6}$$

To obtain $\phi(x,y)$ transform coordinates from $\phi(t,s)$ using $s=s(x,y)$, $t=t(x,y)$ (provided Jacobian $J=x_t y_s - x_s y_t \neq 0$) to finally obtain

$$\phi(x,y) = f(t(x,y)) \tag{9.7}$$

Prescription: To solve (9.4) with (9.5)

(1) Find characteristic equation (9.3) $\frac{\mathrm{d}x}{\mathrm{d}s} = \alpha$, $\frac{\mathrm{d}y}{\mathrm{d}s} = \beta$.

(2) Parametrise initial conditions on B $(x(t), y(t))$ (9.8)

(3) Solve characteristic equation (9.3) to find $x(s,t)$ and $y(s,t)$ subject to (9.8) at $s=0$, $x(0,t) = x(t)$, $y(0,t) = y(t)$.

(4) Solve (9.4) with (9.1)

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = \alpha\phi_x + \beta\phi_y = 0$$

(9.6) $\phi(s,t) = f(t)$ [or $\gamma(s,t)$ on RHS].

(5) Invert relations $s=s(x,y)$, $t=t(x,y)$.

(6) Change coordinates to obtain (9.7), $\phi(x,y)$.

Example. Solve $e^x\phi_x + \phi_y = 0$ with $\phi(x,0) = \cosh x$.

(1) Characteristic equations

$$\frac{\mathrm{d}x}{\mathrm{d}s} = e^x, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = 1 \tag{$*$}$$

(2) Initial conditions $x(t) = t$, $y(t) = 0$ on the x axis (†).

(3) From $(*)$, $\frac{\mathrm{d}x}{e^x} = \mathrm{d}s$, $-e^{-x} = s + c$, $y = s + d$. At $s=0$, $x=0$, $-e^{-t} = c$, $y = 0 = d$.

$$-e^{-x} = e^{-t} - s, \qquad y = s$$

(characteristics).

(4) $\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0 \implies \phi(s,t) = \cosh t$.

(5) $s = y$, $e^{-t} = y + e^{-x} \implies t = -\log(y + e^{-x})$.

(6) So

$$\phi(x,y) = \cosh[-\log(y + e^{-x})]$$

Inhomogeneous 1st order PDE

Want to solve

$$\alpha(x,y)\phi_x + \beta(x,y)\phi_y = \gamma(x,y) \tag{9.9}$$

with Cauchy data $\phi(x(t), y(t)) = f(t)$ on curve B. The characteristic curves C are identical to homogeneous case (9.4) but now (9.1) implies

$$\left.\frac{\mathrm{d}\phi}{\mathrm{d}s}\right|_C = \mathbf{u} \cdot \nabla\phi = \gamma(x, y) \tag{9.10}$$

with $\phi = f(t)$ at $s = 0$ on B, i.e. no longer propagating constantly and we must solve an ODE (9.10). So upgrade point 4 in prescription to integrate $\phi(s, t)$ before reverting to $\phi(x, y)$.

Example. Solve $\phi_x + 2\phi_y = ye^x$ with $\phi = \sin x$ on $y = x$.

(1) Characteristic equation

$$\frac{\mathrm{d}x}{\mathrm{d}s} = 1, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = 2 \tag{$*$}$$

(2) So on $y = x$, take $(x(t), y(t)) = (t, t)$ (†).

(3) From $(*)$, $x = s + c$, $y = 2s + d$. So because of (†), $s = 0$, $x = t = c$, $y = t = d$.

$$x = s + t, \qquad y = 2s + t$$

(4) Solve $\frac{\mathrm{d}\phi}{\mathrm{d}s} = \gamma = ye^x = (2s + t)e^{s+t}$ with $\phi = \sin t$ at $s = 0$. Note $\frac{\mathrm{d}}{\mathrm{d}s}(2se^s) = 2e^s + 2se^s$ so

$$\phi(s, t) = (2s - 2 + t)e^{s+t} + \text{const}$$

But at using $s = 0$ condition we have $\phi(0, t) = \sin t = (t - 2)e^t + \text{const}$ so

$$\phi(s, t) = (2s - 2 + t)e^{s+t} + \sin t + (2 - t)e^t$$

(5) Invert $s = y - x$, $t = 2x - y$.

