Geometry

June 3, 2023

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Start of lecture 1

1 Surfaces (7-8 lectures)

Definition. A *topological surface* is a topological space Σ such that

(a) $\forall \rho \in \Sigma$, there is an open neighbourhood $p \in U \subset \Sigma$ such that U is homeomorphic to $\mathbb{R}^2$, or a disc $D^2 \subset \mathbb{R}^2$, with its usual Euclidean topology.

(b) Σ is Hausdorff and second countable (has a countable base)

Remarks

(1) $\mathbb{R}^2 \cong D(0,1) = \{x \in \mathbb{R}^2 \colon \|x\| < 1\}$

(2) A space X is *Hausdorff* if for $p \neq q$ in X, there exists *disjoint* open sets with $p \in U$ and $q \in V$ in X. A space is *second countable* if it has a countable base, i.e. $\exists \{U_i\}_{i \in \mathbb{N}}$ open sets such that every open set is a union of some of the U_i. Part (a) is the main point of the definition, and part (b) is technical honesty and convenience.

(3) If X is Hausdorff / second countable so are subspaces of X. Euclidean space has these properties. (For second countable, consider open balls $B(c,r)$ with $c \in \mathbb{Q}^n \subset \mathbb{R}^n$ and $r \in \mathbb{Q}_+ \subset \mathbb{R}_+$).

Examples of Topological Surfaces

(i) $\mathbb{R}^2$ the plane

(ii) Any open set in $\mathbb{R}^2$, i.e. $\mathbb{R}^2 \setminus Z$ where Z is closed, for example $Z = \{0\}$, $\mathbb{R}^2 \setminus \{0\}$ is a surface,

$$Z = \{(0,0) \cup \{(0, Y_n) \colon n = 1, 2, \dots\}\}$$

(iii) Graphs: $f \colon \mathbb{R}^2 \to \mathbb{R}$ continuous graph:

$$\Gamma_f = \{(x, y, f(x,y)) \colon (x,y) \in \mathbb{R}^2\} \subset \mathbb{R}^3$$

subspace topology. Then this is a surface.

Recall: if X, Y topological spaces, then the product topology on $X \times Y$ has basic open sets $U \times V$ where $U \subset X$, $V \subset Y$ open.

Feature: $g \colon Z \to X \times Y$ is continuous if and only if

$$\pi_X \circ g \colon Z \to X$$

$$\pi_Y \circ g \colon Z \to Y$$

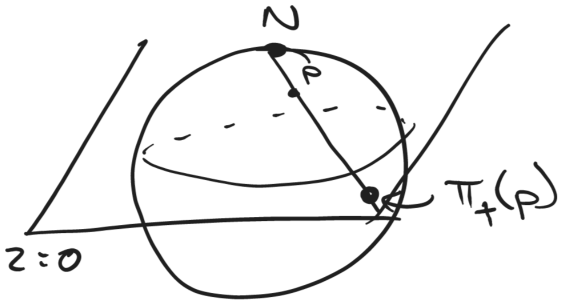
are both continuous (where $\pi_X \colon X \times Y \to X$ and $\pi_Y \colon X \times Y \to Y$ are the canonical projections).

Application: $f \colon X \to Y$ continuous, $\Gamma_f \subset X \times Y$ is homeomorphic to X, $S(x) = (x, f(x))$ is continuous and then $\pi_X \mid_f$ and S are inverse homeomorphisms. So $\Gamma_f \cong \mathbb{R}^2$ for any $f \colon \mathbb{R}^2 \to \mathbb{R}$ continuous, so Γ_f is a topological surface.

(iv) The sphere:

$$S^2 = \{(x, y,) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$$

(subspace topology).

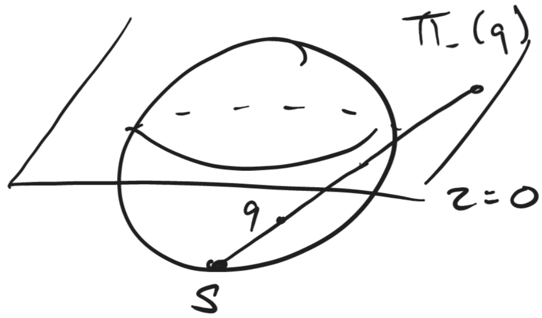


$$\pi_+ \colon S^2 \setminus \{0, 0, 1\} \to \mathbb{R}^2 \quad ((z = 0) \subset \mathbb{R}^3)$$
$$(x, y, z) \mapsto \left(\frac{x}{1 - z}, \frac{y}{1 - z} \right)$$

π_+ is continuous and has an inverse

$$(u, v) \mapsto \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right)$$

so π_+ is a continuous bijection with continuous inverse and hence a homeomorphism. Similarly:



$\pi_i \colon S^2 \setminus \{(0,0,-1)\} \to \mathbb{R}^2\}$

$$(x,y,z) \mapsto \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

Stereographic projection from south pole, also a homeomorphism, so S^2 is a topological surface: the open sets $S^2 \setminus \{(0,0,1)\}$ and $S^2 \setminus \{0,0,-1\}$ cover S^2 is Haudorff and second countable (inherited from $\mathbb{R}^3$). Note: S^2 is compact.

(v) The *real projective plane*. The group $\mathbb{Z}_2$ acts on S^2 by homeomorphisms via the antipodal map: $a \colon S^2 \to S^2$, $a(x,y,z) = (-x,-y,-z)$. $\mathbb{Z}_2 \hookrightarrow \text{Homeo}(S^2)$, non-trivial element $\to a$.

Definition (Real projective plane)**.** The real projective plane is the quotient of S^2 by identifying every point with its antipodal image:

$$\mathbb{RP}^2 = S^2/\mathbb{Z}_2 = S^2/\sim$$

where $x \sim a(x)$.

Start of lecture 2

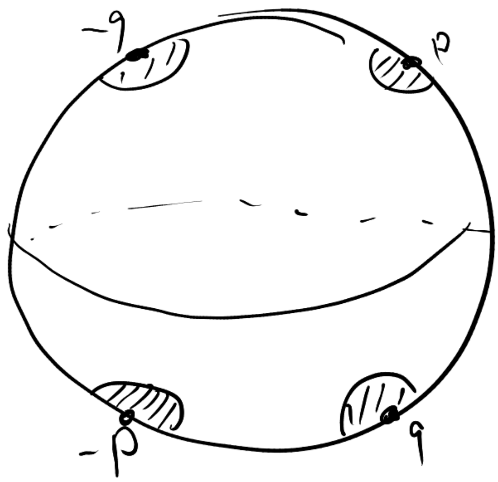
Lemma 1.1. As a set, $\mathbb{RP}^2$ is naturally in bijection with the set of straight lines through 0.

Proof. Any straight line through $0 \in \mathbb{R}^3$ meets S^2 in exactly a pair of antipodal points and each pair determines a straight line. □

Lemma 1.2. $\mathbb{RP}^2$ is a topological surface with the *quotient topology*.

Recall: Quotient topology: $q \colon X \to Y$, $V \subset Y$ is open if and only if $q^{-1}V \subset X$ is open in X (so q is continuous).

Proof. First we check that $\mathbb{RP}^2$ is Hausdorff. If $[p] \neq [q] \in \mathbb{RP}^2$, then $\pm p, \pm q \in \S^2$ are distinct antipodal pairs. Take small open discs centred at p and q and their antipodal image as in picture



This gives open neighbourhood of $[p]$ and $[q]$ in $\mathbb{RP}^2$. $q\colon S^2 \to \mathbb{RP}^2$, $q(B_\delta(p))$ is *open* since

$$r^{-1}(q(B_\delta(p)) = B_\delta(p) \cup (-B_\delta(p))$$

$\mathbb{RP}^2$ is second countable: Let $\mathcal{U}_0$ be a countable base for S^2 and let

$$\overline{\mathcal{U}} = \{q(u)\colon U \in \mathcal{U}\}$$

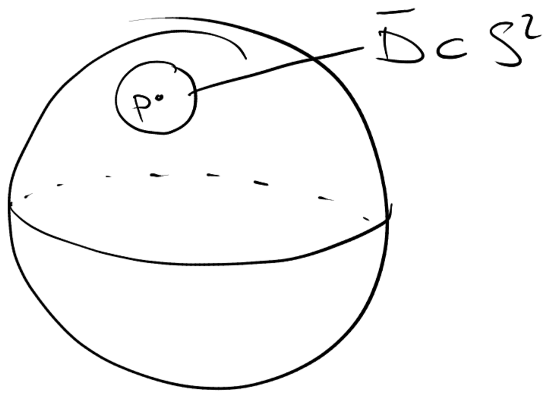
$q(u)$ is open:

$$q^{-1}(qU) = U \cup (-U)$$

$\overline{\mathcal{U}}$ is clearly countable since $\mathcal{U}$ is Take $V \subset \mathbb{RP}^2$ open. By definition, $q^{-1}V$ is open in S^2 hence $q^{-1}V = \bigcup_\alpha U_\alpha$, $U_\alpha \in \mathcal{U}$.

$$V = q(q^{-1}V) = q\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha q(U_\alpha)$$

Finally, let $p \in S^2$ and $[p] \in \mathbb{RP}^2$ its image. Let $\overline{D}$ be a small closed disc neighbourhood of $p \in S^2$.



$q|_{\overline{D}}\colon \overline{D} \to q(\overline{D}) \subset \mathbb{RP}^2$ is injective and continuous, from a compact space to a Hausdorff space. (Recall "Topological inverse function theorem"). A continuous bijection from a compact space to a Hausdorff space is a *homemorphism* (Analysis and Topology). So $q|_{\overline{D}}\colon \overline{D} \to q(\overline{D})$ is a homeomorphism. This induces a homemorphism

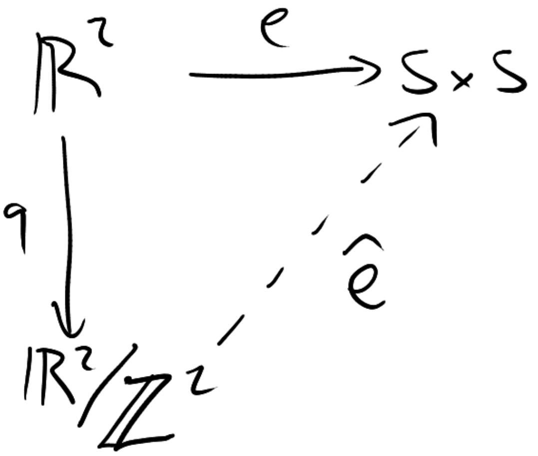
$$q|_D\colon D \to q(D) \subset \mathbb{RP}^2$$

(D is open disc). So $[p] \in q(D)$ has an open neighbourhood in $\mathbb{RP}^2$ homeomorphic to an open disc and we're done. □

(vi) Let $S^1 = \{z \in \mathbb{Z}\colon |z| = 1\}$. The *forms* $S^1 \times S^1$ with the subspace topology of $\mathbb{C}^2$ (check: is the same as product topology).

Lemma 1.3. The forms is a topological surface.

Proof. $\mathbb{R}^2 \xrightarrow{e} S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$, $(s,t) \mapsto (e^{2\pi i s}, e^{2\pi i t})$.



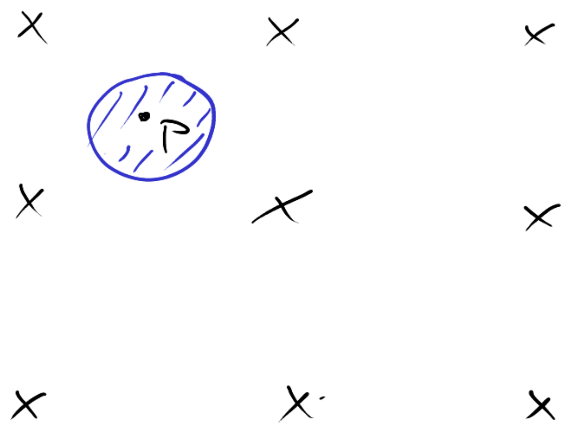
Equivalence relation on $\mathbb{R}^2$ given by translating by $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$. The map

$$[0,1]^2 \hookrightarrow \mathbb{R}^2 \xrightarrow{q} \mathbb{R}^2/\mathbb{Z}^2$$

is onto, so $\mathbb{R}^2/\mathbb{Z}^2$ is compact. Note that $\hat{e}$ is a continuous bijection. By Topological Inverse Function Theorem, $\hat{e}$ is a homeomorphism.

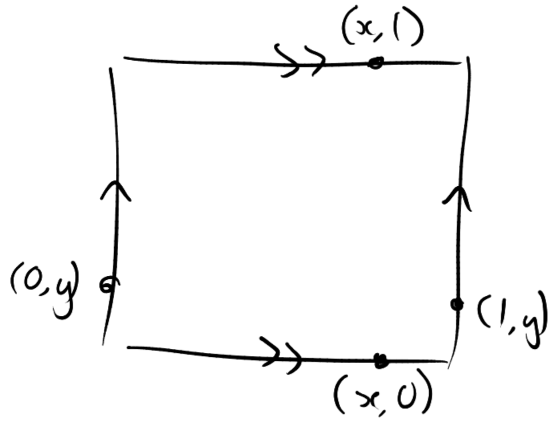
Note. We already know that $S^1 \times S^1$ is compact, Hausdorff and second countable (closed and bounded in $\mathbb{R}^4$).

As for the case of $S^2 \to \mathbb{RP}^2$, pick $[p] \in q(p)$, $p \in \mathbb{R}^2$ and small closed disc $\overline{D}(p) \subset \mathbb{R}^2$ such that $\forall (u,v) \neq (0,0) \in \mathbb{Z}^2$, $\overline{D}(p) \cap (\overline{D}(p) + (u,v)) = \emptyset$.

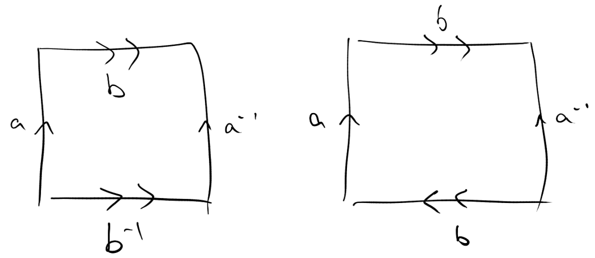


Then $e|_{\overline{D}(p)}$ and $q|_{\overline{D}(p)}$ are injective. Now restricting to the open disc as before, we get an open disc neighbourhood of $[p] \in \mathbb{R}^2/\mathbb{Z}^2$ hence $S^1 \times S^1$ is a topological surface. $\square$

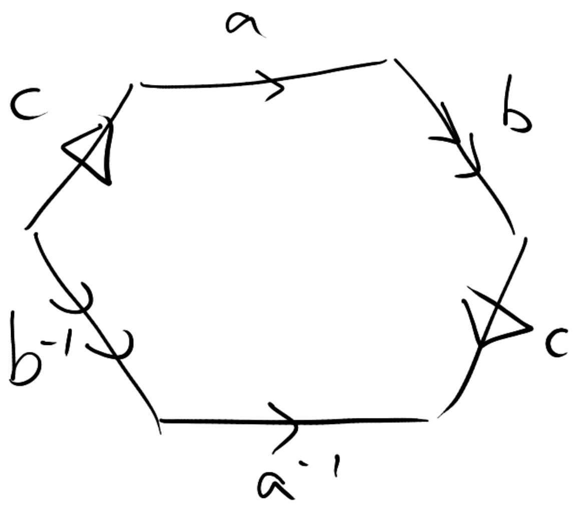
Another viewpoint: $\mathbb{R}^2/\mathbb{Z}^2$ is also given by imposing on $[0,1]^2$ the equivalence relation $(x,0) \sim (x,1)\ \forall 0 \le x \le 1$, $(0,y) \sim (1,y)\ \forall 0 \le y \le 1$.



(vii) Let P be a planar Euclidean polygon (including interior). Assume the edges are *oriented* and *paired* and for simplicity assume Euclidean lengths of e and $\hat{e}$ are equal if $\{e, \hat{e}\}$ are paired.



Label by letters and describe orientation by a sign $\pm$ relative to the clock with orientation of $\mathbb{R}^2$.



If $\{e, \hat{e}\}$ are parallel edges, there is a unique isometry from from e to $\hat{e}$ respecting their orientation, say

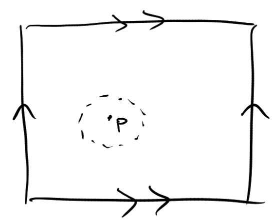
$$f_{e\hat{e}} \colon e \to \hat{e}$$

These maps generate an equivalence relation on P where we identify $x \in \partial P$ with $f_{e\hat{e}}(x)$ whenever $x \in e$.

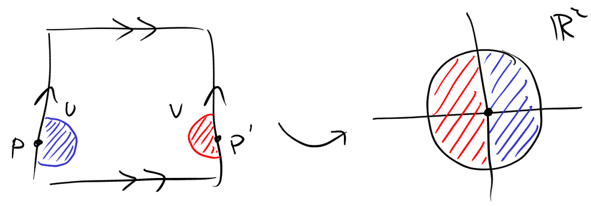
Lemma 1.4. $P/\sim$ (with the quotient topology) is a topological surface.

Start of lecture 3

The torus T^2 as $[0,1]^2/\sim$. $P = [0,1] \times [0,1]$.



If $p \in \text{interior}(P)$ we pick $\delta > 0$ small so that $\overline{B_\delta(p)}$ lies in the interior(P). Now argue as before: the quotient map is injective on $\overline{B_\delta(P)}$ and homeomorphism on its interior. If $p \in \text{edge}(P)$:



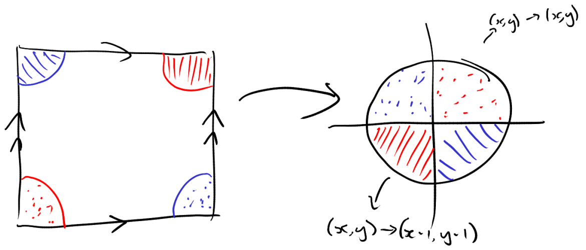
Say $p = (0, y_0) \sim (1, y_0) = p'$. Take δ small enough so half discs of radius δ as shown don't meet the vertices. Define a map as

$$(x, y) \mapsto \begin{cases} (x, y - y_0) & \text{on } U \\ (x - 1, y - y_0) & \text{on } V \end{cases}$$

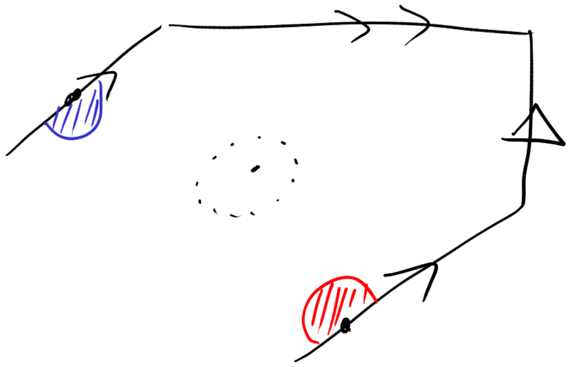
Recall: If $X = A \cup B$, A, B closed and $f\colon A \to Y$ and $g\colon B \to Y$ are continuous, and $g|_{A\cap B} = g|_{A\cap B}$ then they define a continuous map on X.

f_u and f_V are continuous on $U, V \subset [0,1]^2$, they induce continuous maps on q_U and q_V. ($q\colon [0,1]^2 \to [0,1]^2/\sim$). In T^2, $\frac{1}{2}$ discs, q_U and q_V overlap, but our maps agree as they are compatible with the equivalence relation. So f_U and f_V "glue" to give a continuous map on an open neighbourhood of $[p] \in T^2$ to $\mathbb{R}^2$. Now the "usual argument" (pass to closed disc, use Topological Inverse Function Theorem pass back to interior) shows that if $[p] \in T^2$ lies in edge(P), it has a neighbourhood homeomorphic to a disc.