(6) So

$$\phi(x, y) = (y - 2)e^x + (y - 2x + 2)e^{2x-y} + \sin(2x - y)$$

9.4. Second-order PDE classification

In two dimensions, the general 2nd order linear PDE is

$$\mathcal{L} \equiv a(x,y)\frac{\partial^2\phi}{\partial x^2} + 2b(x,y)\frac{\partial^2\phi}{\partial x\partial y} + c(x,y)\frac{\partial^2\phi}{\partial y^2} + d(x,y)\frac{\partial\phi}{\partial x} + e(x,y)\frac{\partial\phi}{\partial y} + f(x,y)\phi(x,y) = 0$$

The *principal part* is given by

$$\begin{aligned}\sigma_p(x, y, k_x, k_y) &\equiv K^\top AK \\ &= (k_x\ k_y)\begin{pmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y)\end{pmatrix}\begin{pmatrix} k_x \\ k_y\end{pmatrix}\end{aligned}$$

The PDE is classified by the eigenvalues of A:

- $b^2 - ac < 0$ *elliptic* (λ_1, λ_2 same sign)
- $b^2 - ac > 0$ *hyperbolic* (λ_1, λ_2 opposite sign)
- $b^2 - ac$ *parabolic* (λ_1 or $\lambda_2 = 0$)

Exercise: show this from $\det(A - \lambda I)$, i.e. $\lambda_\pm = \frac{1}{2}(\mathrm{Tr} \pm \sqrt{\mathrm{Tr}^2 - 4\det})$.

Examples

- Wave equation (3.4)
$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2}$$
$a = \frac{1}{c^2}$, $b = 0$, $c = -1$ is hyperbolic.
- Heat equation (4.3) $a = 0$, $b = 0$, $c = -D$ is parabolic.
- Laplace equation (5.1) $a = 1$, $b = 0$, $c = 1$ is elliptic.

Characteristic curves

A curve defined by $f(x, y) = 0$ will be a characteristic curve if

$$(f_x \ f_y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} f_x \\ f_y \end{pmatrix} = 0 \tag{9.12}$$

(generalisation 1st order $\nabla f \cdot \mathbf{u} = 0$, $\mathbf{u} = (\alpha, \beta)$). The curve can be written as $y = y(x)$ where

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \implies \frac{f_x}{f_y} = -\frac{\mathrm{d}y}{\mathrm{d}x} \tag{9.13}$$

Substituting into (9.12) we obtain a quadratic with solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a} \tag{9.14}$$

(exercise).

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- Hyperbolic if $b^2 - ac > 0$, then *2 solutions*
- Parabolic if $b^2 - ac = 0$, then *1 solution*
- Elliptic if $b^2 - ac < 0$, *no real solutions*

Transforming to characteristic coordinates (u, v) would set $a = c = 0$ in (9.11) so the PDE takes *canonical form*

$$\frac{\partial^2 \phi}{\partial u \partial v} + \cdots = 0 \tag{9.15}$$

where the dots would be lower-order terms ϕ_u, ϕ_v, ϕ (Refer to section 9.4 in R Josza lecture notes).

Example. Consider

$$-y\phi_{xx} + \phi_{yy} = 0 \tag{$*$}$$

With $a = -y$, $b = 0$, $c = 1$, $b^2 - ac = y$. So hyperbolic for $y > 0$ (elliptic for $y < 0$, parabolic for $y = 0$). Find characteristics for $y > 0$ satisfying (9.14)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a} \pm \frac{1}{\sqrt{y}} \implies \sqrt{y}\mathrm{d}y = \pm\mathrm{d}x$$

$$\implies \frac{2}{3}y^{3/2} \pm x = c_{\pm}$$

so characteristic curves are

$$u = \frac{2}{3}y^{3/2} + x, \qquad v = \frac{2}{3}y^{3/2} - x$$

Derivatives are $u_x = 1$, $u_y = y^{1/2}$, $v_x = -1$, $v_y = y^{1/2}$. Hence

$$\phi_x = \phi_u u_x + \phi_v v_x = \phi_u - \phi_v$$

$$\phi_y = y^{1/2}(\phi_u + \phi_v)$$

$$\phi_{xx} = \phi_{uu} - 2\phi_{uv} + \phi_{vv}$$

$$\phi_{yy} = y(\phi_{uu} + 2\phi_{uv} + \phi_{vv}) + \frac{1}{2y^{1/2}}(\phi_u + \phi_v)$$

From $(*)$

$$-y\phi_{xx} + \phi_{yy} = y(4\phi_{uv} + \frac{1}{2y^{3/2}}(\phi_u + \phi_v) = 0$$