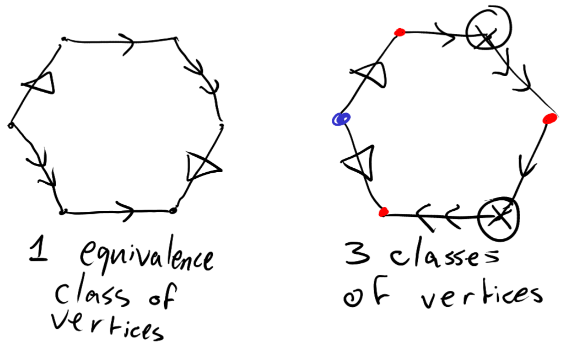
Finally at a vertex of $[0,1]^2$



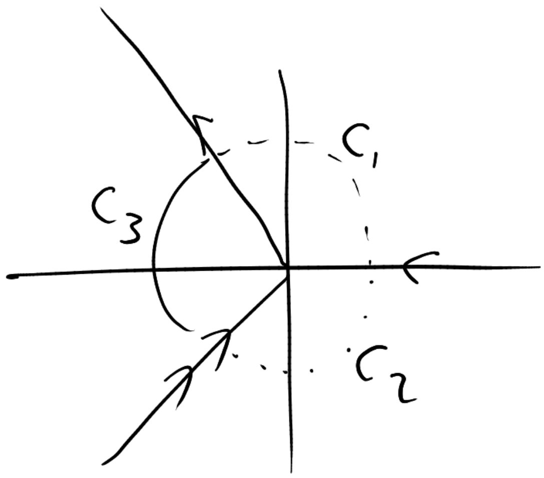
and analogously we get that a vertex has a neighbourhood homeomorphic to a sic and $[0,1]^2/\sim$ is a *topological surface.*



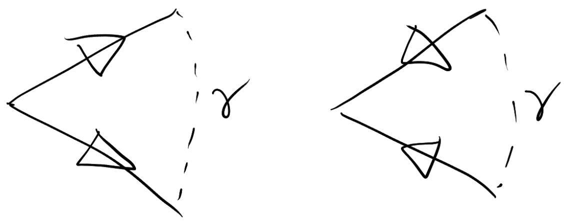
For a general polygon: *similar idea*. Same situation as T^2 for interior points and points on edges. How about vertices?



If $v \in \text{vertex}(P)$ has k vertices in its equivalence class, there exist k sectors in P. Any sector can be identified with our favourite sector (for example a quarter circle is homemorphic to a semi circle). In $(*)$ we get an open disc neighbourhood of v (red dots) via



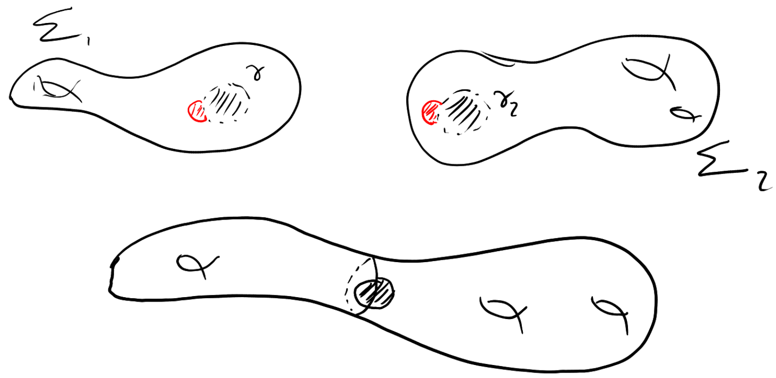
If we have $k = 1$, we must have either



But this quotient space is homemorphic to $D^2 \subset \mathbb{R}^2$. These open neighbourhoods of points in $P/\sim$ show that P is locally homeomorphic to a disc.

Exercise: Check that $P/\sim$ is Hausdorff and second countable.

(viii) Connected sums: Given topological surfaces Σ_1 and Σ_2 we can remove an open disc from each and glue the resulting boundary circles



Explicitly take $\Sigma_1 \setminus D_1 \amalg \Sigma_2 \setminus D_2$ and impose a quotient relation $\theta \in \partial D_1 \sim \theta \in \partial D_2$ where θ parametrisation. $S' = \partial D_i$, ∂D_i is the boundary of ∂D_i. The result $\Sigma_1 \# \Sigma_2$ is called the connected sum of Σ_1 and Σ_2. In principle this depends on many choices and it takes some effort to prove that it is *well-defined*.

Lemma 1.5. The connected sum $\Sigma_1 \# \Sigma_2$ is a topological surface (no proof in this course).

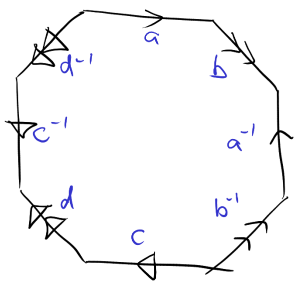
If you want to learn more:

- Introduction to topological manifolds by Jack Lee.
- Introduction to smooth manifolds by Jack Lee.

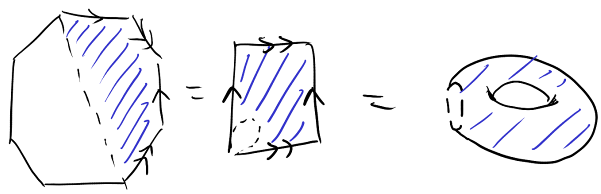
Start of lecture 4

More Examples

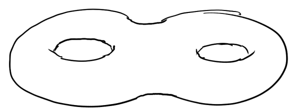
(1) Octagon P:



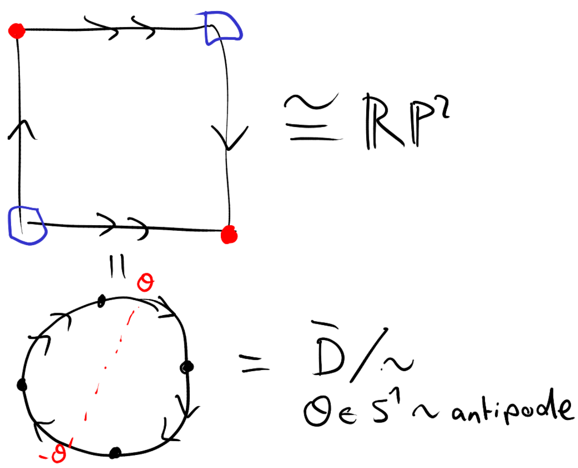
Cut in half:



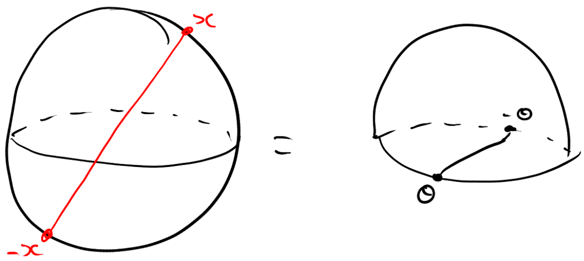
So $P/\sim$ looks like:



(2) Now projective plane example:



$\mathbb{RP}^2 = S^2/\pm 1 =$ (closed upper hemisphere)$/(\theta \sim -\theta)$.



1.1 Triangulations and Euler Characteristic

Definition. A subdivision of compact topological surface Σ comprises

(i) a finite set V of *vertices*.

(ii) a finite collection of edges $E = \{e_i : [0,1] \to \Sigma\}$ such that:

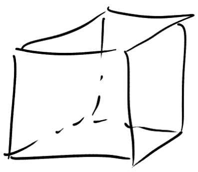
- For all i, e_i is a continuous injection on its interior and $e_i^{-1}V = \{0,1\}$
- e_i and e_j have disjoint image except perhaps at their endpoints in V.

(iii) We *require* that each connected component of $\Sigma \setminus (\bigcup_i e_i[0,1] \cup V)$ is homeomorphic to an open disc called a *face*. (so the closure of a face has boundary $\overline{F} \setminus F$ lying in $\bigcup_i e_i[0,1] \cup V$).

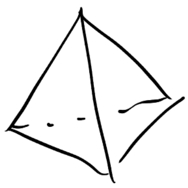
A subdivision is a *triangulation* if every closed face (closure of a face) contains exactly 3 edges and two closed faces are distinct, meet in exactly one edge or just one vertex.

Examples

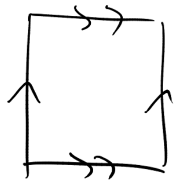
A subdivision of S^2:



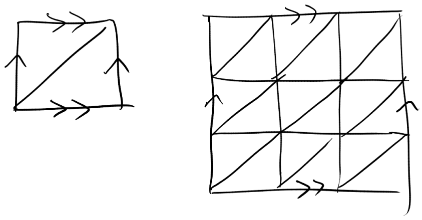
A triangulation of S^2:



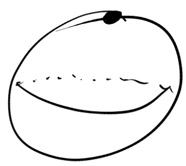
Subdivision of T^2 (1 vertex, 2 edges and 1 face):



Here the left drawing is not a triangulation of T^2 but the right one is:



A very degenerate subdivision of S^2 (1 vertex, 0 edges, 1 face):



Definition. The Euler characteristic of a subdivision is

$$\#V - \#E + \#F$$

Theorem 1.6. (i) Every compact topological surface admits subdivisions and triangulation.

(ii) The Euler characteristic, denote by $\chi(\Sigma)$, does *not* depend on the choice of subdivision and defines a topological invariant of the surface (depends on on the homeomorphism type of Σ).

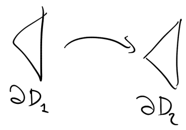
Remark. Hard to prove particularly (i). For (ii) there are cleaner approaches (Algebraic Topology part II).

Examples

(1) $\chi(S^2) = 2$.

(2) $\chi(T^2) = 0$.

(3) Σ_1, Σ_2 compact topological surfaces, and we form $\Sigma_1 \# \Sigma_2$. We remove open discs $D_i \subset \Sigma_i$ which is a face of a triangulation in each surface.



$$\implies \chi(\Sigma_1 \# \Sigma_2) = \chi(\Sigma_1) + \chi(\Sigma_2) - 2$$

In particular if Σ_g is a surface with g holes ($\Sigma = \#_{c=1}^{g} T^2$, then $\chi(\Sigma_g) = 2 - 2g$, g is called the *genus*.

1.2 Abstract Smooth Surfaces

Σ topological surface.

Definition. Let (U, φ) be a pair where $U \subset \Sigma$ is open and $\varphi\colon U \to V \subset \mathbb{R}^2$ is a homeomorphism (with V open). Then this pair is called a *chart*. THe inverse $\sigma = \varphi^{-1}\colon V \to U \subset \Sigma$ is called a *local parametrisation* for Σ.

Definition. A collection of charts

$$\{(U_i, \varphi_i)_{i \in I}\}$$

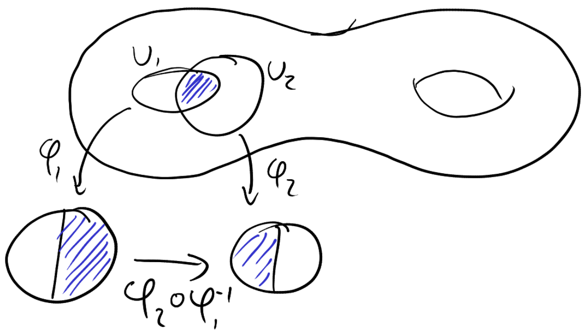
such that $\bigcup_{i \in I} U_i = \Sigma$ is called an *atlas* for Σ.

Examples

(1) If $z \subset \mathbb{R}^2$ closed then $\mathbb{R}^2 \setminus z$ is a topological surface with an atlas with *one* chart: $(\mathbb{R}^2 \setminus z, \varphi = \mathrm{id})$.

(2) For S^2, we have an atlas with 2 charts: the 2 stereographic projections.

Definition. Let (U_i, φ_i), $i = 1, 2$ be two charts containing $p \in \Sigma$. The map $\varphi_2 \circ \varphi_1^{-1}|_{\varphi_1(U_1 \cap U_2)}$ is called a *transition map* between charts.



$\varphi_1(U_1 \cap U_2) \xrightarrow{\varphi_2 \circ \varphi_1^{-1}} \varphi_2(U_1 \cap U_2)$ is a homeomorphism.

Recall: If $V \subset \mathbb{R}^n$ and $V' \subset \mathbb{R}^m$ are open, a map $f\colon V \to V'$ is called *smooth* if it is infinitely differentiable, i.e. it has continuous partial derivatives of all orders. A homeomorphism $f\colon V \to V'$ is called a *diffeomorphism* if it is smooth and it has a smooth inverse.

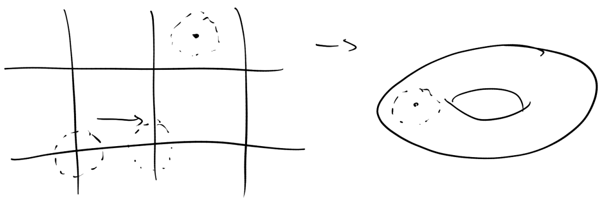
Definition. An abstract smooth surface Σ is a topological surface with an atlas of charts $\{(U_i, \varphi_i)\}_{i \in I}$ such that all transition maps are *diffeomorphisms.*

Start of lecture 5

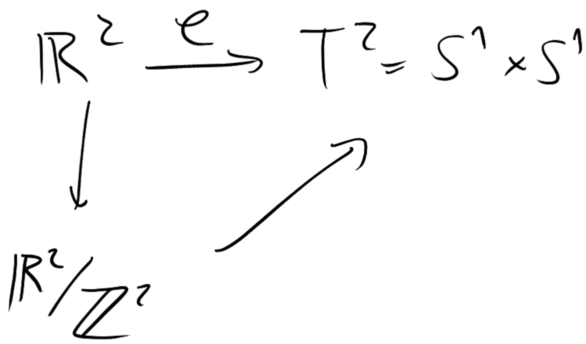
Examples

(1) (See Example Sheet 1). The atlas of 2 charts with stereographic projections gives S^2 the structure of an abstract smooth surface.

(2) The torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Recall that we obtained charts from (the inverses of) the projection restricted to small discs in $\mathbb{R}^2$.



The transition maps are *translations* so T^2 inherits the structure of a smooth surface.



$e(t,s) = (e^{2\pi it}, e^{2\pi is})$. Consider an atlas

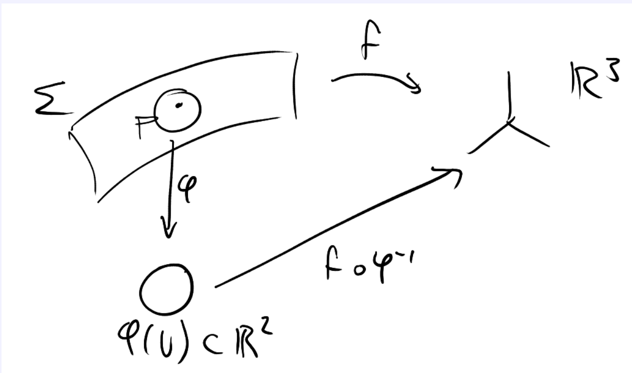
$$\{(e(D_\varepsilon(x,y)), e^{-1} \text{ on its image})\}$$

$\varepsilon < \frac{1}{3}$.

Definition. Let Σ be an abstract smooth surface and $f : \Sigma \to \mathbb{R}^n$ a map. We say that f is smooth at $p \in \Sigma$ if whenever (U, φ) is a chart at p, belonging to the smooth atlas for Σ, the map

$$f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^n$$

is smooth at $\varphi(p) \in \mathbb{R}^2$.

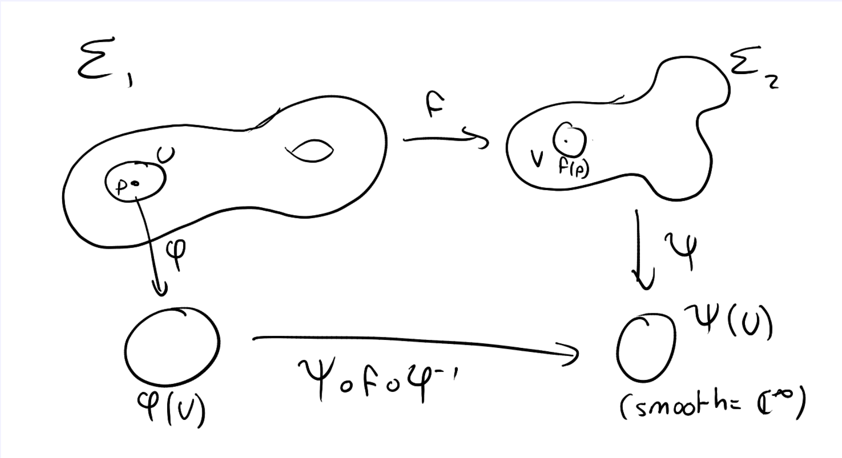


Note. If it holds for one chart at p, it holds for *all* charts at p:

$$f \circ \varphi^{-1} = f \circ \varphi_2^{-1} \circ \underbrace{(\varphi_2 \circ \varphi_1^{-1})}_{\text{diffeomorphism}}$$

Chain rule!

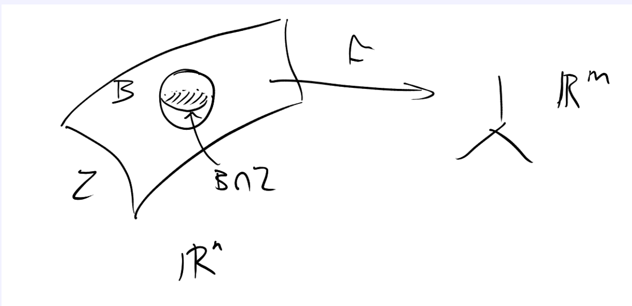
Definition. Σ_1, Σ_2 abstract smooth surfaces. A map $f : \Sigma_1 \to \Sigma_2$ is *smooth* if it is smooth in local charts: there are charts (U, φ) at p and (V, ψ) at $f(p)$ with $f(U) \subset V$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth at $\varphi(p)$.



Again, if f is smooth at p, then smoothness of the local representation of f at p will hold for all charts at p and $f(p)$ on the given atlas.

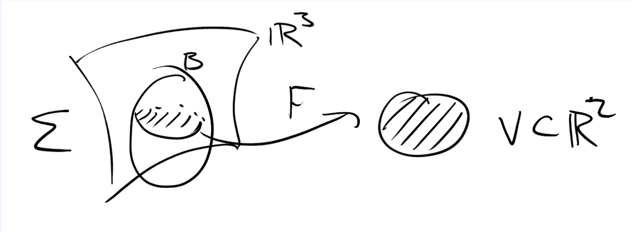
Definition. Σ_1 and Σ_2 are diffeomorphic if there exists $f : \Sigma_1 \to \Sigma_2$ that is smooth with smooth inverse.

Definition. If $Z \subset \mathbb{R}^n$ is an arbitrary subset, we say that $f : Z \to \mathbb{R}^n$ is smooth near $p \in Z$ if there exists open B, $p \in B \subset \mathbb{R}^n$ and a *smooth* $F : B \to \mathbb{R}^n$ such that $F|_{B \cap Z} = f|_{B \cap Z}$, i.e. f is locally the restriction of a smooth map defined on an open set.



Definition. If $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are subsets, we say that X and Y are diffeomorphic if there exists $f : X \to Y$ smooth with smooth inverse.

Definition. A *smooth surface* in $\mathbb{R}^3$ is a subset $\Sigma \subset \mathbb{R}^3$ such that $\forall p \in \Sigma$ there exists an open set $p \in U \subset \Sigma$ such that U is diffeomorphic to an open set in $\mathbb{R}^2$. In other words, for all $p \in \Sigma$ there exists an open ball B such that $p \in B \subset \mathbb{R}^3$ and $F : B \to V \subset \mathbb{R}^2$ smooth such that $F|_{B\cap\Sigma} : B \cap \Sigma \to V$ is a homeomorphism with inverse $V \to B \cap \Sigma$ smooth.



So we have 2 notions: one abstract and one taking advantage of the ambient space $\mathbb{R}^3$.

Theorem 1.7. For a subset $\Sigma \subset \mathbb{R}^3$, the following are equivalent (TFAE):

(a) Σ is a smooth surface in $\mathbb{R}^3$

(b) Σ is locally the graph of a smooth function over one of the coordinate planes, i.e. $\forall p \in \Sigma$ there exists open $p \in B \subset \mathbb{R}^3$ and open $V \subset \mathbb{R}^2$ such that

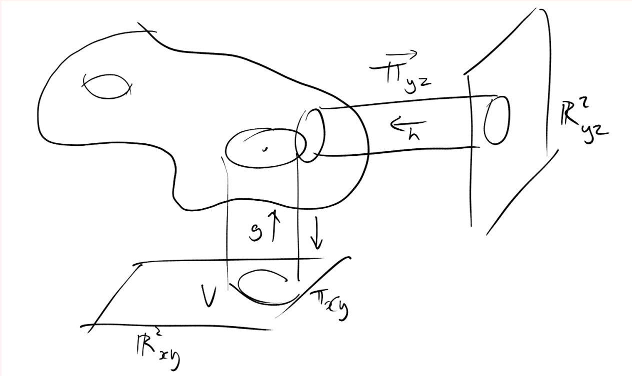
$$\Sigma \cap B = \{(x, y, g(x, y)) : (g : V \to \mathbb{R}), g \text{ smooth}\}$$

(or the graph over the xz or yz plane locally).