Now using $u + v = \frac{4}{3}y^{3/2}$ and $y > 0$, the canonical form is

$$\phi_{uv} + \frac{1}{6(u+v)}(\phi_y + \phi_v) = 0$$

9.5. General solution for Wave Equation (D'Alembert)

Solve (3.4)

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 0$$

with initial conditions

$$\phi(x, 0) = f(x), \qquad \phi_t(x, 0) = g(x) \tag{9.16}$$

With $a = \frac{1}{c^2}$, $b = 0$, $c = -1$ the characteristic equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{-0 \pm \sqrt{0 + \frac{1}{c^2}}}{\frac{1}{c^2}} = \pm c$$

so choose $u = x - ct$ and $v = x + Ct$, which yields simple canonical form:

$$\frac{\partial^2 \phi}{\partial u \partial v} = 0 \tag{9.17}$$

Integrate with respect to u, $\frac{\partial \phi}{\partial v} = F(v)$ and then with respect to u

$$\phi = G(u) + \int^v F(y)\mathrm{d}y = G(u) + H(v)$$

Impose our initial conditions at $t = 0$ when $u = v = x$,

$$\phi(x, 0) = G(x) + H(x) = f(x) \tag{$*$}$$

$$\phi_t(x, 0) = -cG'(x) + cH'(x) = g(x) \tag{$\dagger$}$$

Differentiating ($*$):

$$G'(x) + H'(x) = f'(x) \tag{$\ddagger$}$$

So ($\dagger$) and ($\ddagger$)

$$\implies H'(x) = \frac{1}{2}(f'(x) + \frac{1}{c}g(x))$$

Integrate

$$H(x) = \frac{1}{2}(f(x) - f(0)) + \frac{1}{2c}\int_0^x g(y)\mathrm{d}y$$

and from ($*$)

$$G(x) = \frac{1}{2}(f(x) - f(0)) - \frac{1}{2c}\int_0^x g(y)\mathrm{d}y$$

Putting together:

$$\begin{aligned}\phi(x, t) &= G(x - ct) + H(x + ct) \\ &= \frac{1}{2}(f(x - ct) + f(x + ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\mathrm{d}y\end{aligned}$$

10. Solving PDEs with Green's Functions

10.1. Diffusion equation and Fourier transform

Recall heat equation (4.3) for a conducting wire

$$\frac{\partial \theta}{\partial t}(x,t) - D\frac{\partial^2 \theta}{\partial t^2}(x,t) = 0 \tag{10.1}$$

with initial conditions $\theta(x,0) = h(x)$ with $\theta \to 0$ as $x \to \pm\infty$. Take the Fourier Transform with respect to x using (8.13)

$$\frac{\partial}{\partial t}\tilde{\theta}(k,t) = -Dk^2\tilde{\theta}(k,t)$$

Integrate $\tilde{\theta}(k,t) = Ce^{-Dk^2t}$ with initial conditions $\tilde{\theta}(k,0) = \tilde{h}(k)$, we have

$$\tilde{\theta}(k,t) = \tilde{h}(k)e^{-Dk^2t}$$

Now invert

$$\begin{aligned}
\theta(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{h}(k) \underbrace{e^{-Dk^2t}}_{\text{Gaussian (8.5)}} e^{ikx}\mathrm{d}k \\
&= \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} h(u)\exp\left(\frac{-(x-u)^2}{4Dt}\right)\mathrm{d}u \qquad \text{by convolution theorem (8.17)} \\
&\equiv \int_{-\infty}^{\infty} h(u)S_d(x-u,t)\mathrm{d}u \qquad (10.2)
\end{aligned}$$

where the *fundamental solution* is

$$S_d(x,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-x^2/4Dt} \tag{10.3}$$

(Fourier transform is $\tilde{S}_d(k,t) = e^{-Dk^2t}$). Also known as diffusion kernel or source.

Note. With localised initial conditions $\theta(x,0) = \theta_0\delta(x)$ then

$$\theta(x,t) = \theta_0 S_d(x,t) = \frac{\theta_0}{\sqrt{4\pi Dt}}e^{-\eta^2} \tag{10.4}$$

where $\eta = \frac{x}{2\sqrt{Dt}}$ is the similarity parameter. Initial condition $t \geq 0$ spreads smoothly as a Gaussian.