(c) Σ is locally cut out by a smooth function with non-zero derivative, i.e. $\forall p \in \Sigma$, there exists $p \in B \subset \mathbb{R}^n$ (B open) and $f : B \to \mathbb{R}$ such that $\Sigma \cap B = f^{-1}(0)$ and $Df|_x \neq 0$ for all $x \in B$.

(d) Σ is locally the image of an *allowable parametrisation* i.e. if $p \in \Sigma$, there exists open $p \in U \subset \Sigma$ and $\sigma : V \to U$ smooth such that σ is a homeomorphism and $D\sigma|_x$ has rank 2 for all $x \in V$.

Remark. (b) says that if Σ is a smooth surface in $\mathbb{R}^3$, each $p \in \Sigma$ belongs to a chart (U, φ) where φ is the restriction of π_{xy}, π_{yz}, π_{xz} from $\mathbb{R}^3$ to $\mathbb{R}^2$.



The transition map

$$(x, y) \mapsto (x, y, f(x, y)) \to (y, g(x, y))$$

has inverse

$$(y, z) \to (h(y, z), y, z) \to (h(y, z), y)$$

are *clearly smooth*. This gives Σ the structure of an abstract smooth surface.

Start of lecture 6

Theorem (Inverse function theorem)**.** Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^n$ be a continuously differentiable map. Let $p \in U$, $f(p) = q$ and suppose $Df|_p$ is invertible. Then there is an open neighbourhood V of q and a differentiable map $g : V \to \mathbb{R}^n$, $g(q) = p$ with image an open neighbourhood $U' \subset U$ of p such that $f \circ g = \mathrm{id}_V$ and $g \circ f = \mathrm{id}_U$. If f is smooth, then so is g.

Remark. $Df|_q = (Df|_p)^{-1}$ by the chain rule.

If we have a map $f : \mathbb{R}^n \to \mathbb{R}^m$ where $n > m$, what can we say if $Df|_p$ is surjective?

$$Df|_p = \left(\frac{\partial f_i}{\partial x_j}\right)_{m \times n}$$

having full rank means that permuting coordinates if necessary, we can assume that the first (or last) m columns are linearly independent.

Theorem (Implicit function theorem)**.** Let $p = (x_0, y_0) \in U$, $U \subset \mathbb{R}^k \times \mathbb{R}^l$ open and let $f : U \to \mathbb{R}^l$, $p \mapsto 0$ be a continuously differentiable map with $\left(\frac{\partial f_i}{\partial y_j}\right)_{l \times l}$ an isomorphism at p. Then there exists an open neighbourhood $x_0 \in V \subset \mathbb{R}^k$ and a continuously differentiable map $g : V \to \mathbb{R}^l$ such that if $(x, y) \in U \cap (V \times \mathbb{R}^l)$, then

$$f(x,y) = 0 \iff y = g(x)$$

If f is smooth so is g.

Proof. Introduce $F : U \to \mathbb{R}^k \times \mathbb{R}^l$, $(x, y) \mapsto (x, f(x, y))$. Then

$$DF = \begin{pmatrix} I & * \\ 0 * \frac{\partial f_i}{\partial y_j} & \end{pmatrix}$$

So $DF|_{(x_0,y_0)}$ is an isomorphism. So the inverse function theorem says that F is locally invertible near

$$F(x_0, y_0) = (x_0, f(x_0, y_0)) = (x_0, 0)$$

Take a product open neighbourhood $(x_0, 0) \in V \times V'$, $V \subset \mathbb{R}^t$ open, $V' \subset \mathbb{R}^l$ open and the continuously differentiable inverse $G : V \times V' \to U' \subset U \subset \mathbb{R}^k \times \mathbb{R}^l$ such that $F \circ G = \mathrm{id}_{V \times V'}$. Write

$$G(x, y) = (\varphi(v, y), \psi(x, y))$$

Then

$$\begin{aligned} F \circ G(x, y) &= (\varphi(x, y), f(\varphi(x, y), \psi(x, y)) \\ &= (x, y) \\ \implies \varphi(x, y) &= x \\ \text{and } f(x, \psi(x, y)) &= y \end{aligned}$$

when $(x, y) \in V \times V'$. Thus

$$f(x,y) = 0 \iff y = \psi(x0)$$

Define $g : V \to \mathbb{R}^l$ as $x \to \psi(x, 0)$, $v_0 \mapsto y_0$ and does what we want. $\square$

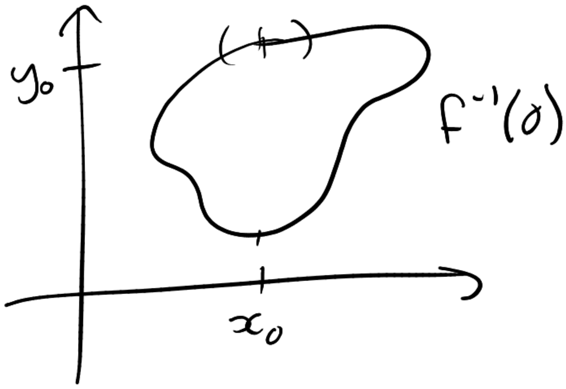
Examples

(1) $f : \mathbb{R}^2 \to \mathbb{R}$ smooth, $f(x_0, y_0) = 0$. Assume $\frac{\partial f}{\partial y}|_{(x_0,y_0)} \neq 0$. Then there exists smooth $g : (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$ such that

$$g(x_0) = g(y_0)$$

and

$$f(x,y) = 0 \iff y = g(x)$$



Since $f(x,g(x))=0$, chain rule implies

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}g'(x)=0$$
$$\implies g'(x)=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

(2) Let $f:\mathbb{R}^3\to\mathbb{R}$ smooth and $f(x_0,y_0,z_0)=0$, and assume $Df|_{(x_0,y_0,z_0)}\neq 0$. Permuting coordinates if necessary we may assume that $\left.\frac{\partial f}{\partial z}\right|_{(x_0,y_0,z_0)}\neq 0$. Then there exists an open neighbourhood $(x_0,y_0)\in V\subset\mathbb{R}^2$ and a smooth $g:V\to\mathbb{R}$, $g(x_0,y_0)=z_0$ such that for an open set $U\ni(x_0,y_0,z_0)$

$$f^{-1}(0)\cap U=\text{graph}(g)$$

$(\{(x,y,g(x,y)):(x,y)\in V\})$

We return to Theorem 1.7. Recall:

Theorem. For a subset $\Sigma\subset\mathbb{R}^3$, the following are equivalent (TFAE):

(a) Σ is a smooth surface in $\mathbb{R}^3$

(b) Σ is locally a graph over a coordinate plane.

(c) Σ is locally $f^{-1}(0)$, f smooth and $Df|_p\neq 0$

(d) Σ is locally the image of an allowable parametrisation $\sigma:V\to\Sigma\subset\mathbb{R}^3$, σ smooth, σ homeomorphism onto $\sigma(V)$ and $D\sigma$ injective.

Proof. (1) (b) implies all others.

If Σ is locally $\{(x, y, g(x, y)) : (x, y) \in V\}$ then we get a chart from projection Π_{xy} which is smooth and defined on an open neighbourhood of Σ, hence (b) implies (c).

Also, it is cut out by

$$f(x, y, z) = z - g(x, y)$$

Clearly $\frac{\partial f}{\partial z} = 1 \neq 0$, (b) implies (c). Also $\sigma(x, y) = (x, y, f(x, y))$ is allowable and smooth:

$$\sigma_x = (1, 0, g_x)$$
$$\sigma_y = (0, 1, g_y)$$

are linearly independent, (b) implies (d).

(2) (a) implies (d) is part of the definition of being a smooth surface in $\mathbb{R}^3$ and hence locally diffeomorphic to $\mathbb{R}^2$ (the inverse of the local diffeomorphism is the allowable parametrisation).

(3) (c) implies (b) was example number 2 above for the implicit function theorem.

(4) We'll show that (d) implies (b) and we're done. Let $p \in \Sigma$, $\sigma : V \to U \subset \Sigma$, $\sigma(0) = p \in U$.

$$\sigma = (\sigma_1(u, v), \sigma_2(u, v), \sigma_3(u, v))$$
$$D\sigma = \begin{pmatrix} \frac{\partial \sigma_1}{\partial u} & \frac{\partial \sigma_1}{\partial v} \\ \frac{\partial \sigma_2}{\partial u} & \frac{\partial \sigma_2}{\partial v} \\ \frac{\partial \sigma_3}{\partial u} & \frac{\partial \sigma_3}{\partial v} \end{pmatrix}$$

So there exists 2 rows defining an invertible matrix. Suppose first two rows

$$\Pi_{xy} \circ \sigma : V \to \mathbb{R}^2, D(\Pi_{xy} \circ \sigma)|_0$$

isomorphism. Then inverse function theorem implies locally invertible, let $\phi = \Pi_{xy} \circ \sigma$. Σ is now the graph of

$$(x, y, \sigma(\phi^{-1}(x, y)) = (\sigma_1(u, v), \sigma_2(u, v), \sigma_3(u, v)) \in \Sigma \qquad \square$$

Start of lecture 7

Examples

(1) The *ellipsoid* $E \subset \mathbb{R}^3$ is $f^{-1}(0)$, for $f : \mathbb{R}^3 \to \mathbb{R}$

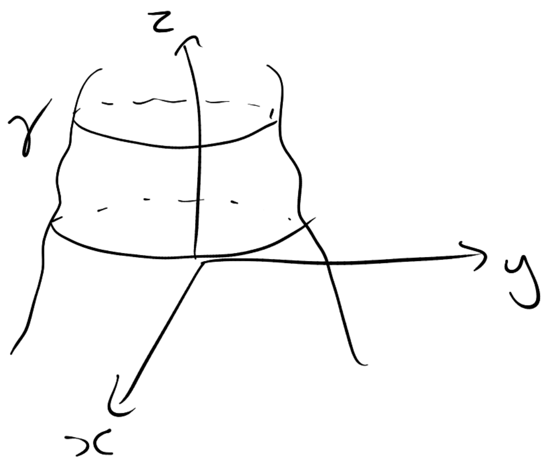
$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

$\forall p \in E = f^{-1}(0)$, $Df|_p \neq 0$, so E is a smooth surface in $\mathbb{R}^2$.

(2) Surfaces of revolution.
Let $\gamma : [a, b] \to \mathbb{R}^3$ be a smooth map with image in the xz-plane

$$\gamma(t) = (f(t), 0, g(t))$$

Assume γ is injective, $\gamma'(t) \neq 0 \forall t$ and $f > 0$. Rotate this curve around the z-axis.



The associated surface of revolution as *allowable* parametrization

$$\sigma(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$$

$$(u, v) \subseteq (a, b) \times (\theta, \theta + 2\pi)$$

$\theta \in [0, 2\pi)$ fixed. σ homemorphic onto its image (check!)

$$\sigma_u = (f' \cos v, f' \sin v, g')$$

$$\sigma_v = (-f \sin v, f \cos v, 0)$$

and $\|\sigma_u \times \sigma_v\|^2 = f^2(f'^2 + g'^2) \neq 0$.

1.3 Orientability

$V, V' \subset \mathbb{R}^2$ open, $f : V \to V'$ a diffeomorphism. Then at any $x \in V$, $Df|_x \in \mathrm{GL}_2(\mathbb{R})$. Let $\mathrm{GL}_2^+(\mathbb{R}) \subset \mathrm{GL}_2(\mathbb{R})$ be the subgroup of matrices of *positive* determinant.

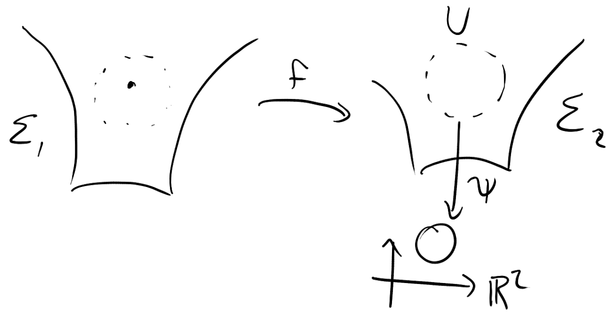
Definition. We say that f is *orientation preserving* if $Df|_x \in \mathrm{GL}_2^+(\mathbb{R})$ for all $x \in V$.

Definition. An abstract smooth surface Σ is *orientable* if it admits an atlas such that the transition maps are orientation preserving diffeomorphisms of open sets of $\mathbb{R}^2$.

A choice of such an atlas is an *orientation* of Σ and we say that Σ is oriented.

Lemma 1.8. If Σ_1 and Σ_2 are abstract smooth surfaces and they are diffeomorphic then Σ_1 is orientable if and only if Σ_2 is orientable.

Proof. Suppose $f : \Sigma_1 \to \Sigma_2$ is a diffeomorphism and Σ_2 is orientable and equipped with an oriented atlas



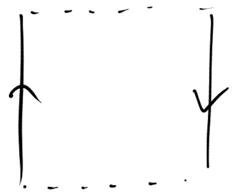
Consider the atlas on Σ_1 given by $(f^{-1}U, \psi \circ f|_{f^{-1}U})$, where (U, ψ) is a chart of Σ_2. The transition function between two such charts is exactly the transition function in the Σ_2 atlas. □

Remarks

(1) There is no sensible classification of all smooth or topological surfaces, for example $\mathbb{R}^2 \setminus Z$ for Z closed realises all kinds of different homeomorphic types.

By contrast, *compact* smooth surfaces up to diffeomorphism are classified by Euler characteristic.

(2) There's a definition of orientation preserving homeomorphism that needs some algebraic topology. The Möbius band is the surface:



It turns out that abstract smooth surface is orientable $\iff$ it contains no sub-surface homeomorphic to a Möbius band.

(3) We get other structure by demanding transition maps to be such that

$$D(\varphi_1\varphi_2^{-1})|_x \in G \subset \mathrm{GL}_2(\mathbb{R})$$

$G = \mathrm{GL}_1(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{R}) \to$ Riemann surfaces.

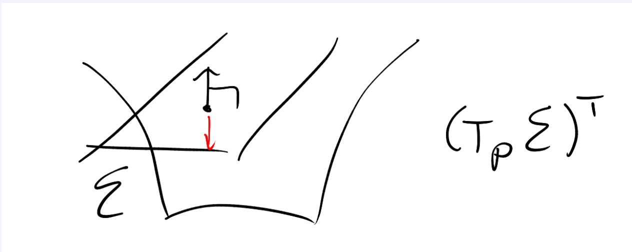
Examples

(1) S^2 with the atlas of 2 stereographic projections, you computed the transition map $(u,v) \mapsto \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$ on $\mathbb{R}^2 \setminus \{0\}$. Check: this is orientation preserving.

(2) T^2, transition maps are translations of $\mathbb{R}^2$, so T^2 is orientated.

For surfaces in $\mathbb{R}^3$ we'd like to have orientability dictated by some "ambient feature".

Definition. $\Sigma \subset \mathbb{R}^3$ smooth surface, $p \in \Sigma$. Fix an allowable parametrization $\sigma : V \to U \subset \Sigma$, $\sigma(0) = p$. Then the *tangent plane* $T_p\Sigma$ of Σ at p is $\mathrm{Im}(D\sigma|_0) \subset \mathbb{R}^3$ a 2D vector subspace of $\mathbb{R}^2$. The *affine* tangent plane of Σ at p, is $p + T_p\Sigma \subset \mathbb{R}^3$.



Lemma 1.9. $T_p\Sigma$ is well-defined, i.e. it is independent of the choice of allowable parametrisation near p.

Start of lecture 8

Proof. $\sigma : V \to U \subset \Sigma$, $\sigma(0) = p$, $\tilde{\sigma} : \tilde{V} \to \tilde{U} \subset \Sigma$, $\tilde{\sigma}(0) = p$. Transition map $\sigma^{-1} \circ \tilde{\sigma}$ implies:

$$\tilde{\sigma} = \sigma \circ (\sigma^{-1} \circ \tilde{\sigma})$$

So $D(\sigma^{-1} \circ \tilde{\sigma})|_0$ isan isomorphism. Chain rule implies

$$\mathrm{Im}(D\tilde{\sigma}|_0) = \mathrm{Im}(D\sigma|_0)$$

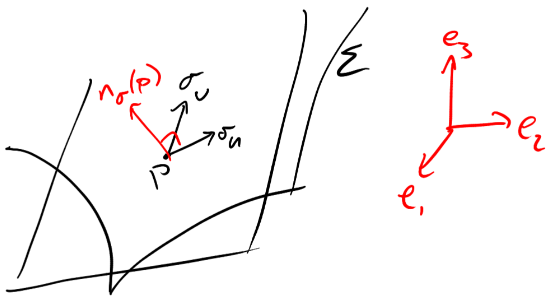
□

Definition. $\Sigma \subset \mathbb{R}^3$. The normal direction at p is $(T_p\Sigma)^\perp$ (Euclidean orthogonal ocmplement to $T_p\Sigma$). At each $p \in \Sigma$ we have two unit normal vectors.

Definition. A smooth surface in $\mathbb{R}^3$ is *two-sided* if it admits a continuous global choice of unit normal vectors.

Lemma 1.10. A smooth surface in $\mathbb{R}^3$ is orientable with its abstract smooth surface structure if and only if it is two-sided.

Proof. Let $\sigma : V \to U \subset \Sigma$ be allowable. Define the *positive* unit normal with respect to σ at p as the unique $n_\sigma(p)$ such that $\{\sigma_u, \sigma_v, n_\sigma(p)\}$ and $\{e_1, e_2, e_3\}$ induce the same orientation in $\mathbb{R}^3$ (ie they are related by a choice of basis matrix with positive determinant).



Explicitly:

$$n_\sigma(p) = \frac{\sigma_n \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

Let $\tilde{\sigma}$ be another allowable parametrisation.

$$\tilde{\sigma} : \tilde{V} \to \tilde{U} \subset \Sigma$$

and suppose Σ is orientable as an abstract smooth surface with σ and $\tilde{\sigma}$ belonging to the same oriented atlas.

$$\sigma = \tilde{\sigma} \circ \varphi \qquad \varphi = \tilde{\sigma}^{-1} \circ \sigma$$

and

$$D\varphi|_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Chain rule:

$$\sigma_u = \alpha \tilde{\sigma}_{\tilde{w}} + \gamma \tilde{\sigma}_{\tilde{v}} \sigma_v \qquad\qquad = \beta \tilde{\sigma}_{\tilde{w}} + \delta \tilde{\sigma}_{\tilde{v}}$$

and

$$\sigma_u \times \sigma_v = \underbrace{\det(D\varphi|_0)}_{>0} \tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}} \tag{$*$}$$

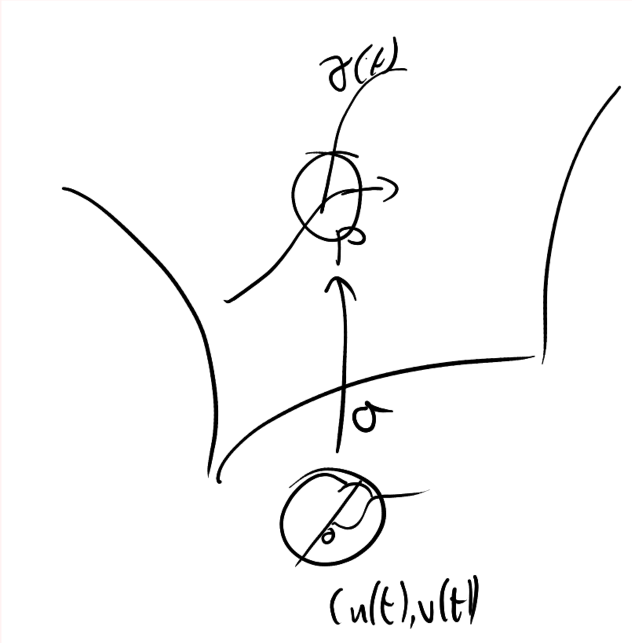
So the positive unit normal at p was intrinsic and does not depend on the parametrisation. Since

$$\frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

is continuous, Σ is 2-sided.