Example (Gaussian pulse)**.** Suppose initially

$$f(x) = \sqrt{\frac{a}{\pi}}\theta_0 e^{-ax^2}$$

$$\begin{aligned}\theta(x,t) &= \frac{\theta_0\sqrt{a}}{\sqrt{4\pi^2 Dt}}\int \exp\left[-au^2 - \frac{(x-u)^2}{4Dt}\right]\mathrm{d}u \\ &= \frac{\theta_0\sqrt{a}}{\sqrt{4\pi^2 Dt}}\int_{-\infty}^{\infty} \exp\left[\frac{(1+4aDt)u^2 - 2xu + x^2}{4Dt}\right]\mathrm{d}u \\ &= \frac{\theta_0\sqrt{a}}{\sqrt{4\pi^2 Dt}}\int_{-\infty}^{\infty} \exp\left[-\frac{(1+4aDt)}{4Dt}\left(u - \frac{x}{1+4Dt}\right)^2\right]\mathrm{d}u \times \exp\left[\frac{-ax^2}{1+4aDt}\right] \\ &= \theta_0\sqrt{\frac{a}{\pi(1+4Dt)}}\exp\left[\frac{-ax^2}{1+4aDt}\right] \qquad (10.5)\end{aligned}$$

Here, width spreads as standard deviation $\propto \sqrt{t}$ with area constant (i.e. heat energy conserved).

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10.2. Forced heat (diffusion) equation

Consider

$$\frac{\partial}{\partial t}\theta(x,t) - D\frac{\partial^2}{\partial x^2}\theta(x,t) = f(x,t) \qquad (10.6)$$

with homogeneous boundary conditions $\theta(x,0) = 0$. Construct a 2D Green's function $G(x,t;\xi,\tau)$ such that

$$\frac{\partial G}{\partial t} - D\frac{\partial^2 G}{\partial x^2} = \delta(x,\xi)\delta(t-\tau) \qquad (10.7)$$

with $G(x,0;\xi,\tau) = 0$. Take Fourier Transform with respect to x using (8.23)

$$\frac{\partial \tilde{G}}{\partial t} + Dk^2\tilde{G} = e^{-ik\xi}\delta(t-\tau)$$

Using multiplicative factor e^{Dk^2t}

$$\frac{\partial}{\partial t}[e^{Dk^2t}\tilde{G}] = e^{ik\xi + Dk^2t}\delta(t-\tau)$$

Integrate with respect to t using $G = 0$, at $t = 0$

$$\begin{aligned}e^{Dk^2t}\tilde{G} &= e^{-ik\xi}\int_0^t e^{Dk^2t'}\delta(t'-\tau)\mathrm{d}t' \qquad \text{by (6.7)} \\ &= e^{-ik\xi}e^{Dk^2\tau}H(t-\tau)\end{aligned}$$

$$G(x,t;\xi,\tau) = H(t-\tau)e^{-ik\xi}e^{-Dk^2(t-\tau)}$$

So inverting we get Green's function

$$\begin{aligned}
G(x,t;\xi,\tau) &= \frac{H(t-\tau)}{2\pi}\int_{-\infty}^{\infty} e^{-k(x-\xi)}e^{-Dk^2(t-\tau)}\mathrm{d}k \\
&= \frac{H(t')}{2\pi}\int_{-\infty}^{\infty} e^{ikx'}e^{-Dk^2t'}\mathrm{d}k && t' = t-\tau, x' = x-\xi \\
&= \frac{H(t')}{\sqrt{4\pi Dt'}}e^{-x^2/4Dt'} && \text{see section 8.1} \\
&= H(t-\tau)S_d(x-\xi), t-\tau) && (10.8)
\end{aligned}$$

where S_d is the fundamental solution (10.3). General solution is

$$\begin{aligned}
\theta(x,t) &= \int_0^{\infty}\int_{-\infty}^{\infty} G(x,t;\xi,\tau)f(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau \\
&= \int_0^{t}\int_{-\infty}^{\infty} f(u,\tau)S_d(x-u,t-\tau)\mathrm{d}u\mathrm{d}\tau && (10.9)
\end{aligned}$$

This is an example of Duhamel's principle relating (i) solution of forced PDE with homogeneous boundary conditions (10.6) to (ii) solutions of homogeneous PDE with inhomogeneous boundary conditions (10.1).
Recall solutions of (10.1) with initial conditions at $t = \tau$

$$\theta(x,t) = \int_{-\infty}^{\infty} f(u)S_d(x-u,t-\tau)\mathrm{d}u \qquad (t > \tau)$$

So forcing term $f(x,t)$ at $t = \tau$ acts as an initial condition for subsequential evolution. The integral (10.9) is a superposition of all these initial condition effects for $0 < \tau < t$.

Duhamel's principle

Let $\mathcal{L}$ be a linear differential operator involving no time derivatives, and D a spatial domain D in $\mathbb{R}^n$. Let $P^s f$ denote the solution to the homogeneous problem:

$$\begin{cases} u_t - \mathcal{L}u = 0 & (x,t) \in D \times (s,\infty) \\ u = 0 & \text{on } \partial D \\ u(x,s) = f(x,s) & x \in D \end{cases}$$

Then the solution to the forced problem:

$$\begin{cases} u_t - \mathcal{L}u = 0 & (x,t) \in D \times (0,\infty) \\ u = 0 & \text{on } \partial D \\ u(x,0) = 0 & x \in D \end{cases}$$

is given by

$$u(x,t) = \int_0^t (P^s f)(x,t)\mathrm{d}s$$

10.3. Forced wave equation

Consider

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} = f(x,t) \tag{10.10}$$

with $\phi(x,0) = 0$, $\phi_t(x,0) = 0$. Construct Green's function

$$\frac{\partial^2 G}{\partial t^2} - c^2 \frac{\partial^2 G}{\partial x^2} = \delta(x-\xi)\delta(t-\tau)$$

with $G = 0$, $G_t = 0$ at $t = 0$. Take Fourier transform with respect to x

$$\frac{\partial^2 \tilde{G}}{\partial t^2} + c^2 k^2 \tilde{G} = e^{-ik\xi}\delta(t-\tau)$$

Recall section 7.4 for IVP Green's function (7.26)

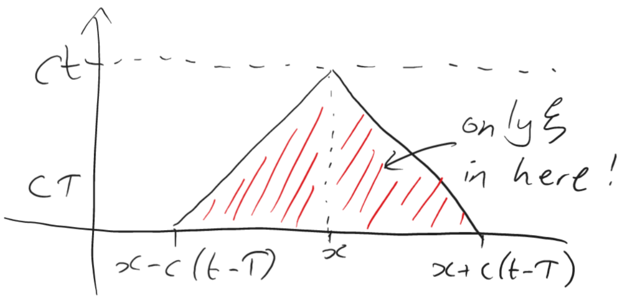
$$\tilde{G} = \begin{cases} 0 & t < \tau \\ e^{-ik\xi}\frac{\sin kc(t-\tau)}{kc} & t > \tau \end{cases} = e^{-ik\xi}\frac{\sin kc(t-\tau)}{kc}H(t-\tau)$$

Invert Fourier Transform

$$\begin{aligned} G(x,t;\xi,\tau) &= \int_{-\infty}^{\infty} \frac{e^{ik\overbrace{(x-\xi)}^{A}} \sin k \overbrace{c(t-\tau)}^{B}}{k} \mathrm{d}k \\ &= \frac{H(t-\tau)}{2\pi c} \cdot 2 \int_0^\infty \frac{\cos kA \sin kB}{k} \mathrm{d}k \\ &= \frac{H(t-\tau)}{2\pi c} \int_0^\infty \frac{\sin k(A+B) - \sin(A-B)}{k} \mathrm{d}k \\ &= \frac{H(t-\tau)}{4c}[\operatorname{sgn}(A+B) - \operatorname{sgn}(A-B)] \\ &= \frac{H(t-\tau)}{4c}[2H(B-|A|)] \end{aligned}$$

Now with $H(t-\tau) \implies B = c(t-\tau) > 0$ so only non-zero if $|A| < B$, i.e. $|x-\xi| < c(t-\tau)$. So Green's function or *causal fundamental solution* is

$$G(x,t;\xi,\tau) = \frac{1}{2c}H(c(t-\tau) - |x-\xi|) \tag{10.11}$$



The solution is

$$\phi(x,t) = \int_0^\infty \int_{-\infty}^\infty f(\xi,\tau)G(x,t;\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau$$
$$= \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau \tag{10.12}$$

Exercise: relation (10.12) to D'Alembert's solution with initial conditions (9.18) at $t = 0$, $\phi = 0$, $\phi_t = g(x)$ as an example of Duhamel's principle.