Conversely, if Σ is 2-sided we have a global choice of n, so we can consider the subatlas of the smooth atlas such that we have a chart $(U, \varphi) = \varphi^{-1} = \sigma$ and $\{\sigma_u, \sigma_v, u\}$ is an orientable basis of $\mathbb{R}^3$. $(*)$ shows that transition maps between such charts are orientation preserving. Hence Σ is orientable. □

Remark. Given $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ smooth with $\mathrm{Im}(\gamma) \subset \Sigma$ and $\gamma(0) = p$



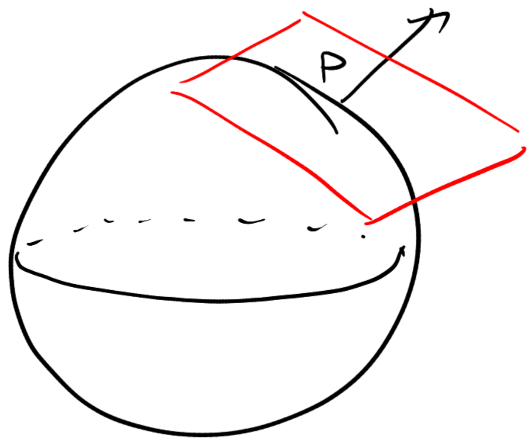
$$\gamma(t) = \sigma(u(t), v(t))$$
$$\gamma'(0) = D\sigma|_0(u'(0), v'(0)) \in T_p\Sigma$$

This gives that

$$\begin{aligned} T_p\Sigma &= \{\gamma'(0) : \text{with } \gamma \text{ as above}\} \\ &= \text{"tangent vector to curves in } \Sigma\text{"} \end{aligned}$$

Examples

(1) $S^2 \subset \mathbb{R}^3$

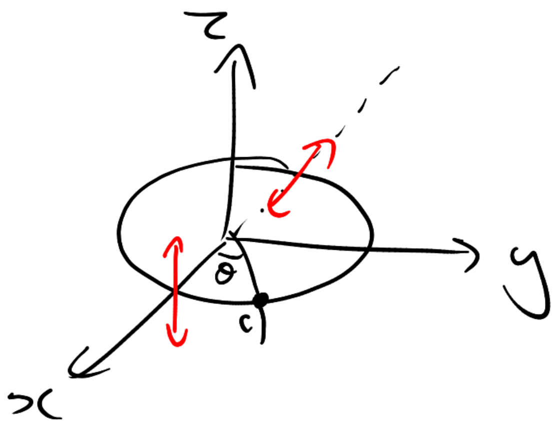


Take any $\gamma : (-\varepsilon, \varepsilon) \to S^2$, $\gamma(0) = p$, $|\gamma(t)|^2 = 1$. Differentiation at $t = 0$

$$\langle \gamma'(0), p \rangle = 0$$
$$\implies (T_p S^2)^\perp = \mathbb{R}p$$
$$\implies n(p) = p$$

S^2 is 2-sided.

(2) The Möbius band, start with unit circle in the xy-plane.



and take an open interval of length 1. Rotate this line in the cz-plane s we move around the circle such that it has rotate by $\frac{\theta}{2}$ after moving an angle θ in the circle (see picture). After a full turn the segment returns to its original position by iwth end points inverted. We can describe the surface with

$$\sigma(t,\theta) = \left(\left(1 - t \sin \frac{\theta}{2}\right) \cos\theta, \left(1 - t\sin\frac{\theta}{2}\right)\sin\theta, t\cos\frac{\theta}{2} \right)$$

where (t, θ) belongs to

$$V_1 = \left\{ t \in \left(-\frac{1}{2}, \frac{1}{2} \right), \theta \in (0, 2\pi) \right\}$$

or

$$V_2 = \left\{ t \in \left(-\frac{1}{2}, \frac{1}{2} \right), \theta \in (-\pi, \pi) \right\}$$

Check: if σ_i is σ on V_i then σ_i is allowable. A computation shows

$$\sigma_t \times \sigma_\theta(0, \theta) = \left(-\cos\theta\cos\frac{\theta}{2}, -\sin\theta\cos\frac{\theta}{2}, -\sin\frac{\theta}{2} \right) := n_\theta$$

As $\theta \to 0^+$, $h_\theta \to (-1, 0, 0)$. As $\theta \to 2\pi^-$, $n_\theta \to (1, 0, 0)$. Hence the Möbius band is *not* 2-sided.

2 Surfaces in 3-space

$\gamma : (a, b) \to \mathbb{R}^3$ smooth, the length of γ is

$$L(\gamma) = \int_a^b \|\gamma'(t)\| \mathrm{d}t$$

If $s : (A, B) \to (a, b)$ is monotone increasing and we let $\tau(t) = \gamma(s(t))$, then

$$\begin{aligned} L(\tau) &= \int_A^B \|\tau'(t)\| \mathrm{d}t \\ &= \int_A^B \|\gamma'(s(t))\| |\underbrace{s'(t)}_{\ge 0}| \mathrm{d}t \\ &= \int_a^b \|\gamma'(s)\| \mathrm{d}s \\ &= L(\gamma) \end{aligned}$$

Lemma 2.1. If $\gamma : (a, b) \to \mathbb{R}^3$ and $\gamma'(t) \neq 0$ for all t, then γ can be paramaterised by *arc-length*, i.e. a parameter such that $\|\gamma'(s)\| = 1$ for all s.

Start of lecture 9

$\Sigma \subset \mathbb{R}^3$, $\sigma : V \to U \subset \Sigma$ allowable. If $\gamma : (a, b) \to U$ is smooth, write

$$\boxed{\gamma(t) = \sigma(u(t), v(t))}$$

with $(u, v) : (a, b) \to V$ smooth. Then

$$\gamma'(t) = \sigma_u u'(t) + \sigma_v v'(t)$$

then

$$\|\gamma'(t)\|^2 = E(u'(t))^2 + 2Fu'(t)v'(t) + G(v'(T))^2$$

where

$$\begin{aligned} E &= \langle \sigma_u, \sigma_u \rangle = \|\sigma_u\|^2 \\ F &= \langle \sigma_u, \sigma_v \rangle = \langle \sigma_v, \sigma_u \rangle \\ G &= \langle \sigma_v, \sigma_v \rangle = \|\sigma_v\|^2 \end{aligned}$$

(smooth functions on V). They depend only on σ and *not* γ.

Definition. The *first fundamental form* (FFF) in the parametrisation σ is the expression

$$E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2$$

The notation is designed to remind you that

$$L(\gamma) = \int_a^b \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\mathrm{d}t,$$

where $\gamma(t) = \sigma(u(t), v(t))$.

Remark. The FFF is sometimes defined as the quadratic form in $T_p\Sigma$ given by the restriction of the standard inner product in $\mathbb{R}^3$.

$$I_p(w) = |w|^2 = \langle w, w\rangle_{\mathbb{R}^3}, w \in T_p\Sigma$$

After picking σ, $\sigma(0) = p$ and after writing $w = D\sigma|_0(u', v')$ we have

$$I_p(w) = Eu'^2 + 2Fu'v' + Gv'^2$$

This is an example of a *Riemannian metric*.

Examples

(1) The xy-plane $\mathbb{R}^3$:

$$\sigma(u, v) = (u, v, 0)$$

so

$$\sigma_u = (1, 0, 0), \qquad \sigma_v = (0, 1, 0)$$

and FFF is $\mathrm{d}u^2 + \mathrm{d}v^2$, $E = G = 1$, $F = 0$. In polar coordinates

$$\sigma(r, \theta) = (r\cos\theta, r\sin\theta, 0)$$

$r \in (0, \infty$, $(0, 2\pi)$, we have

$$\sigma_r = (\cos\theta, \sin\theta, 0)$$
$$\sigma_\theta = (-r\sin\theta, r\cos\theta, 0)$$

and FFF is $\mathrm{d}r^2 + r^2\mathrm{d}\theta^2$, $E = 1$, $F = 0$, $G = r^2$.

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

Definition. $\Sigma, \Sigma' \subset \mathbb{R}^3$ smooth surfaces. We say that Σ and Σ' are *isometric* if there exists $f : \Sigma \to \Sigma'$ diffeomorphic such that for every smooth curve $\gamma : (a, b) \to \Sigma$,

$$L_\Sigma(\gamma) = L_{\Sigma'}(f \circ \gamma)$$

Example. If $\Sigma' = f(\Sigma)$ where $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a "rigid motion" i.e. $f(x) = Ax + b$, $A \in O(3)$, $b \in \mathbb{R}^3$ (so f preserves $\langle \bullet, \bullet \rangle_{\mathbb{R}^3}$). $f : \Sigma \to \Sigma'$ *isometry* because

$$\begin{aligned} |(f \circ \gamma)'(t)| &= |A\gamma'(t)| \\ &= |\gamma'(t)| \end{aligned}$$

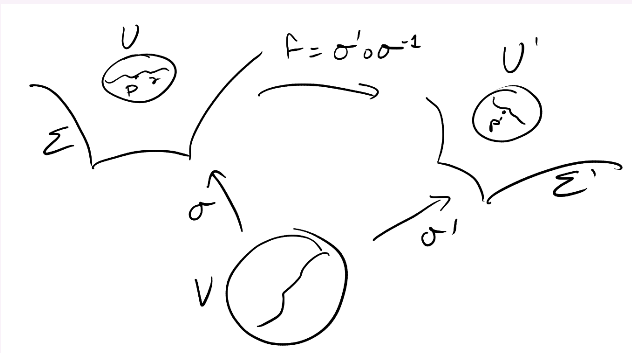
hence lengths of curves are preserved. Often we're interested in local statements.

Definition. Σ, Σ' are *locally isometric* (near points $p \in \Sigma$ and $p' \in \Sigma'$) if there exists open neighbourhoods $p \in U \subset \Sigma$ and $p' \in U' \subset \Sigma'$ which are isometric.

Lemma 2.2. $\Sigma, \Sigma' \subset \mathbb{R}^3$ are locally isometric near $p \in \Sigma$ and $p' \in \Sigma'$ if and only if there exists allowable parametrisation

$$\begin{aligned} \sigma : V \to U \subset \Sigma \\ \sigma' : V \to U' \subset \Sigma' \end{aligned}$$

for which the FFFs are equal in V ($E = E'$, $F = F'$, $G = G'$).



Proof. We know (by definition) that the FFF of σ determines the lengths of all curves on $\sigma(V) = U$. If we have σ and σ' as in the lemma, then

$$\sigma' \circ \sigma^{-1} : U \to U'$$

is an isometry since

$$\sigma^{-1}(\gamma(t)) = (u(t), v(t))$$

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\sigma' \circ \sigma^{-1}}_{f} \circ \gamma\right|^2 &= \left|\frac{\mathrm{d}}{\mathrm{d}t}\sigma'(u(t), v(t))\right|^2 \\
&= E'\dot{u}^2 + 2F'\dot{u}\dot{v} + G'\dot{v}^2 \\
&= E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 \\
&= \left|\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\right|^2 \implies l(\sigma' \circ \sigma^{-1} \circ \gamma) &= L(\gamma)
\end{aligned}$$

For the converse, we'll show first that the lengths of the curves in U determine the FFF of σ.

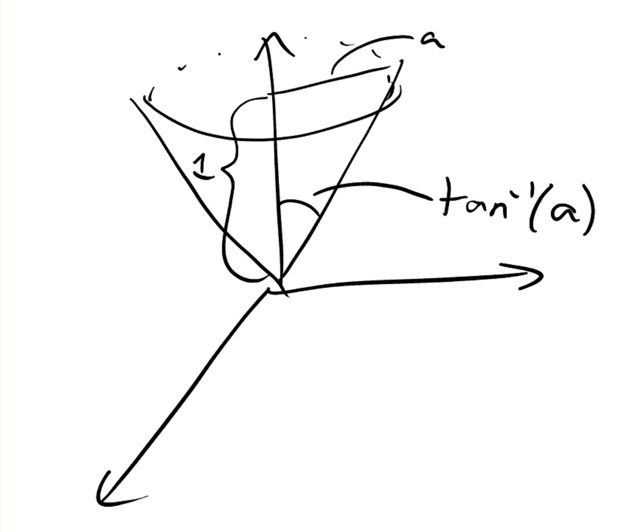
$$\sigma : (0, f) \to U \subset \Sigma, \sigma(0) = p$$

$\gamma_\varepsilon : [0, \varepsilon] \to U$, $t \mapsto \sigma(t, 0)$. Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}L(\gamma_\varepsilon) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_0^\varepsilon \sqrt{E(t,0)}\mathrm{d}t \\
&= \sqrt{E(\varepsilon, 0)}
\end{aligned}$$

So length of curve determine $E(0,0)$. Similarly, $\chi_\varepsilon : [0, \varepsilon] \to U$, $t \mapsto \sigma(0,t)$ determine $G(0,0)$. Finally considering $\lambda_\varepsilon : [0, \varepsilon] \to U$, $t \mapsto \sigma(t,t)$ we get $\sqrt{(E + 2F + G)(0,0)}$, so knowing E and G we get F. So if $f : U \to U'$ is a local isometry take any allowable parametrization $\sigma' : V \to U'$, then $\sigma = f^{-1} \circ \sigma'$ is such that the FFFs of σ and σ' agree. □

Example (Cone)**.** $u > 0$, $v \in (0, 2\pi)$.

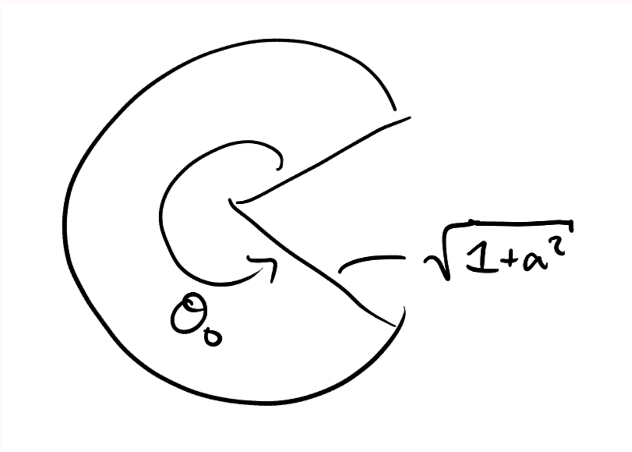


$$\sigma(u, v) = (au \cos v, au \sin v, u)$$

parametrizes the complement of the line of the cone. FFF:

$$(1 + a^2)\mathrm{d}u^2 + a^2u^2\mathrm{d}v^2$$

If we cut open the cone and unfold it we get a plane such that



$\theta_0 = \frac{2\pi a}{\sqrt{1+a^2}}$. Parametrize the plane sector by

$$\sigma(r, \theta) = \left(\sqrt{1+a^2} r \cos\left(\frac{a\theta}{\sqrt{1+a^2}}\right), \sqrt{1+a^2} r \sin\left(\frac{a\theta}{\sqrt{1+a^2}}\right), 0\right)$$

$r > 0$, $\theta \in (0, 2\pi)$ Check: FFF: $(1 + a^2)\mathrm{d}r^2 + r^2a^2\mathrm{d}\theta^2$, $V = (0, \infty) \times (0, 2\pi)$. So the cone is locally isometric to the plane!

> **Remark.** The cone can't be *globally isometric* to the plane, since they are not even homeomorphic

Start of lecture 10

$\Sigma \underbrace{\overset{3}{\mathbb{R}}}$, $p \in \Sigma$. Take allowable parametrisations:

$$\begin{aligned} \sigma &: V \to U \subset \Sigma, \qquad \sigma(0) = p \\ \tilde{\sigma} &: \tilde{V} \to U \subset \Sigma, \qquad \tilde{\sigma}(0) = p \end{aligned}$$

Transition map $f : \tilde{\sigma}^{-1} \circ \sigma : V \to \tilde{V}$. We have FFFs:

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \qquad \text{for } \sigma$$

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} \qquad \text{for } \tilde{\sigma}$$

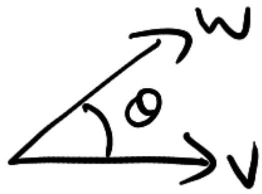
> **Lemma 2.3.** If $f = \tilde{\sigma}^{-1} \circ \sigma$ is a transition map, then
>
> $$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = (Df)^\top \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} Df$$
>
> ($^\top$ is transpose)

Proof.

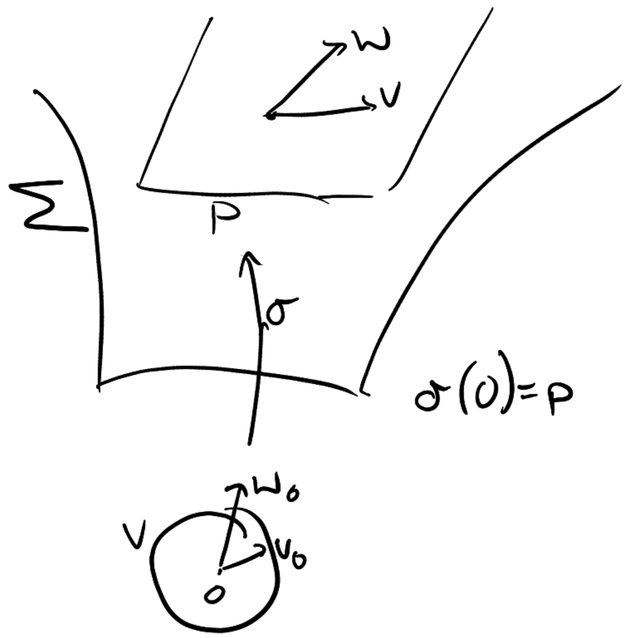
$$\begin{aligned} \begin{pmatrix} E & F \\ F & G \end{pmatrix} &= \begin{pmatrix} \sigma_u \cdot \sigma_u & \sigma_u \cdot \sigma_v \\ \sigma_v \cdot \sigma_u & \sigma_v \cdot \sigma_v \end{pmatrix} \\ &= \begin{pmatrix} \sigma_u & \sigma_u \\ \sigma_v & \sigma_v \end{pmatrix} \begin{pmatrix} \sigma_u & \sigma_v \\ \sigma_u & \sigma_v \end{pmatrix} \\ &= (D\sigma)^\top (D\sigma) \\ &= (D\tilde{\sigma} Df)^\top (D\tilde{\sigma} Df) && \text{using} \sigma = \tilde{\sigma} \circ f \\ &= (Df)^\top (D\tilde{\sigma})^\top D\tilde{\sigma} Df \\ &= (Df)^\top \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} Df \end{aligned}$$

□

FFF and Angles



$v, w \in \mathbb{R}^3$, $v \cdot w = |v||w| \cos\theta$. If $v, w \in T_p\Sigma$, $\cos\theta = \frac{v \cdot w}{|v||w|}$ $(*)$.



$$w = D\sigma|_0(w_0)$$
$$v = D\sigma|_0(v_0)$$
$$v \cdot w = v_0^\top \begin{pmatrix} E & F \\ F & G \end{pmatrix} w_0$$

So using $(*)$ we can compute angles using the FFF of σ.

Lemma 2.4. σ is *conformal* (preserves angles) exactly when $E = G$ and $F = 0$.

Proof. Consider curves

$$\alpha(t) = (u(t), v(t))$$
$$\tilde{\alpha}(t) = (\tilde{u}(t), \tilde{v}(t))$$

$\alpha(0) = \tilde{\alpha}(0) \in V$. The curves $\sigma \circ \alpha$ and $\sigma \circ \tilde{\alpha}$ meet at p with angle θ given by

$$\cos\theta = \frac{E\dot{u}\dot{\tilde{u}} + F(\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v}) + G\dot{v}\dot{\tilde{v}}}{(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2}(E\dot{\tilde{u}}^2 + 2F\dot{\tilde{u}}\dot{\tilde{v}} + G\dot{\tilde{v}}^2)^{1/2}} \qquad (\dagger)$$

If σ is conformal and $\alpha(t) = (t, 0)$, $\tilde{\alpha}(t) = (0, 1)$ meeting at angle $\frac{\pi}{2}$ in V, we get using ($\dagger$)

$$0 = F$$

Similarly using $\alpha(t) = (t, t)$ and $\tilde{\alpha}(t) = (t, -t)$ we get $E = G$.

Conversely, if σ is such that $E = G$ and $F = 0$ then with respect to the FFF is just

$$\rho(\mathrm{d}u^2 + \mathrm{d}v^2),$$

where $\rho = E = G : V \to \mathbb{R}$. From ($\dagger$) we see that

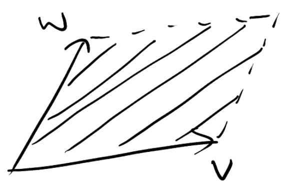
$$\cos\theta = \frac{\dot{u}\dot{\tilde{u}} + \dot{v}\dot{\tilde{v}}}{(\dot{u}^2 + \dot{v}^2)^{1/2}(\dot{\tilde{u}}^2 + \dot{\tilde{v}}^2)^{1/2}}$$

i.e. angles do *not* change. □

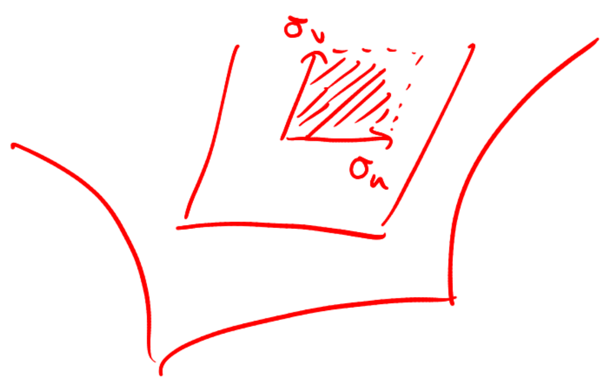
Areas

Recall that the area of a parallelogram spanned by vectors v and w is

$$|v \times w| = (|v|^2|w|^2 - (v \cdot w)^2)^{1/2}$$



Suppose we have $\sigma : V \to U \subset \Sigma$, $\sigma(0) = p$ and consider $\sigma_u, \sigma_v \in T_p\Sigma$



They span a parallelogram in $T_p\Sigma$ of area:

$$(|\sigma_u|^2|\sigma_v|^2 - (\sigma_u \cdot \sigma_v)^2)^{1/2} = \sqrt{EG - F^2}$$

Definition. $\mathrm{Area}(U) = \int_V \sqrt{EG - F^2}\mathrm{d}u\mathrm{d}v$.

Note. Suppose $\sigma : V \to U$, $\tilde{\sigma} : \tilde{V} \to U$ allowable parametrisations. $\tilde{\sigma} = \sigma \circ \varphi$, $\varphi = \sigma^{-1} \circ \tilde{\sigma}$, $\varphi : \tilde{V} \to V$ transition map. By Lemma 2.3:

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = (D\varphi)^\top \begin{pmatrix} E & F \\ F & G \end{pmatrix} D\varphi$$

Hence (taking determinants)

$$\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2} = |\det(D\varphi)|\sqrt{EG - F^2}$$

Note the change of variables formula for integration shows that

$$\int_{\tilde{V}} \sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}\mathrm{d}\tilde{u}\mathrm{d}\tilde{v} = \int_V \sqrt{EG - F^2}\mathrm{d}u\mathrm{d}v$$

so $\mathrm{Area}(U)$ is *instrinsic* and well-defined.

Example. Consider the graph

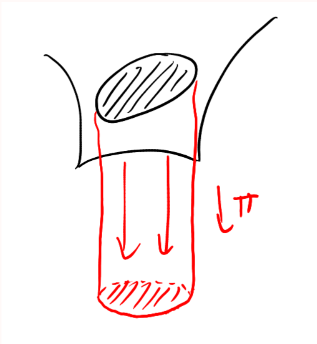
$$\Sigma = \{(u, v, f(u, v)) : (u, v) \subset \mathbb{R}^2\}$$

with $f : \mathbb{R}^2 \to \mathbb{R}$ smooth.

$$\begin{aligned}\sigma(u, v) &= (u, v, f(u, v)) \\ \sigma_u &= (1, 0, f_u) \\ \sigma_v &= (0, 1, f_v) \\ EG - F^2 &= 1 + f_u^2 + f_v^2\end{aligned}$$

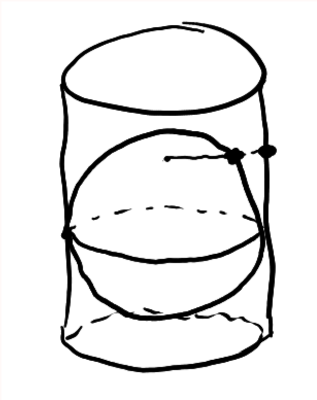
If $U_R \subset \Sigma$ is $\sigma(B(0, R))$ then

$$\begin{aligned}\text{Area}(U_R) &= \int_{B(0,R)} \sqrt{1 + f_u^2 + f_v^2} \mathrm{d}u\mathrm{d}v \\ &\geq \pi R^2\end{aligned}$$



With equality only when $f_u = f_v = 0$ in $B(0, R)$, i.e. only when U_R is a subset of a $z =$ const plane. So projection from Σ to $\mathbb{R}^2_{xy}$ is *not* area preserving unless Σ is a plane parallel to $\mathbb{R}^2_{xy}$.

Example. Contrast (Archimedes).

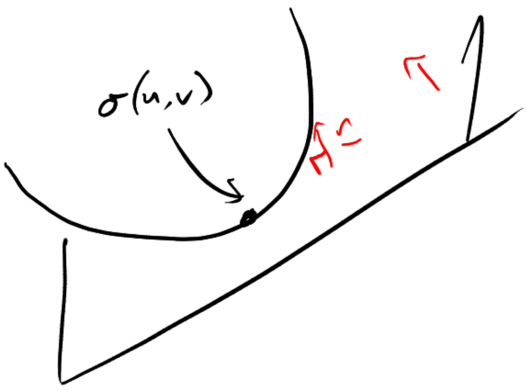


The horizontal radial projection (with centre the z-axis) from S^2 to the cylinder is *area-preserving* (see Example Sheet 2).

Start of lecture 11

The Second Fundamental Form

Let's try to measure how much $\Sigma \subset \mathbb{R}^3$ deviates from its own tangent planes.



Let's take $\sigma : V \to U \subset \Sigma$. Using Taylor's theorem

$$\sigma(u+h, v+l) = \sigma(u,v) + h\sigma_u(u,v) + l\sigma_v(u,v) + \frac{1}{2}(h^2\sigma_{uu}(u,v) + 2hl\sigma_{uv}(u,v) + l^2\sigma_{vv}(u,v)) + O(h^3, l^3)$$

where (h, l) are small enough so that (u, v) and $(u+h, v+l) \in V$. Let's take projection in the normal direction:

$$\langle n, \sigma(u+h), v+l) - \sigma(u,v)\rangle = \frac{1}{2}(\langle n, \sigma_{uu}\rangle h^2 + 2\langle n, \sigma_{ub}\rangle hl + \langle n, \sigma_{vv}\rangle l^2) + O(h^3, l^3)$$

Definition. The *second fundamental form* of $\Sigma \subset \mathbb{R}^3$ in the parametrisation σ is the quadratic form:

$$L\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2$$

where

$$\begin{aligned} L &= \langle n, \sigma_{uu} \rangle \\ M &= \langle n, \sigma_{uv} \rangle \\ N &= \langle n, \sigma_{vv} \rangle \end{aligned}$$

and $n = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$.

Lemma 2.5. Let V be connected and $\sigma : V \to U \subset \Sigma$ such that second FFF vanishes identically. Then U lies in an affine plane in $\mathbb{R}^3$.

Proof. Recall $\langle n, \sigma_u \rangle = \langle n, \sigma_v \rangle = 0$. Hence

$$\begin{aligned} \langle n_u, \sigma_u \rangle + \langle n, \sigma_{uu} \rangle &= 0 \\ \langle n_v, \sigma_v \rangle + \langle n, \sigma_{vv} \rangle &= 0 \\ \langle n_v, \sigma_u \rangle + \langle n, \sigma_{uv} \rangle &= 0 \end{aligned}$$

$$\begin{aligned} L &= \langle n, \sigma uu \rangle = -\langle n_u, \sigma_u \rangle \\ M &= \langle n, \sigma_{uv} \rangle = -\langle n_v, \sigma_u \rangle = -\langle n_u, \sigma_v \rangle \\ N &= \langle n, \sigma_{vv} \rangle = -\langle n_v, \sigma_v \rangle \end{aligned}$$

So if second FFF vanishes then n_u is orthogonal to σ_u and σ_v. Also $\langle n, n \rangle = 1$ implies (differentiate with respect to u) $\langle n, n_u \rangle = 0$. Hence n_u is orthogonal to $\{n, \sigma_u \sigma_v\}$, hence $n_u \equiv 0$. Similarly $n_u \equiv 0$. So n is *constant* (V connected and mean value inequality). This implies that $\langle \sigma, n \rangle =$ constant (since $\langle \sigma, n_u \rangle = \langle \sigma_u, n \rangle = \langle \sigma, n_v \rangle = \langle \sigma_v, n \rangle = 0$) and U is contained in a plane. $\square$

Remark. Recall that FFF in the parametrisation σ has

$$(D\sigma)^\top D\sigma = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \sigma_u \cdot \sigma_v & \sigma_u \cdot \sigma_v \\ \sigma_v \cdot \sigma_u & \sigma_v \cdot \sigma_v \end{pmatrix}$$

Analogously the second FFF:

$$-(Dn)^\top D\sigma = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} n_u \cdot \sigma_u & n_u \cdot \sigma_v \\ n_v \cdot \sigma_u & n_v \cdot \sigma_v \end{pmatrix}$$

(using the alternative exrpessions for L, M and N in the previous proof). So if $\sigma : V \to U \subset \Sigma$, $\tilde{\sigma} : \tilde{V} \to U \subset \Sigma$ are 2 parametrisations with transition map

$$\varphi : \tilde{V} \to V, \qquad \varphi = \sigma^{-1} \circ \tilde{\sigma}$$

($\tilde{\sigma} = \sigma\varphi$) then

$$n_{\tilde{\sigma}}(\tilde{u}, \tilde{v}) = \pm n_\sigma(\varphi(\tilde{u}, \tilde{v}))$$

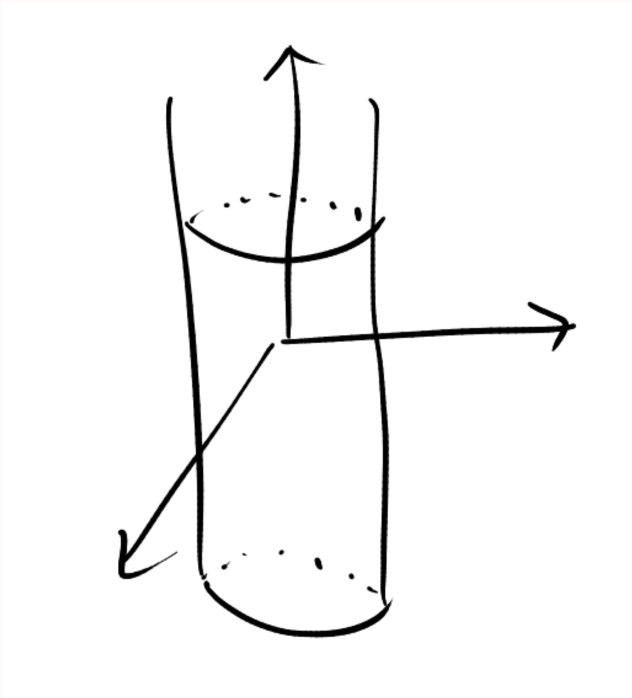
(use the $-$ sign if $\det(D\varphi) < 0$). Recall the discusssion on orientabioilty Lemma 1.10).

$$\begin{aligned} \begin{pmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{pmatrix} &= -(Dn_{\tilde{\sigma}})^\top D\tilde{\sigma} \\ &= \pm(D\varphi)^\top \begin{pmatrix} L & M \\ M & N \end{pmatrix} D\varphi \end{aligned}$$

(use $-$ sign if φ is orientation preserving).

Example. The cylinder has

$$\sigma(u, v) = (a \cos u, \sin u, v)$$



Note $\sigma_{uv} = \sigma_{vv} = 0$ hence $M = N = 0$. Check that second FFF is

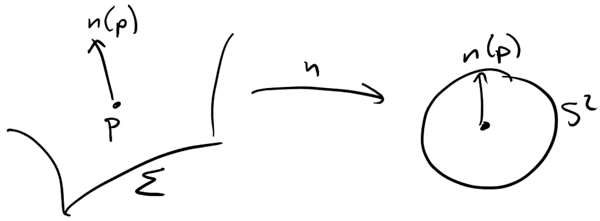
$$\begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix}$$

$(-adu^2)$.

Definition (The gauss map)**.** Let $\Sigma \subset \mathbb{R}^3$ be a smooth *oriented* surface. The *Gauss map* is

$$n : \Sigma \to S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$$

is the map $p \mapsto n(p)$, the unit normal vector at p (well-defined by the orientation of Σ).

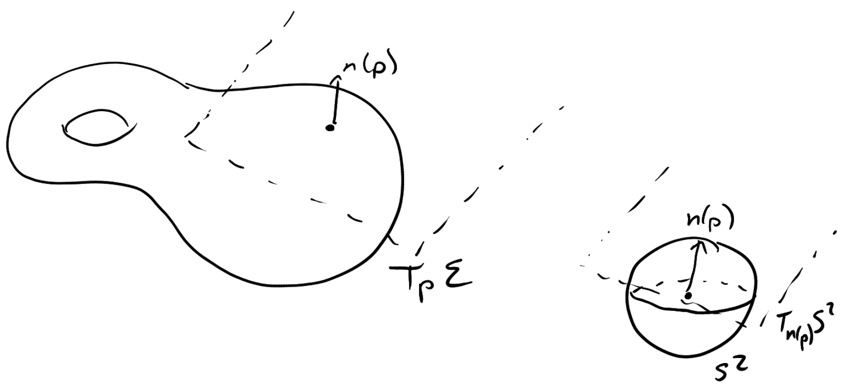


Lemma 2.6. The Gauss map $n : \Sigma \to S^2$ is *smooth*.

Proof. Smoothness can be checked locally. If $\sigma : V \to U \subset \Sigma$ is allowable and compatible with the orientation, then at $\sigma(u, v) = p \in \Sigma$

$$n(\sigma(u, v)) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

(smooth since σ is). $n \circ \sigma : V \to S^2 \subseteq \mathbb{R}^3$. □



Note. $T_p\Sigma = T_{n(p)}S^2$. Thus we can view

$$Dn|_p : T_p\Sigma \to T_{n(p)}S^2 = T_p\Sigma$$

(recall Q9, Example Sheet 1).

We can also view $Dn|p$ acting on tangent vectors in terms of curves

$$\gamma : (-\varepsilon, \varepsilon) \to \Sigma, \qquad \gamma(1) = p, \quad \gamma'(0) = v$$

$$Dn|_p(v) = Dn|_p(\gamma'(0)) = (n \circ \gamma)'(0)$$

(by chain rule).

Start of lecture 12

Recall FFF $I_p : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$, $I_p(v, w) = \langle v, w \rangle_{\mathbb{R}^3}$.

Lemma 2.7. $Dn|_p : T_p\Sigma \to T_p\Sigma$ is self-adjoint with respect to I_p, i.e.

$$I_p(Dn|_p(v), w) = I_p(v, Dn|_p(w))$$

Proof. Take σ a parametrisation with $\sigma(0) = p$. Then $\{\sigma_u, \sigma_v\}$ is a basis of $T_p\Sigma$. To prove self-adjoint if suffices to check that

$$\langle n_u, \sigma_v \rangle = \langle \sigma_u, n_v \rangle$$

(because $n_u = Dn|_p(\sigma_u)$ and $n_v = Dn|_p(\sigma_v)$). Note that

$$\langle n, \sigma_u \rangle = \langle n, \sigma_v \rangle = 0$$

Differentiate the first term with respect to v:

$$\langle n_v, \sigma_u \rangle + \langle n, \sigma_{uv} \rangle = 0$$

Similarly differentiate the second:

$$\langle n_u, \sigma_v \rangle + \langle n, \sigma_{uv} \rangle = 0$$

So $\langle n_u, \sigma_v \rangle = \langle n_v, \sigma_u \rangle$ as desired. □

(Recall $M = -\langle n_v, \sigma_u \rangle = -\langle n_u, \sigma_v \rangle$).

Let's try to find the matrix of $Dn|_p$ in the basis $\{\sigma_u, \sigma_v\}$.

$$\begin{aligned} n_u &= Dn|_p(\sigma_u) = a_{11}\sigma_u + a_{21}\sigma_v \\ n_v &= Dn|_p(\sigma_v) = a_{12}\sigma_u + a_{22}\sigma_v \end{aligned}$$

Taking products of the above with σ_u and σ_v (check!):

$$-\underbrace{\begin{pmatrix} L & M \\ M & N \end{pmatrix}}_{Q} = \underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A}$$

$$Q = -PA = A^\top P$$

If $V = D\sigma|_0(\hat{v})$, $w = D\sigma|_0(\hat{w})$, then

$$\begin{aligned} -\hat{v}^\top \begin{pmatrix} L & M \\ M & N \end{pmatrix} \hat{w} &= -v^\top \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \hat{w} \\ &= I_p(v, -Dn|_p(w)) \\ &= I_p(-Dn|_p(v), w) \end{aligned}$$

Then the second FF has an intrinsic form given by the *symmetric* bilinear form

$$\mathbb{I} : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$$

given by

$$\mathbb{I}(v, w) = I_p(-Dn_p(v), w)$$

Definition. Let $\Sigma \subset \mathbb{R}^3$ smooth surface. The *Gauss curvature* $\kappa : \Sigma \to \mathbb{R}$ of Σ is the function

$$p \mapsto \det(Dn|_p)$$

Remark. This is always *well-defined* even if Σ is not oriented. We can always choose a local expression for n. If we replace it by $-n$, the determinant will *not* change (this is because $\det(A) = \det(-A)$ if A is 2×2).

Computing κ: If we pick σ using (†) we see that taking det:

$$LN - M^2 = (EG - F^2)\kappa$$

hence

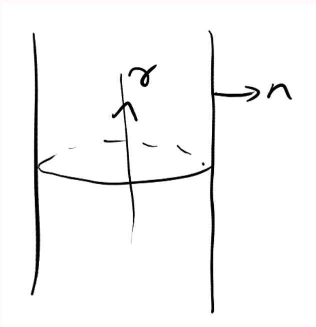
$$\kappa = \det(A) = \frac{LN - M^2}{EG - F^2}$$

Example. Cylinder we saw last time. We computed second fundamental form is

$$\sigma(u, v) = (a \cos u, a \sin u, v)$$

$$\begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix} \implies \kappa(p) = 0$$

for all p.



$n : \Sigma \to \text{Equator} \subset S^2$. So if $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ is a vertical curve, then

$$Dn|_p(\gamma'(0)) = (n \circ \gamma)'(0)$$

$$\implies \det(Dn|_p) = 0$$

Definition. Σ is said to be *flat* if $\kappa \equiv 0$ on Σ.

Example. If Σ is a graph of a smooth function f, then it is easy to check that

$$
\begin{aligned}
E &= 1 + f_u^2 \\
G &= 1 + f_v^2 \\
F &= f_u f_v \\
EG - F^2 &= 1 + f_u^2 + f_v^2 \\
L &= \frac{f_{uu}}{\sqrt{EG - F^2}} \\
M &= \frac{f_{uv}}{\sqrt{EG - F^2}} \\
N &= \frac{f_{vv}}{\sqrt{EG - F^2}} \\
\kappa &= \frac{f_{uu} f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2}
\end{aligned}
$$

So depends on the *Hessian* of f.

Definition. $\Sigma \subset \mathbb{R}^3$, $p \in \Sigma$. We say that p is:

- *elliptic* if $\kappa(p) > 0$
- *hyperbolic* if $\kappa(p) < 0$
- *parabolic* if $\kappa(p) = 0$

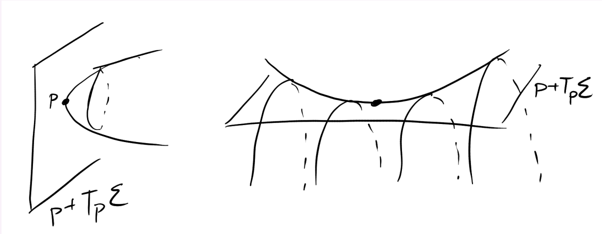
Graphs

(1) $f(u, v) = \frac{u^2+v^2}{2}$ at $(0, 0)$, $\kappa(0, 0, 0) = 1$.