10.4. Poisson's Equation

$$\nabla^2\phi = -\rho(\mathbf{x}) \tag{10.13}$$

on domain $\mathcal{D}$ with Dirichlet boundary conditions $\phi = 0$ on ∂D.
Fundamental solution: The $\delta(\mathbf{x})$ function in $\mathbb{R}^3$ has the following properties:

$$\delta(\mathbf{x}-\mathbf{x}') = 0, \quad \forall \mathbf{x} \neq \mathbf{x}'$$

$$\int_{\partial\mathcal{D}} \delta(x-\mathbf{x}')\mathrm{d}\mathbf{x} = \begin{cases} 1 & \mathbf{x}' \in \mathcal{D} \\ 0 & \text{otherwise} \end{cases} \tag{10.14}$$

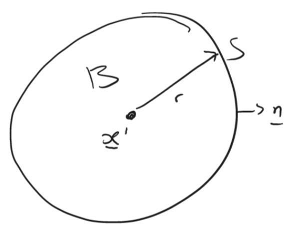
Sampling property

$$\int_{\mathcal{D}} f(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')\mathrm{d}\mathbf{x} = f(\mathbf{x}')$$

The *free-sapce Green's function* is defined to be

$$\nabla^2 G(\mathbf{x};\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}') \tag{10.15}$$

with homogeneous boundary conditions on $\mathbb{R}^3$, $G \to 0$ as $\mathbf{x} \to \infty$.



This is spherically symmetric about $\mathbf{x}'$, so the fundamental solution can only depend on the scalar distance $G(\mathbf{x};\mathbf{x}') = G(|\mathbf{x}-\mathbf{x}'|) = G(r)$. WLOG $\mathbf{x}' = 0$. Integrate (10.15) over ball B radius r around $\mathbf{x}' = 0$.

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$$\begin{aligned}
LHS &= \int_B \nabla^2 G \mathrm{d}\mathbf{x} \\
&= \int_S \nabla G \cdot \mathbf{n} \mathrm{d}S \\
&= 4\pi r^2 \frac{\mathrm{d}G}{\mathrm{d}r} \\
RHS &= \int_B \delta(\mathbf{x}) \mathrm{d}\mathbf{x} \\
&= 1 && \text{by (10.14)}
\end{aligned}$$

So

$$\frac{\mathrm{d}G}{\mathrm{d}r} = \frac{1}{4\pi r^2} \implies G = -\frac{1}{4\pi r} + c$$

But $G \to 0$ as $r \to \infty$, so $c = 0$. Free-space Green's function:

$$\boxed{G(\mathbf{x}; \mathbf{x}') = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}} \tag{10.16}$$

General solution in $\mathbb{R}^3$

$$\phi(\mathbf{x}) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \mathrm{d}\mathbf{x}'$$

Exercise: Similarly in $\mathbb{R}^2$ derive

$$G_{2\mathrm{D}}(\mathbf{x}; \mathbf{x}') = \frac{1}{2\pi} \log(|\mathbf{x} - \mathbf{x}'|) + c_2$$

Green's Identities

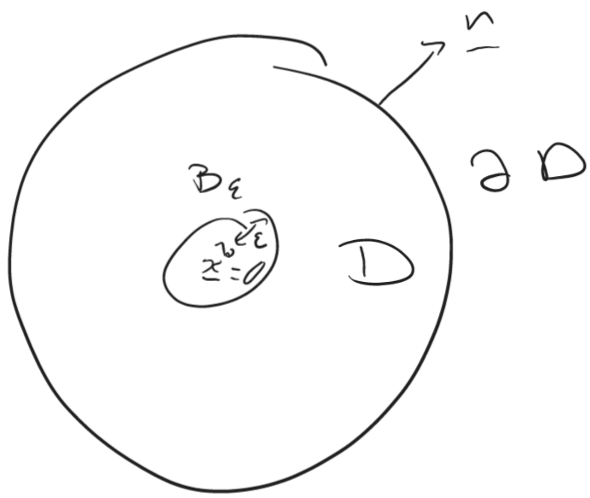
Consider two scalar functions ϕ, φ twice differentiable on $\mathcal{D}$.

$$\begin{aligned}
\int_{\mathcal{D}} \nabla \cdot (\varphi \nabla \psi) \mathrm{d}\mathbf{x} &= \int_{\mathcal{D}} (\phi \nabla^2 \phi + \nabla \phi \cdot \nabla \psi) \mathrm{d}\mathbf{x} \\
&= \int_{\partial \mathcal{D}} \phi \nabla \psi \cdot \hat{\mathbf{n}} \mathrm{d}S
\end{aligned} \tag{10.17}$$

This is *Green's first identity* $\phi \leftrightarrow \psi$ and subtract from (10.17), then *Green's second identity*

$$\int_{\partial \mathcal{D}} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) \mathrm{d}S = \int_{\mathcal{D}} (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \mathrm{d}\mathbf{x} \tag{10.18}$$

Now consider a small spherical ball B_ε (radius ε) about $\mathbf{x}$ (WLOG $\mathbf{x}' = 0$) Take ϕ in (10.18) such that $\nabla^2 \phi = -\rho(\mathbf{x})$ and $\psi = G(\mathbf{x}; \mathbf{x}')$ ($\nabla^2 G = \delta(\mathbf{x} - \mathbf{x}')$)



$$RHS = \int_{\mathcal{D}-B_\varepsilon} (\phi \underbrace{\nabla^2 G}_{=0} - G \underbrace{\nabla^2 \phi}_{=\rho}) \mathrm{d}\mathbf{x}$$
$$= \int_{\mathcal{D}-B_\varepsilon} G\rho \mathrm{d}\mathbf{x}$$
$$LHS = \int_{\partial\mathcal{D}} \left(\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right)\mathrm{d}S + \int_{S_\varepsilon} \underbrace{\left(\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right)}_{(*)}\mathrm{d}S$$

Second integral on small sphere S_ε, $\varepsilon \to 0$ (outward normal on S_ε points in $-$ direction)

$$\int_{S_\varepsilon} (*)\mathrm{d}S = \left(\overline{\phi}\left(-\frac{1}{4\pi\varepsilon^2}\right) - \frac{1}{4\pi\varepsilon^2}\frac{\partial\overline{\phi}}{\partial r}\right) 4\pi\varepsilon^2 = -\phi(0)$$

($\overline{\phi}$ denotes the average value, because we are on S_ε) Combining (with arbitrary $\mathbf{x}'$ now) we get *Green's third identity*

$$\phi(\mathbf{x}') = \int_{\mathcal{D}} G(\mathbf{x};\mathbf{x}')(-\rho(\mathbf{x}))\mathrm{d}\mathbf{x} + \int_{\partial\mathcal{D}} \left(\phi(\mathbf{x})\frac{\partial G}{\partial n}(\mathbf{x};\mathbf{x}') - G(\mathbf{x};\mathbf{x}')\frac{\partial \phi}{\partial n}(\mathbf{x})\right)\mathrm{d}S \quad (10.19)$$

Dirichlet Green's function:
Solve $\nabla^2\phi = -\rho$ on $\mathcal{D}$ with inhomogeneous boundary conditions $\phi(\mathbf{x}) = h(\mathbf{x})$ on $\partial\mathcal{D}$. Dirichlet Green's function satisfies

(i) $\nabla^2 G(\mathbf{x};\mathbf{x}') = 0$, $\forall \mathbf{x} \neq \mathbf{x}'$

(ii) $G(\mathbf{x};\mathbf{x}') = 0$ on $\partial\mathcal{D}$.

(iii) $G(\mathbf{x};\mathbf{x}') = G_{\mathrm{FS}}(\mathbf{x};\mathbf{x}') + H(\mathbf{x};\mathbf{x}')$ with $\nabla^2 H(\mathbf{x};\mathbf{x}') = 0$ $\forall \mathbf{x} \in \mathcal{D}$.

Green's second identity (10.18) with $\nabla^2\phi = -\rho$, $\nabla^2 H = 0$

$$\int_{\partial\mathcal{D}} \left(\phi \frac{\partial H}{\partial n} - H \frac{\partial \phi}{\partial n} \right) \mathrm{d}S = \int_{\mathcal{D}} H\rho \mathrm{d}\mathbf{x} \tag{$\dagger$}$$

Now we use $G_{\mathrm{FS}} = G - H$ in Green's third identity (10.19)

$$\phi(\mathbf{x}') = \int_{\mathcal{D}} (G-H)(-\rho)\mathrm{d}\mathbf{x} + \int_{\partial\mathcal{D}} \left(\phi \frac{\partial (G-H)}{\partial n} - (G-H)\frac{\partial \phi}{\partial n} \right) \mathrm{d}S$$

Subtract H terms above in ($\dagger$) ($G = 0$, $\phi = h$ on $\partial\mathcal{D}$)

$$\boxed{\phi(\mathbf{x}') = \int_{\mathcal{D}} G(\mathbf{x};\mathbf{x}')(-\rho(\mathbf{x}))\mathrm{d}\mathbf{x} + \int_{\partial\mathcal{D}} h(\mathbf{x}) \frac{\partial G}{\partial n}(\mathbf{x};\mathbf{x}')\mathrm{d}S} \tag{10.20}$$

Exercise: Use (10.18) to show that GF is symmetric (third identity), $G(\mathbf{x};\mathbf{x}') = G(\mathbf{x}'; xbf)$, $\forall \mathbf{x} \neq \mathbf{x}'$.