(2) $f(u, v) = \frac{(u^2-v^2)}{2}$, $\kappa(0, 0, 0) = -1$

Lemma 2.8. (a) In a sufficiently small neighbourhood of an elliptic point p, Σ lies entirely on one side of the affine tangent plane $p + T_p\Sigma$.

(b) In a sufficiently small neighbourhood of a hyperbolic point, Σ meets both sides of its affine tangent plane.



Proof. Take a parametrisation σ near p.

$$\kappa = \frac{LN - M^2}{EG - F^2}$$

and $EG - F^2 > 0$. Recall also that if

$$w = n\sigma_n + l\sigma_v \in T_p\Sigma,$$

then $\frac{1}{2}\mathbb{I}_p(w, w)$ measured the signed distance from $\sigma(h, l)$ to $p + T_p\Sigma$ ($\sigma(0) = p$) measured via the inner product with positive normal:

$$\frac{1}{2}(Lh^2 + 2Mhl + Nl^2) + O(h^3, l^3)$$

p elliptic implies

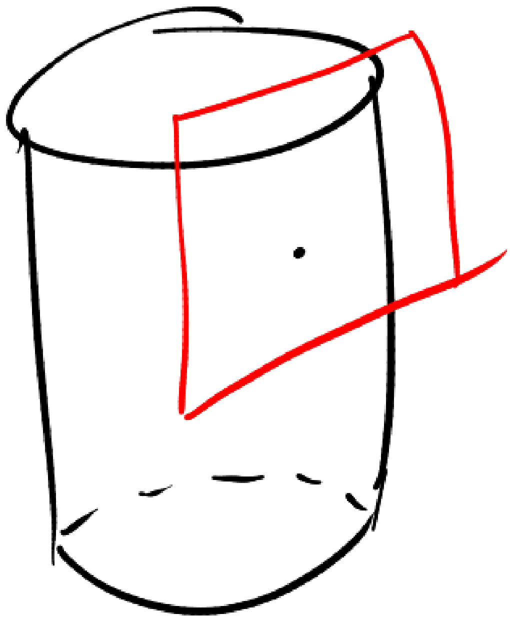
$$\begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

has eigenvalues of the same sign, so positive or negative definite at p, so in a neighbourhood of p, κ is signed distance only has one sign locally.

If p is hyperbolic, then $\mathbb{I}_p$ is indefinite, so Σ meets both sides of $p + T_p\Sigma$. □

Remark. If p is parabolic we can't conclude either.

Start of lecture 13

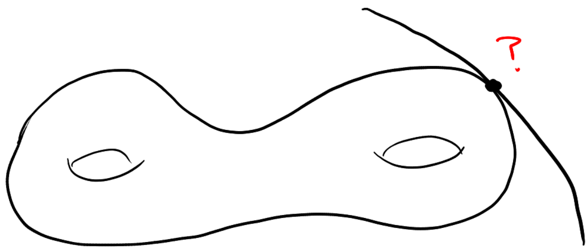


All points lie on one side of the tangent plane.

$$\sigma(u,v) = (u, v, u^3 - 3v^2u)$$

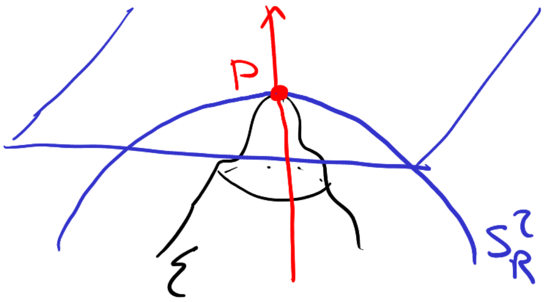
at $p = \sigma(0,0)$, $\kappa(p) = 0$, but locally Σ meets both sides of its tangent plane. Monkey saddle: see picture.

Proposition 2.9. Let Σ be a compact surface in $\mathbb{R}^3$. Then Σ has an elliptic point.



Proof. Σ compect implies $\Sigma \subset \overline{B(0,R)}$ for some R large enough. Decrease R to the minimal such value. Up to applying a rotation and translation we may assume that the point of contact is on the z axis. Locally near p view Σ as the graph of a smooth function f such that

$$f - \sqrt{\kappa^2 - u^2 - v^2} \leq 0$$



$f : V \to \mathbb{R}$, V open in $\mathbb{R}^2$. Since f has a local maximum at $(0,0) \implies f_v = f_0 = 0$ at $(0,0)$, $f(0,0) = p$.

$$F(u,v) = f(u,v) - \sqrt{R^2 - u^2 - v^2} \leq 0 \qquad (*)$$

An easy computation shows: $F_u = F_v = 0$ at $(0,0)$,

$$F_{uu} = f_{uu} + \frac{1}{R} \quad (\text{at } (0,0)) F_{uv} \quad = f_{uv} \quad (\text{at } (0,0)) F_{vv} = f_{vv} + \frac{1}{R} \quad (\text{at } (0,0))$$

$(*) \implies$ (Taylor expansion and use that 0 is a local maximum):

$$\implies \left(f_{uu} + \frac{1}{R}\right) h^2 + 2f_{uv}hl + \left(f_{vv} + \frac{1}{R}\right) l^2 \leq 0$$

for (h,l) small enough.

$$\implies f_{uu}h^2 + 2f_{uv}hl + f_{vv}l^2 \leq -\frac{1}{R}(h^2 + l^2)$$

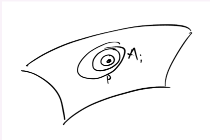
$$\implies \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}$$

Is negative definite at $(0,0)$. At $(0,0$, $E = G = 1$, $F = 0$, hence $\kappa(p) > 0$. $\square$

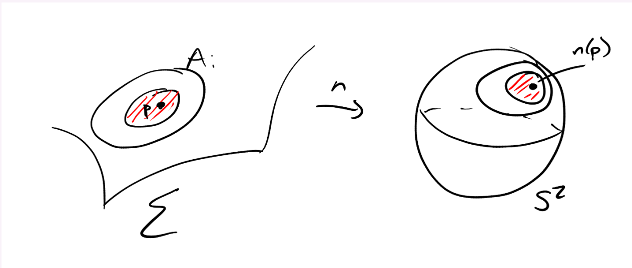
Gauss Curvature and Area

Theorem 2.10. $\Sigma \subset \mathbb{R}^3$, $p \in \Sigma$, $\kappa(p) \neq 0$. Let $U \subset \Sigma$ be a small open neighbourhood of p and consider a sequence $p \in A_i \subset U \subset \Sigma$ (A_i open) such that A_i "shrink to p" in the sense that $\forall \varepsilon > 0$, $A_i \subset B(p, \varepsilon) \subset \mathbb{R}^3$ for all i large enough. Then

$$|\kappa(p)| = \lim_{i \to \infty} \frac{\text{Area}_{S^2}(n(A_i))}{\text{Area}_\Sigma(A_i)}$$



i.e. the Gaussian curvature is an i-finiteseimal measure of how much the Gauss map n *distorts areas.*



Proof. κ is all "local", so take $\sigma : V \to U \subset \Sigma$ with $\sigma(0) = p$ and let $V_i = \sigma^{-1}(A_i) \subset V$ open. Since A_i shrinks to p,

$$\bigcap_{i \geq 1} V_i = \{(0,0)\}$$

$$\begin{aligned} \text{Area}_\Sigma(A_i) &= \int_{V_i} \sqrt{EG - F^2} \text{d}u\text{d}v \\ &= \int_{V_i} \|\sigma_u \times \sigma_v\| \text{d}u\text{d}v \end{aligned}$$

Now $n \circ \sigma : V \to S^2 \subset \mathbb{R}^2$. $Dn|_p \circ D\sigma|_0$ has rank 2 since $\kappa(p) \neq 0$. This $n \circ \sigma$ defines an allowable parametrisation in an open neighbourhood of $n(p) \subset S^2$ by the inverse function theorem.

$$\text{Area}_{S^2}(n(A_i)) = \int_{V_i} \|n_u \times n_v\| \text{d}u\text{d}v$$

(Same source as $(*)$ but for S^2!)

$$\|n_u \times n_v\| = \|Dn(\sigma_u) \times Dn(\sigma_v)\|$$

Recall from last lecture

$$\begin{aligned} Dn(\sigma_u) &= a_{11}\sigma_u + a_{21}\sigma_v \\ Dn(\sigma_v) &= a_{12}\sigma_u + a_{22}\sigma_v \\ \implies Dn(\sigma_u) \times Dn(\sigma_v) &= (a_{11}\sigma_u + a_{21}\sigma_v) \times (a_{12}\sigma_u + a_{22}\sigma_v) \\ &= \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{\det(Dn)=\kappa(p)} \sigma_u \times \sigma_v \\ \implies \text{Area}_{S^2}(n(A_i)) &= \int_{V_i} \|n_u \times n_v\| \\ &= \int_{V_i} |\det Dn| \|\sigma_u \times \sigma_v\| \mathrm{d}u\mathrm{d}v \\ &= \int_{V_i} |\kappa(u,v)| \|\sigma_u \times \sigma_v\| \mathrm{d}u\mathrm{d}v \end{aligned}$$

Since κ is continuous, given $\varepsilon > 0$, $\exists \delta > 0$ such that $|\kappa(u,v) - \kappa(0,0)| < \varepsilon$ for all $(u,v) \in B((0,0),\delta) \subset V$. So if $i \geq i_0$ we have

$$|\kappa(p)| - \varepsilon \leq |\kappa(u,v)| \leq |\kappa(p)| + \varepsilon \qquad \forall (u,v) \in V_i$$

$$\begin{aligned} \implies (|\kappa(p)| - \varepsilon) \int_{V_i} \|\sigma_u \times \sigma_v\| \mathrm{d}u\mathrm{d}v & \\ &\leq \int_{V_i} |\kappa(u,v)| \|\sigma_u \times \sigma_v\| \mathrm{d}u\mathrm{d}v \\ &\leq (|\kappa(p)| + \varepsilon) \underbrace{\int_{V_i} \|\sigma_u \times \sigma_v\| \mathrm{d}u\mathrm{d}v}_{\text{Area}_\Sigma(A_i)} \\ \implies |\kappa(p)| - \varepsilon &\leq \frac{\text{Area}_{S^2}(n(A_i))}{\text{Area}_\Sigma(A_i)} \\ &\leq |\kappa(p)| + \varepsilon \end{aligned}$$

for all $i \geq i_0$. □

Start of lecture 14

$\kappa(p) = \det(Dn|_p)$ ($\kappa = \frac{LN-M^2}{EG-F^2}$) The Gauss urvature is constraint by 2 *amazing theorems*.

The first is called "theorema egregium" (remarkable theorem)

Theorem (Theorema Egregium)**.** The Guass curvature of a smooth surface in $\mathbb{R}^3$ is an *isometry invariant* i.e. if $f : \Sigma_1 \to \Sigma_2$ is an isometry, then

$$\kappa_1(p) = \kappa_2(f(p)) \qquad \forall p \in \Sigma_1$$

In fact, κ can be computed exclusively in terms of I_p even though it was defined using I_p and $\mathbb{I}_p$.

The second result is a *global result*:

Theorem (Gauss-Bonnet Theorem)**.** If Σ is a compact smooth surface in $\mathbb{R}^3$, then

$$\int_\Sigma \kappa \mathrm{d}A_\Sigma = 2\pi\chi(\Sigma)$$

This is an amazing result because the left side of this equation is a quantity involving a lot of complicated geometric notions, but the right hand side is purely a topological property!

The proofs of these theorems will be in Part II Differential Geometry.

3 Geodesics

Recall, if $\gamma : [a,b] \to \mathbb{R}^3$ is smooth then

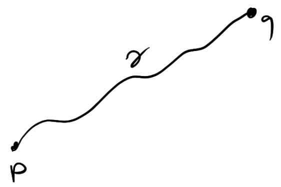
$$\text{length}(\gamma) = L(\gamma) = \int_a^b |\gamma'(0)| \mathrm{d}t$$

Definition. The *energy* of γ is

$$E(\gamma) := \int_a^b |\gamma'(t)|^2 \mathrm{d}t$$

Think

$$\Omega_{pq} = \{\text{all smooth curves } \gamma : [a,b] \to \mathbb{R}^3,\ \gamma(a) = p,\ \gamma(b) = q\}$$



$E : \Omega_{pq} \to \mathbb{R}$. In fact what we really want is given $\Sigma \subset \mathbb{R}^3$, $\gamma : [a,b] \to \Sigma$ "Variational Principles."

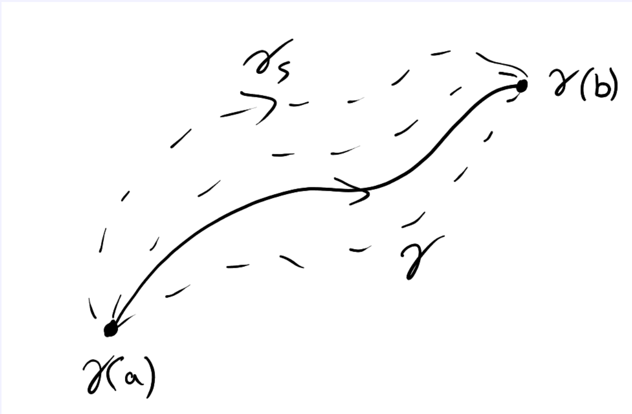
Definition. Let $\gamma : [a,b] \to \Sigma \subset \mathbb{R}^3$ be smooth. A one-parameter variation (with fixed end points) of γ is a smooth map

$$\Gamma : (-\varepsilon, \varepsilon) \times [a,b] \to \Sigma$$

such that $\gamma_s := \Gamma(s, \bullet)$, then

(a) $\gamma_0(t) = \gamma(t)$ for all t,

(b) $\gamma_s(a)$, $\gamma_s(b)$ are independent of s.



Definition. A smooth curve $\gamma : [a,b] \to \Sigma$ is a *geodesic* if for every variation γ_s of γ with fixed end points we have:

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\gamma_s) = 0$$

i.e. γ is a "critical point" of the energy functional on curves from $\gamma(a)$ to $\gamma(b)$.

Suppose γ has image contained in the image of a parametrisation σ and we write

$$\gamma_s(t) = \sigma(u(s,t), v(s,t))$$

FFF is $E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2$.

$$R := E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

(E, F, G are functions of $u(s,t)$, $v(s,t)$) where $\dot{u} = \frac{\partial u}{\partial t}$, $\dot{v} = \frac{\partial v}{\partial t}$. R depends on s

$$E(\gamma_s) = \int_a^b R\mathrm{d}t$$

$$\frac{\partial R}{\partial s} = (E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)\frac{\partial u}{\partial s} + (E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)\frac{\partial v}{\partial s}$$
$$+ 2(E\dot{u} + F\dot{v})\frac{\partial \dot{u}}{\partial s} + 2(F\dot{u} + G\dot{v})\frac{\partial \dot{v}}{\partial s}$$

so $\frac{\mathrm{d}}{\mathrm{d}s}E(\gamma_s) = \int_a^b \frac{\partial R}{\partial s}\mathrm{d}t$

Note. $\frac{\partial \dot{u}}{\partial s} = \frac{\partial^2 u}{\partial s \partial t}$, $\frac{\partial \dot{v}}{\partial s} = \frac{\partial^2 v}{\partial s \partial t}$, and we can integrate by parts and note $\frac{\partial u}{\partial s}$, $\frac{\partial v}{\partial s}$ vanish at the end points of a and b. We get

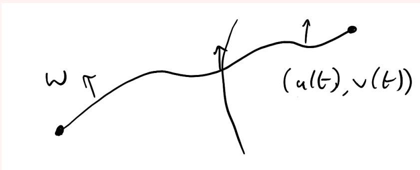
$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\gamma_s) = \int_a^b \left[A\frac{\partial u}{\partial s} + B\frac{\partial v}{\partial s}\right]\mathrm{d}t \qquad (*)$$

where

$$A = E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2 - 2\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v})$$
$$B = E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2 - 2\frac{\mathrm{d}}{\mathrm{d}t}(F\dot{u} + G\dot{v})$$

Note that we have absolute freedom for choosing the "rotational vector field"

$$W(t) = \left(\frac{\partial u}{\partial s}(0,t), \frac{\partial v}{\partial s}(0,t)\right)$$



Going back to $(*)$ we see that γ is a geodesic if and only if $A = B = 0$. That is γ is a geodesic if and only if $\gamma(t) = \sigma(u(t), v(t))$, then we have the *geodesic equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)$$

Remarks

(1) If $W(t)$ with $W(a) = W(b) = 0$ then

$$\gamma_s(t) = \sigma((u(t), v(t)) + sw(t))$$

for s small enough is a variation of γ with fixed and points and vertical field W.

(2) Recall Q10 in Example Sheet 4 of IA Analysis:

$$\int_a^b f(x)g(x)\mathrm{d}x = 0$$

for all $g : [a, b] \to \mathbb{R}$, $g(a) = g(b)$, then we had to prove that $f \equiv 0$.

Start of lecture 15

Recall from last time the quadratic equations:

$$(*) \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2) \\ \frac{\mathrm{d}}{\mathrm{d}t}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2) \end{cases}$$

Probably the best way to think about these is via the *Euler-Lagrange* (E-L) equations of the Lagrangian:

$$L(u, v, \dot{u}, \dot{v}) = \frac{1}{2}(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)$$

(purely kinetic energy). Recall from Variational Principles that the (E-L) equations are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i}$$

and here $q_1 = u$, $q_2 = v$. $(\dagger) = (*)$.

Proposition 3.1. Let $\Sigma \subset \mathbb{R}^3$ a smooth surface. A smooth curve $\gamma : [a, b] \to \Sigma$ is a geodesic if and only if $\ddot{\gamma}(t)$ is *everywhere* normal to Σ.

Proof. The statement is purely local so we can work in a parametrisation $\sigma : V \to U \subset \Sigma$ and as usual, $\gamma(t) = \sigma(u(t), v(t))$, $\dot{\gamma}(t) = \sigma_u\dot{u} + \sigma_v\dot{v}$. So $\ddot{\gamma}(t)$ is normal to Σ exactly when it is orthogonal to $T_{\gamma(t)}\Sigma$ spanned by $\{\sigma_u, \sigma_v\}$. In other words:

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\sigma_u\dot{u} + \sigma_v\dot{v}), \sigma_u \right\rangle = 0 \qquad (*_1)$$

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\sigma_u\dot{u} + \sigma_v\dot{v}), \sigma_v \right\rangle = 0 \qquad (*_2)$$

$(*_1)$ is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \sigma_u \dot{u} + \sigma_v \dot{v}, \sigma_u \rangle - \left\langle \sigma_u \dot{u} + \sigma_v \dot{v}, \frac{\mathrm{d}}{\mathrm{d}t}\sigma_u \right\rangle = 0$$

Noting that $E = \langle \sigma_u, \sigma_u \rangle$ and $F = \langle \sigma_u, \sigma_v \rangle$, this is:

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) - \langle \sigma_u \dot{u} + \sigma_v \dot{v}, \sigma_{uu}\dot{u} + \sigma_{vv}\dot{v} \rangle = 0$$

This second term is:

$$\dot{u}^2 \langle \sigma_u, \sigma_{uu} \rangle + \dot{u}\dot{v}(\langle \sigma_u, \sigma_{uv} \rangle + \langle \sigma_v, \sigma_{uu} \rangle) + \dot{v}^2 \langle \sigma_u, \sigma_{uv} \rangle$$

But

$$\begin{aligned} E &= \langle \sigma_u, \sigma_u \rangle & &\implies & E_u &= 2\langle \sigma_u, \sigma_{uu} \rangle \\ F &= \langle \sigma_u, \sigma_v \rangle & &\implies & F_u &= \langle \sigma_u, \sigma_{uv} \rangle + \langle \sigma_v, \sigma_{uu} \rangle \\ G &= \langle \sigma_v, \sigma_v \rangle & &\implies & G_u &= 2\langle \sigma_v, \sigma_{uv} \rangle \end{aligned}$$

Thus $(*_1)$ becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2)$$

which is the first geodesic equation. Similarly for $(*_2)$. □

Corollary 3.2. If $\gamma : [a, b] \to \Sigma$ is a geodesic, then $|\dot{\gamma}(t)|$ is constant.