For Neumann BCs specifying

$$\frac{\partial \phi}{\partial n} = K(\mathbf{x})$$

on $\partial\mathcal{D}$ we have

$$\phi(\mathbf{x}') = \int_{\partial\mathcal{D}} G(\mathbf{x};\mathbf{x}')(-\rho(\mathbf{x}))\mathrm{d}\mathbf{x} + \int_{\partial\mathcal{D}} G(\mathbf{x};\mathbf{x}')(-K(\mathbf{x}))\mathrm{d}S$$

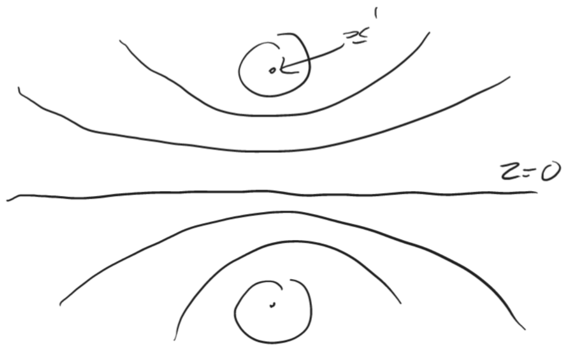
(see RJ lecture notes)

10.5. Method of images

For symmetric domains $\mathcal{D}$ we can construct Green's functions with $G = 0$ on $\partial\mathcal{D}$ by cancelling the Boundary non-zero values by placing "an image" of Green's function outside $\mathcal{D}$.

Laplace's equation on half-space

Solve $\nabla^2\phi = 0$ on $\mathcal{D} = \{(x, y, z) : z > 0\}$ with $\phi(x, y, z = 0) = h(x, y)$ and $\phi \to 0$ as $\mathbf{x} \to \infty$. Now fundamental solution $G(\mathbf{x};\mathbf{x}') \to 0$ as $|\mathbf{x} \to \infty$, but $G \neq 0$ at $z = 0$. So for G at $\mathbf{x}' = (x', y', z')$ subtract "image" G at $\mathbf{x}'' = (x', y', -z')$. $G(\mathbf{x};\mathbf{x}') = -\frac{1}{4\pi|-\mathbf{x}'|} - \left(\frac{1}{4\pi|\mathbf{x}-\mathbf{x}''|}\right)$



$$G(\mathbf{x};\mathbf{x};) = -\frac{1}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \frac{1}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$= 0$ if $z = 0$, i.e. satisfies the Dirichlet BCs on all $\partial\mathcal{D}$. Contribution from the Boundary

$$\begin{aligned}
\left.\frac{\partial G}{\partial n}\right|_{z=0} &= -\left.\frac{\partial G}{\partial z}\right|_{z=0} \\
&= -\frac{1}{4\pi}\left(\frac{z-z'}{|\mathbf{x}-\mathbf{x}'|^3} - \frac{z+z'}{|\mathbf{x}-\mathbf{x}''|^3}\right)\Bigg|_{z=0} \\
&= \frac{z'}{2\pi}((x-x')^2+(y-y')^2+z'^2)^{-3/2} \qquad (10.22)
\end{aligned}$$

Solution is then from (10.20) (no sources)

$$\phi(x',y',z') = \frac{z'}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}((x-x')^2+(y-y')^2+z'^2)^{-3/2}h(x,y)\mathrm{d}x\mathrm{d}y \qquad (10.23)$$

Wave equation for $x > 0$

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial n^2} = f(x,t)$$

BCs $\phi(0,t) = 0$ Dirichlet BCs. Create matching Green's function from (10.11) with opposite sign centred at $x = -\xi$

$$G(x,t;\xi,\tau) = \frac{1}{2c}H(c(t-\tau)-|x-\xi|) - \frac{1}{2c}H(c(t-\tau)-|x+\xi|)$$

Similarly, for a homogeneous Neumann BC at $x = 0$ $\frac{\partial G}{\partial n}\big|\,|_{x=0} = 0$ for all t the appropriate Green's function is

$$G(x,t;\xi,\tau) = \frac{H(c(t-\tau)-|x-\xi|)}{2c} + \frac{H(c(t-\tau)-|x+\xi|)}{2c}$$

> **Note.** Image has the *same* sign.

For small $x > 0$, $|-\xi| = \xi - x$, $|x+\xi| = x + \xi$. For all t

$$\left.\frac{\partial G}{\partial n}\right|_{x=0} = \frac{1}{2c}(\delta(c(t-\tau) - |x-\xi|) + \delta(c(t-\tau) - |x+\xi|)(-1))_{x=0} = 0$$