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \dot{\gamma}, \dot{\gamma} \rangle = 2\langle \ddot{\gamma}, \dot{\gamma} \rangle = 0$$

Since $\ddot{\gamma} \perp T_{\gamma(t)}\Sigma$ and $\dot{\gamma}(t) \in T_{\gamma(t)}\Sigma$. □

Thus geodesics are parametrised with *constant speed* (i.e. proporitonal to arc-length).

Length vs Energy

Energy *is* sensitive to *parametrisations*. Given $\gamma : [a, b] \to \mathbb{R}^3$, we always have:

$$(L(\gamma))^2 \leq (b - a)E(\gamma)$$

with *equality* if and only if $|\dot{\gamma}|$ is constant. Indeed, using Cauchy-Schwarz:

$$\left(\int_a^b \underbrace{|\dot{\gamma}(t)|}_{=|\dot{\gamma}(t)\cdot 1} \mathrm{d}t \right)^2 \leq \underbrace{\left(\int_a^b |\dot{\gamma}(t)|^2 \mathrm{d}t \right)}_{=E(\gamma)} \underbrace{\left(\int_a^b 1 \mathrm{d}t \right)}_{=(b-a)}$$

with equality if and only if $|\dot{\gamma}(t)|$ is constant.

Corollary 3.3. A smooth curve $\gamma : [a, b] \to \Sigma \subset \mathbb{R}^3$ that minimises length and has constant speed is a *geodesic*.

Proof. Need to prove γ is a critical point of E. $\tau : [a, b] \to \Sigma$ any other curve connecting $\gamma(a)$ to $\gamma(b)$.

$$\begin{aligned} E(\gamma) &= \frac{(L(\gamma))^2}{b-a} && |\dot{\gamma}| \text{ constant} \\ &\leq \frac{(L(\tau))^2}{b-a} && \gamma \text{ minimises length} \\ &\leq E(\tau) \end{aligned}$$

Hence γ is critical for E and hence a geodesic. □

But geodesics might not be global minimisers (examples coming up) but they are always *local* minimisers. (No proof in this course, see Wilson's book).

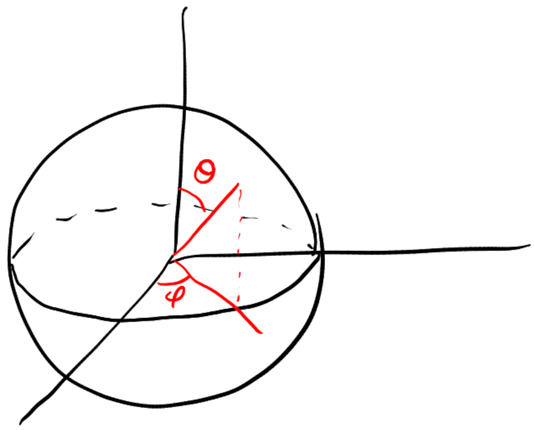
Examples

(1) The plane $\mathbb{R}^2$ has $\sigma(u, v) = (u, v, 0)$ and FFF $\mathrm{d}u^2 + \mathrm{d}v^2$. Geodesic equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{u}) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(\dot{v}) = 0$$

So $u(t) = \alpha t + \beta$, $v(t) = \gamma t + \delta$, which is a straight line parametrised by constant speed.

(2) Take the unit sphere with σ given by spherical coordinates:



$$
\begin{aligned}
\sigma(\varphi, \theta) &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\
\sigma_\varphi &= (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0) \\
\sigma_\theta &= (\cos\theta\cos\varphi, -\cos\theta\sin\varphi, -\sin\theta) \\
\implies E &= \sin^2\theta, F = 0, G = 1
\end{aligned}
$$

$L = \frac{1}{2}(\sin^2\theta\dot{\varphi}^2 + \dot{\theta}^2)$, $L(\theta, \varphi, \dot{\theta}, \dot{\varphi})$. Euler-Lagrange:

$$\frac{\partial L}{\partial \dot{\theta}} = \dot{\theta}, \qquad \frac{\partial L}{\partial \dot{\varphi}} = \sin^2\theta\dot{\varphi}$$

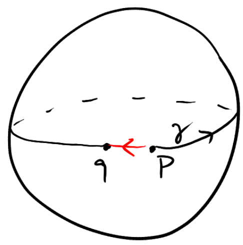
$\frac{\partial L}{\partial \varphi} = 0$.

$$(*) \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\sin^2\theta\dot{\varphi}) = 0 \\ \ddot{\theta} = \sin\theta\cos\theta\dot{\varphi}^2 \end{cases}$$

$(*)$ gives right away that the equator $t \mapsto \left(t, \frac{\pi}{2}\right)$ is a geodesic.

In fact all great circles parametrised with constant speed are geodesics. We can prove this by integrating $(*)$, but we can see this geometrically by notiving that such curves have $\ddot{\gamma} \perp T_{\gamma(t)}S^2$. Since geodesics solve a second order ODE (ordinary differential equation) prescribing $v \in T_p\Sigma$ determines the geodesic completely (proof next time). Thus great circles are *all possible* geodesics.

Note that γ between p and q as in the picture does *not* minimise length.



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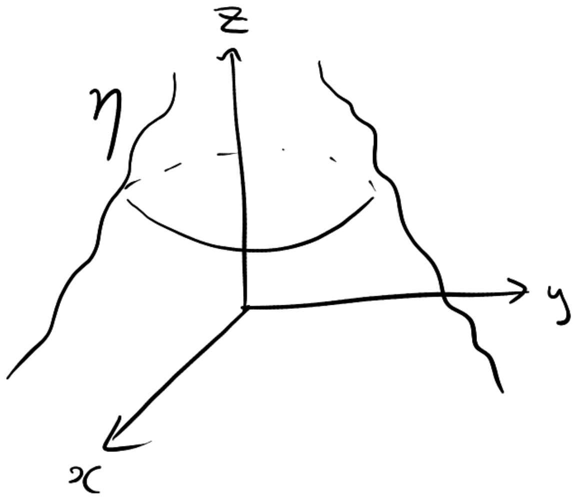
Important example: Surface of revolution again! We take $\eta(u) = (f(u), 0, g(u))$ in the xz-plane and rotate about the z-axis. ($\eta : [a, b] \to \mathbb{R}^3$ smooth, injective, $\eta' \neq 0$, $f > 0$).



Take the usual σ:

$$\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

$a < u < b$, $v \in (0, 2\pi)$. The FFF is

$$(f'^2 + g'^2)\mathrm{d}u^2 + f^2\mathrm{d}v^2$$

WLOG we assume η is parametrised by arc-length so the FFF becomes

$$\mathrm{d}u^2 + f^2\mathrm{d}v^2$$

Then Lagrangian for the geodesics is

$$L = \frac{1}{2}(\dot{u}^2 + f^2\dot{v}^2)$$

(E-L) equations implies $\frac{\partial L}{\partial u} = ff'\dot{v}^2$

$$\frac{\partial L}{\partial \dot{u}} = \dot{u} \implies \boxed{\ddot{u} = ff'\dot{v}^2}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}}$$

$$\frac{\partial L}{\partial v} = 0 \qquad \frac{\partial L}{\partial \dot{v}} = f^2\dot{v}$$

$$(*)\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(f^2\dot{v}) = 0 \\ \ddot{u} = ff'\dot{v}^2 \end{cases}$$

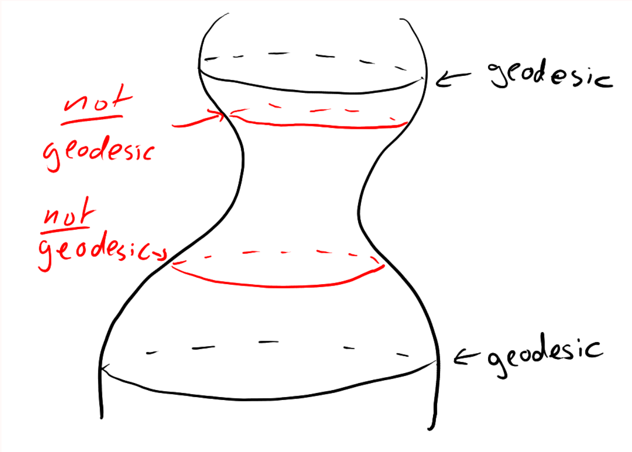
We also know that geodesics travel with constant speed:

$$\begin{cases} \dot{u}^2 + f^2\dot{v}^2 = \text{const} \\ f^2\dot{v} = c \end{cases}$$

Example ("completely integrable"). Meridians: $v = v_0$m if $u(t) = t + u_0$, $t \mapsto (t + u_0, v_0)$ is a geodesic with speed 1 through (u_0, v_0) (check this works).

Parallels: $u = u_0$, $\dot{v} = a \,|_{f(u_0)}$. From $(*)$ we see that we need

$$\boxed{f'(u_0) = 0}$$



Let's look at the conserved quantity $f^2\dot{v}$ in more detail: .image Suppose γ makes an angle θ with a parallel of radius $\rho = f$. Write as usual $\gamma = \sigma(u(t), v(t))$, $\dot{\gamma} = \sigma_u\dot{u} + \sigma_v\dot{v}$ and note that σ_v is tangent to the parallel since $\sigma_v = (-f\sin v, f\cos v, 0)$. Thus

$$\cos\theta = \frac{\langle \sigma_v, \sigma_u\dot{u} + \sigma_v\dot{v}\rangle}{|\sigma_v||\dot{\gamma}|}$$

Assume γ is parametrised by arc-length, so $|\dot{\gamma}| = 1$. Using that $F = 0$ and $G = f^2$ we get

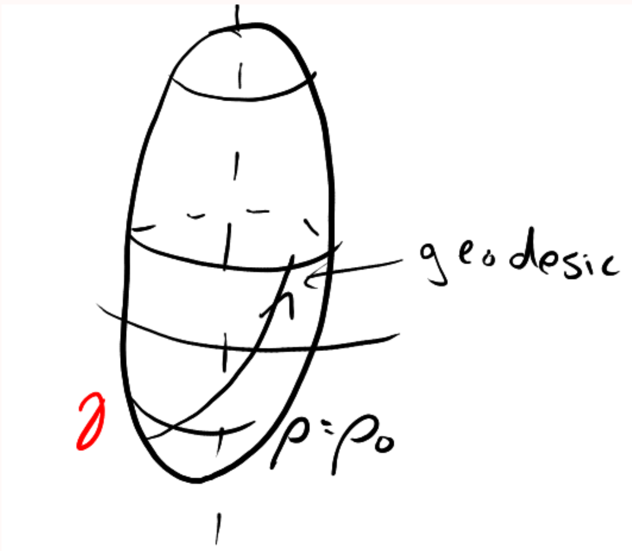
$$\cos\theta = \frac{f^2\dot{v}}{f} = f\dot{v}$$

and therefore if γ is geodesic,

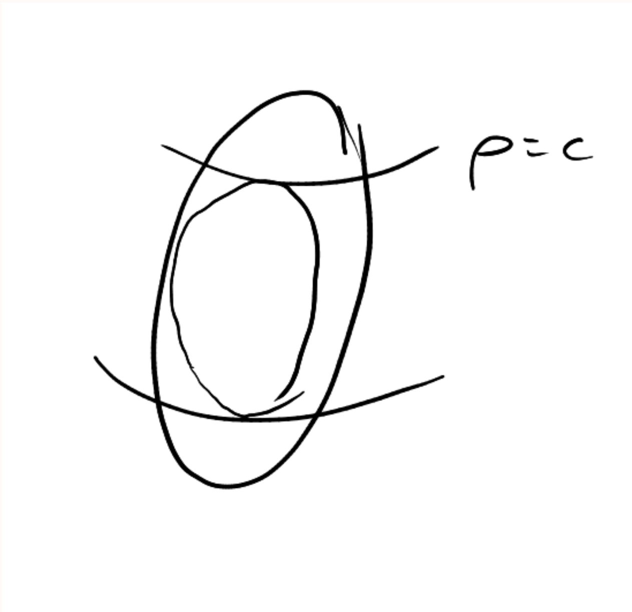
$$\boxed{\rho\cos\theta = \text{constant}}$$

Clairut's relation. This is just another way to write the conservation law arising from $\frac{\partial L}{\partial v} = 0$.

Example. Sub-example: ellipsoid of revolution.



$\rho\cos\theta = c$, $c = \rho_0\cos\theta_0 > 0$, $\theta_0 \in [0, \pi/2)$. $c = \rho\cos\theta \le \rho$. This means that γ must move between the region bounded by the parallels of radius c.



Recall Picard's theorem for ODEs: $I = [t_0 - a, t_0 + a] \subset \mathbb{R}$. $B = \{x : \|x - x_0\| \le b\} \subseteq \mathbb{R}^n$. $f : I \times B \to \mathbb{R}^n$ and Lipschitz in the second variable

$$\|f(t, x_1) - f(t, x_2)\| \le K\|x_1 - x_2\|$$

Then

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

has a unique solution for some interval $|t - t_0| < h$.

Addendum: If f is smooth, then the solution is smooth *and* depends smoothly on the initial condition. In our setting we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \ddot{u} \\ \ddot{v} \end{pmatrix} \qquad = \mathcal{A}(u, \sigma, \dot{u}, \dot{v})$$

Since the matrix is invertible, we can write the geodesics equations as:

$$\ddot{u} = A(u, v, \dot{u}, \dot{v})$$
$$\ddot{v} = B(u, v, \dot{u}, \dot{v})$$

for smooth A and B. We can turn this into a first order system by the usual trick:

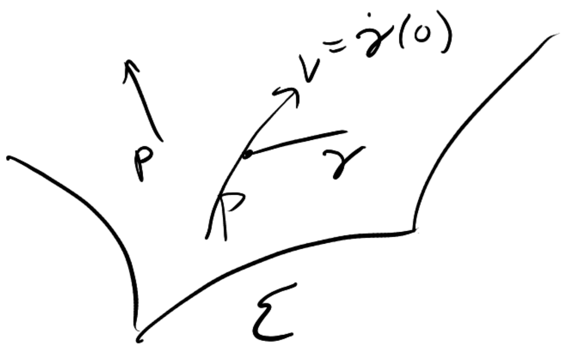
$$\dot{u} = X, \dot{v} = Y$$

Then

$$\begin{cases} \dot{u} = X \\ \dot{v} = Y \\ \dot{X} = A(u, v, X, Y) \\ \dot{Y} = B(u, v, X, Y) \end{cases}$$

So Picard's theorem applies, noting that since A and B are smooth, a local bound on $\|DA\|$ and $\|DB\|$ wil give the Lipschitz conditions. So we get:

Corollary 3.4. Let Σ be a smooth surface in $\mathbb{R}^3$. For $p \in \Sigma$ and $v \in T_p\Sigma$, there is $\varepsilon > 0$ and a unique geodesic $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ with initial conditions $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Moreover, γ depends smoothly on (p, v).



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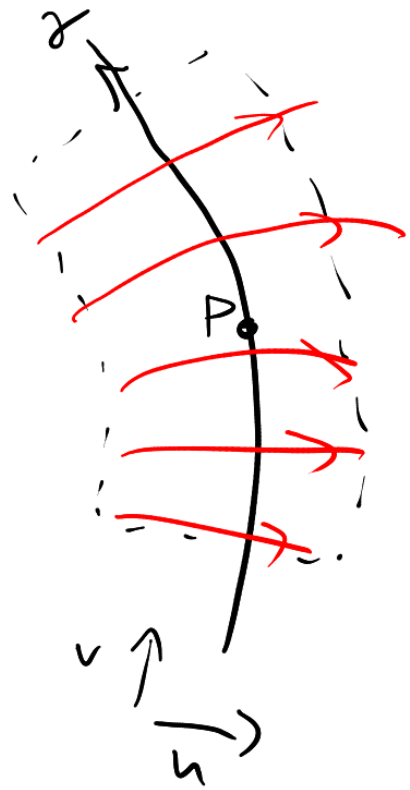
The local existence of geodesics gives rise to parametrisations with very *nice properties.*

Fix $p \in \Sigma$. Consider a geodesic arc γ going through p and parametrised by arc-length. For t small enough, let γ_t be the unique geodesic such that

- $\gamma_t(0) = \gamma(t)$
- $\gamma_t'(0)$ is orthogonal to $\gamma'(t)$ and has unit length.

Define $\sigma(u, v) = \gamma_v(u)$ for $u \in (-\varepsilon, \varepsilon)$, $v \in (-\varepsilon, \varepsilon)$.



Lemma 3.5. For ε and δ sufficiently small, σ defines an allowable parametrisation of an open set in Σ.

Proof. Smoothness follows from Corollary 3.4 (smoothness of geodesics with respect to initial conditions). At $(0, 0)$, σ_u and σ_v are orthogonal and have norm 1 by construction. Thus

$$D\sigma|_0 : \mathbb{R}^2 \to T_p\Sigma$$

is a linear isomorphism. Now applying the inverse function theorem as in Example Sheet 1 Question 9, we deduce that σ is a local diffeomorphism at $(0,0)$ and hence for ε, δ small enough it is an allowable parametrisation. ☐

Proposition 3.6. Any smooth surface Σ in $\mathbb{R}^3$ admits a local parametrisation for which the FFF is of the form

$$\mathrm{d}u^2 + G\mathrm{d}v^2$$

i.e. $E = 1$, $F = 0$.

Proof. Consider $\sigma(u,v) = \delta_v(u)$ as above. If we fix v_0, the curve $u \to \gamma_{v_0}(u)$ is a geodesic parametrised by arc-length. So $E = \langle \sigma_u, \sigma_u \rangle = 1$. Also one of the geodesic equations is

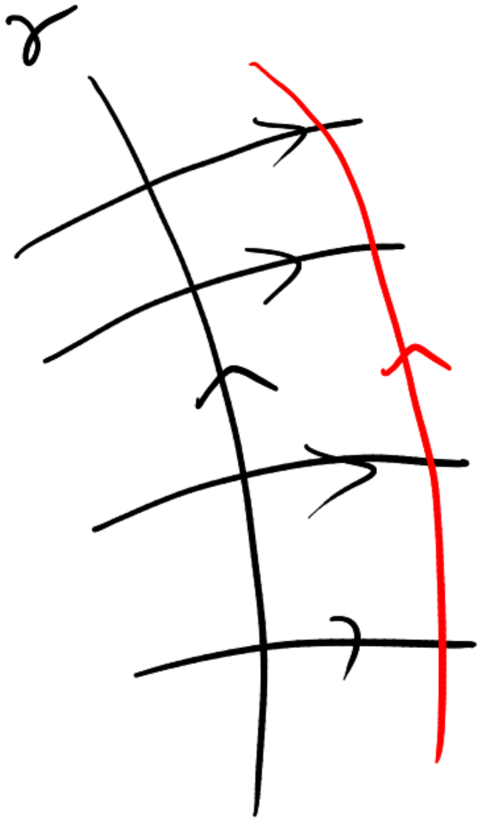
$$\frac{\mathrm{d}}{\mathrm{d}t}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)$$

and $v = v_0$, $u(t) = t$. $\frac{\mathrm{d}}{\mathrm{d}t}(F) = 0$ i.e. $F_u = 0$. So F is independent of u. But when $u = 0$, then by construction of γ_v as orthogonal to γ at $\gamma(v)$, we see that $F = 0$ everywhere. ☐

Remarks

(1) Somtimes these coordinates are called *Fermi coordinates*.

(2) Careful:



u-fixed, v vary is typically *not* a geodesic.

(3) In these coordinates we also have:

(a) $G(0, v) = 1$,

(b) $G_u(0, v) = 0$

(a) holds because σ_v has length 1 at $u = 0$. To see (b), we use that $u = 0$, $v = t$ is a geodesic and

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)$$

because $0 = \frac{1}{2}G_u(0, t)$.

(4) One can show that if $E = 1$ and $F = 0$, then the Gauss curvature is given by

$$\boxed{\kappa = -\frac{(\sqrt{G})_{uu}}{\sqrt{G}}} \qquad (\dagger)$$

(cf Theorem 8.1 in Wilson's book). This proves theorema egregium! The computation that proves (†) is not too hard, but beyond the scope of this course. But we'll use (†) in our next discussion.

Constant Gaussian Curvature

First a general remark / Exercise: If $\Sigma \subset \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}^3$ is dilation by $\lambda \neq 0$:

$$f(x, y, z) = \lambda(x, y, z)$$

then $\kappa_{f(\Sigma)} = \frac{1}{\lambda^2}\kappa_\Sigma$. To check this note that the coefficients E, F, G rescale by λ^2 and L, M, N by λ.

QUestion: What do constant curvature surfaces look like? By dilations it sufficies to understand surfaces of constant curvature $1, -1, 0$.

Proposition 3.7. $\Sigma \subset \mathbb{R}^3$ smooth surface.

(a) If $\kappa_\Sigma \equiv 0$, then Σ is locally isometric to $(\mathbb{R}^2, \mathrm{d}u^2 + \mathrm{d}v^2)$.

(b) If $\kappa_\Sigma \equiv 1$, then Σ is locally isometric to $(S^2, \mathrm{d}u^2 + \cos^2(u)\mathrm{d}v^2)$.

Proof. We know that Σ admits a parametrisation with $E = 1$, $F = 0$ and $G(0, v) = 1$ and $G_u(0, v) = 0$. Also

$$\kappa = -\frac{(\sqrt{G})_{uu}}{\sqrt{G}}$$

If $\kappa \equiv 0$, we get $(\sqrt{G})_{uu} = 0$, hence $\sqrt{G} = A(v)u + B(v)$. Conditions on G give $B = 1$ and $A = 0$. Then the FFF is $\mathrm{d}u^2 + \mathrm{d}v^2$.

If $\kappa = 1$, $(\sqrt{G})_{uu} + \sqrt{G} = 0$, so

$$\sqrt{G} = A(v)\sin u + B(v)\cos u$$

conditions on G gives $A = 0$ and $B \equiv 1$, so FFF is $\mathrm{d}u^2 + \cos^2 u\,\mathrm{d}v^2$. In the parametrisation

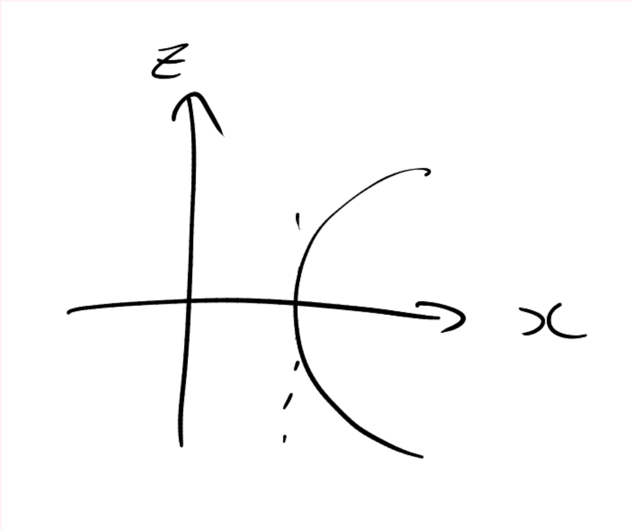
$$\sigma(u, v) = (\cos u \cos v, \cos u \sin v, \sin u)$$

of S^2 this is the FFF. □

Remark. We can certainly do the same for $\kappa = -1$, and we'll get FFF $\mathrm{d}u^2 + \cosh^2(u)\mathrm{d}v^2$. A surface of revolution with FFF $\mathrm{d}u^2 + \cosh^2(u)\mathrm{d}v^2$ is given by rotating

$$\eta(u) = \left(\cosh u, 0, \int_0^h \sqrt{1 - \sinh^2 x}\mathrm{d}x\right)$$

This has $f'^2 + g'^2 = 1$ and hence $\kappa = -f''/f$ (Question 5 on Example Sheet 2).



Or... we forget about $\mathbb{R}^3$ and think in more abstract terms. The change of variables $v = e^v \tanh u$, $w = e^v \mathrm{sech} u$ turns $\mathrm{d}u^2 + \cosh^2(u)\mathrm{d}v^2$ into

$$\frac{\mathrm{d}v^2 + \mathrm{d}w^2}{w^2}$$

which is "the standard presentation" of the hyperbolic plane.

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4 Hyperbolic Surfaces

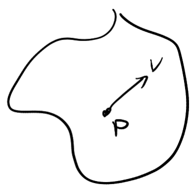
We start by discussing *abstract Riemannian metrics*.

Definition (Riemannian metric)**.** Let $V \subset \mathbb{R}^2$ be an open set. An abstract Riemannian metric on V is a smooth map:

$$V \to \{\text{positive definite symmetric forms}\} \subset \mathbb{R}^4$$

$$p \mapsto \begin{pmatrix} E(p) & F(g) \\ F(g) & G(p) \end{pmatrix} = g(p)$$

$E > 0$, $G > 0$ and $EG - F^2 > 0$.



If v is a vector at $p \in V$, then its norm is:

$$\|v\|_g^2 = v^\top \begin{pmatrix} E(p) & F(p) \\ F(p) & G(p) \end{pmatrix} v$$

and if $\gamma : [a, b] \to V$ is smooth, then its length

$$\begin{aligned} L(\gamma) &= \int_a^b \|\dot{\gamma}(t)\|_g \mathrm{d}t \\ &= \int_a^b (E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)\mathrm{d}t \end{aligned}$$

where $\gamma(t) = (u(t), v(t))$.

Definition. Given (V, g), $(\tilde{V}, \tilde{g})$, we say that they are *isometric* if there exists a diffeomorphism $f : V \to \tilde{V}$ such that

$$\|Df|_p(v)\|_{\tilde{g}} = \|v\|_g \qquad \forall v \in T_pV = \mathbb{R}^2, \forall p \in V \tag{$*$}$$

This is *equivalent* to saying that f preserves the length of curves.

Note. $Df_p : T_pV \to T_{f(p)}\tilde{V}$. Let's spell out this condition $(*)$ using g and $\tilde{g}$.

$$\begin{aligned}\|Df|_p(v)\|^2_{\tilde{g}} &= (Df|_p v)^\top \tilde{g}_{f(p)} Df|_p v \\ &= v^\top (Df|_p)^\top \tilde{g}_{f(p)} Df|_p v \\ &= \|v\|^2_g \\ &= v^\top g v\end{aligned}$$

This holds for all v iff

$$\boxed{(Df|_p)^\top \tilde{g}_{f(p)} Df|_p = g} \qquad (\dagger)$$

Recall that $(\dagger)$ is exactly the transformation law from Lemma 2.3 (Lecture #9).

Definition (Riemannian metric on a surface)**.** Let Σ be an abstract smooth surface, so $\Sigma = \bigcup_{i \in I} U_i$, $U_i \subset \Sigma$ open and $\varphi : U_i \to V_i \subset \mathbb{R}^2$ homeomorphism with V_i open such that

$$\varphi_i \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is smooth for all i, j.

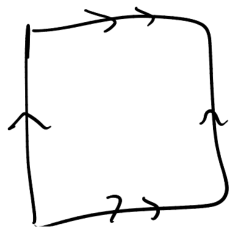
A *Riemannian metric* on Σ usually denoted by g is a choice of Riemannian metrics g_i on each V_i which are compatible in the following sense:

For all i, j, $\varphi_i \varphi_j^{-1}$ is an isometry between $\varphi_j(U_i \cap U_j)$ and $\varphi_i(U_i \cap U_j)$, i.e. if we let $f = \varphi_i \varphi_j^{-1}$, then

$$(Df|_p)^\top \begin{pmatrix} E_i & F_i \\ F_i & G_i \end{pmatrix}_{f(p)} Df|_p = \begin{pmatrix} E_j & F_j \\ F_j & G_j \end{pmatrix}_p \qquad \forall p \in \varphi_j(U_i \cap U_j)$$

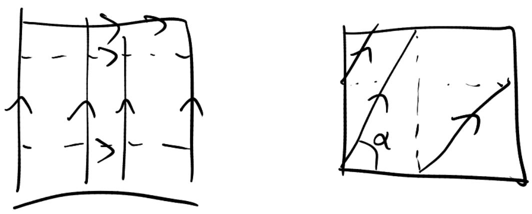
Examples

(1) Recall the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$



We exhibited charts where transition functions were restriction of translations. Equip each $V_i \subset \mathbb{R}^2$ (image of such a chart) with the Euclidean metric $\mathrm{d}u^2 + \mathrm{d}v^2$, i.e. the

$V_i \mapsto \mathrm{id}_{2\times 2}$. If f is a translation then $Df = \mathrm{id}$ so $(Df)^\top IDf = I$ is obvious! So T^2 inherits a global Riemannian metric everywhere locally isometric to $\mathbb{R}^2$ (hence *flat*). Since geodesics are well-defined for abstract Riemannian metrics (Energy only depends on g!) they are also well-defined on T^2 and they are just projections of straight lines in $\mathbb{R}^2$:

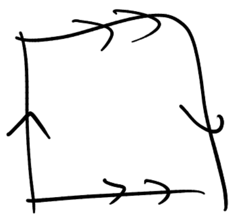


Exercise: Show that there are infinitely many cloesd geodesics and also infinitely many non-closed ones (think about lines with rational / irrational slope).

Note. This flat metric on T^2 is *not* induced by any embedding of T^2 in $\mathbb{R}^3$! (For example because it would have to have an elliptic point).

(2) The real projective plane admits a Riemannian metric with constant curvature $+1$. Inded, in lecture 2 we exhibited an atlas for $\mathbb{RP}^2$ with charts of the form (U, φ) where $U = q\hat{U}$, $q : S^2 \to \mathbb{RP}^2$, $\hat{U} \subset S^2$ open small enough so that $\hat{U}$ subset of open hemisphere and $\varphi : U \to V \subset \mathbb{R}^2$ was $\varphi : \hat{\varphi} \circ q^{-1}|_U$, where $\hat{\varphi} : \hat{U} \to V$ chart of S^2. Transition maps for this atlas were all the identity or induced by the antipodal map. But both are isometries of the round metric in S^2.

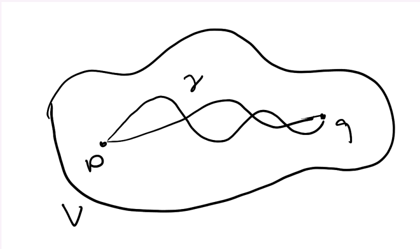
(3) Exercise: The Klein bottle has a flat Riemannian metric induced from $\mathbb{R}^2$.



Proposition 4.1. Given a Riemannian metric g on a *connected* open set $V \subset \mathbb{R}^2$, define the length metric

$$d_g(p,q) = \inf_{\gamma} L(\gamma)$$

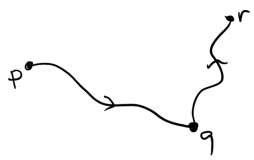
where γ varies over all piece-wise smooth paths in V from p to q and $L(\gamma)$ is computed using g. Then d_g is a metric in V (in the sence of metric spaces!)



Remarks / Examples

(1) Given $p, q \in V$, there is always a piece-wise smooth path connecting p to q.

(2) $d_g(p,q) \geq 0$. Also $d_g(p,q) = d_g(q,p)$ and $d_g(p,r) \leq d_g(p,q) + d_g(q,r)$



The only non-trivial claim is that $d_g(p,q) = 0$ implies $p = q$ (proof next lecture).

(3) All this works on *any* abstract smooth *connected* surface (Σ, g) equipped with a Riemmanian metric g.

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Proposition 4.2. d_g is a metric.

Proof. We only show that $d_g(p,q)$ for $p \neq q$ for $p \neq q$. (See remarks from last lecture). Since

$$g = \begin{pmatrix} E(p) & F(p) \\ F(p) & G(p) \end{pmatrix}$$

is positive definite, there is ε sufficiently small such that

$$\begin{pmatrix} E(p) - \varepsilon^2 & F(p) \\ F(p) & G(p) - \varepsilon^2 \end{pmatrix}$$

is also positive definite. Moreover, the matrix

$$\begin{pmatrix} E(p') - \varepsilon^2 & F(p') \\ F(p') & G(p') - \varepsilon^2 \end{pmatrix}$$

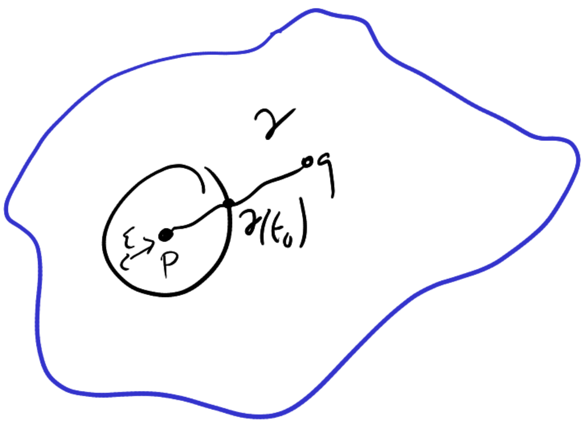
remains positive definite $\forall p' \in B(p, \delta) \subset V$. (Euclidean ball). Thus, for any $p' \in B(p, \delta)$ and $v = (v_1, v_2) \in \mathbb{R}^2$ we have

$$\begin{aligned} \|v\|_{p'}^2 &= E(p')v_1^2 + 2F(p')v_1 v_2 + G(p')v_2^2 \\ &\geq \varepsilon^2(v_1^2 + v_2^2) \end{aligned}$$

Hence if γ is a curve in $B(p, \delta)$ then we have

$$L_g(\gamma) \geq \varepsilon L_{\text{Euclidean}}(\gamma)$$

Given $p \neq q$, let $\gamma : [a, b] \to V$ be any curve connecting p to q. If γ is not contained in $B(p, \delta)$ then there exists $t_0 \in [a, b]$ such that $\gamma|_{[a,t_0]}$ is in $B(p, \delta)$, but $\gamma(t_0)$ is on the boundary of the ball.



Thus $L_g(\gamma) \geq L_g(\gamma|_{[a,t_0]}) \geq \varepsilon\delta$. If however γ is contained in $B(p, \delta)$, then

$$L_g(\gamma) \geq \varepsilon d_{\text{Euclidean}}(p, q)$$

Taking inf over all such γ we get

$$d_g(p, q) \geq \varepsilon \min\{\delta, d_{\text{Euclidean}}(p, q)\} > 0$$ □

Remark. d_g gives the same topology that $V \subset \mathbb{R}^2$ inherits from $\mathbb{R}^2$ (can check this as an exercise).

4.1 Hyperbolic Geometry

Definition. We define an abstract Riemannian metric on the disc

$$D = B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$$

by

$$g_{\text{hyp}} = \frac{4(\mathrm{d}u^2 + \mathrm{d}v^2)}{(1-u^2-v^2)^2} = \frac{4|\mathrm{d}z|^2}{(1-|z|^2)^2}.$$

In other words,

$$E = G = \frac{4}{(1-u^2-v^2)^2} \qquad F = 0$$

Recall the Möbius group

$$\text{Möb} = \left\{ z \mapsto \frac{az+b}{cz+d} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C}) \right\}$$

acts on $\mathbb{C} \cup \{\infty\}$.

Lemma 4.3.

$$\text{Möb}(D) = \{T \in \text{Möb} : T(D) = D\} = \left\{ z \mapsto e^{i\theta}\frac{z-a}{1-\bar{a}z} : |a| < 1 \right\}$$

Proof. First we note:

$$\begin{aligned}
\left|\frac{z-a}{1-\bar{a}z}\right| = 1 &\iff (z-a)(\bar{z}-\bar{a}) = (1-\bar{a}z)(1-a\bar{z}) \\
&\iff z\bar{z} - \cancel{a\bar{z}} - \cancel{\bar{a}z} + a\bar{a} = 1 - \cancel{a\bar{z}} - \cancel{\bar{a}z} + a\bar{a}z\bar{z} \\
&\iff |z|^2(1-|a|^2) = 1 - |a|^2 \\
&\iff |z| = 1
\end{aligned}$$

So $z \mapsto e^{i\theta}\frac{z-a}{1-\bar{a}z}$ preserves $|z| = 1$ and maps a to 0, thus it belongs to $\text{Möb}(D)$. To show that they are all of this form, pick $T \in \text{Möb}(D)$. If $a = T^{-1}(0)$ and $Q(z) = \frac{z-a}{1-\bar{a}z} \in \text{Möb}(D)$, then $TQ^{-1}(0) = 0$ and preserves $|z| = 1$ and hence (check!) it must be of the form $z \mapsto e^{i\theta}z$. □

Lemma 4.4. The Riemannian metric g_{hyp} is invariant under $\text{Möb}(D)$, i.e. it acts by hyperbolic isometries.

Proof. Möb(D) is generated by $z \mapsto e^{i\theta}z$ and $z \mapsto \frac{z-a}{1-\overline{a}z}$, $|a| < 1$. The first (rotation) clearly preserves $g_{\text{hyp}} = \frac{4|\mathrm{d}z|^2}{(1-|z|^2)^2}$. For the second type, let

$$w = \frac{z-a}{1-\overline{a}z},$$

so

$$\begin{aligned}
\mathrm{d}w &= \frac{\mathrm{d}z}{1-\overline{a}z} + \frac{(z-a)}{(1-\overline{a}z)^2}\overline{a}\mathrm{d}z \\
&= \frac{\mathrm{d}z(1-|a|^2)}{(1-\overline{a}z)^2} \\
\frac{|\mathrm{d}w|}{1-|w|^2} &= \frac{|\mathrm{d}z|(1-|a|^2)}{|1-\overline{a}z|^2\left(1-\left|\frac{-a}{1-\overline{a}z}\right|^2\right)} \\
&= \frac{|az|(1-|a|^2)}{|1-\overline{a}z|^2 - |z-a|^2} && \text{(check)} \\
&= \frac{|\mathrm{d}z|}{1-|z|^2}
\end{aligned}$$

□

"Another view"

$$g_{\text{hyp}} = \lambda \text{id}$$

$\lambda(z) = \frac{4}{(1-|z|^2)^2}$, $f(z) = \frac{z-a}{1-\overline{a}z}$, $|a| < 1$. To check isometry:

$$(Df|_z)^\top \underbrace{(g_{\text{hyp}})_{f(z)}}_{\lambda(f(z))\text{id}} Df|_z = \lambda(z)\text{id}$$

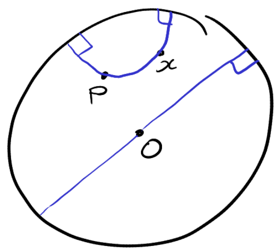
i.e.

$$\lambda(f(z))(Df|_z)^\top Df|_z = \lambda(z)\text{id}$$
$$\lambda(f(z))|f'(z)|^2 = \lambda(z)$$

and this is checked as previously!

Lemma 4.5. (i) Every pair of points in (D, hyp) is joined by a unique geodesic (up to reparametrisation).

(ii) The geodesics are diameters of the discs and circular arcs orthogonal to ∂D.



Geodesics in the hyperbolic disc. The whole geodesics are called hyperbolic lines (defined on all $\mathbb{R}$).

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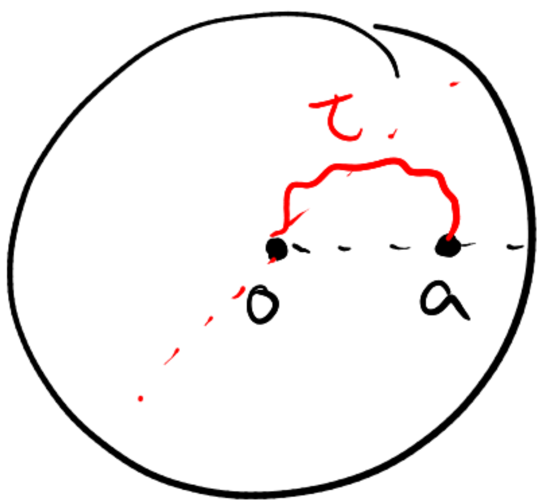
Proof. Let $a \in \mathbb{R}_+ \cap D$ and τ a smooth path from 0 to a, say

$$\tau(t) = (u(t), v(t)), \qquad t \in [0,1]$$



$$\begin{aligned}
L(\tau) &= \int_0^1 \frac{2|\dot{\tau}(t)|}{1-|\tau(w)|^2}\mathrm{d}t \\
&= \int_0^1 \frac{2\sqrt{\dot{u}^2+\dot{v}^2}}{1-u^2-v^2}\mathrm{d}t \\
&\ge \int_0^1 \frac{2|\dot{u}(t)|\mathrm{d}t}{1-u^2} \\
&\ge \int_0^1 \frac{2\dot{u}}{1-u^2}\mathrm{d}t \\
&= \int_0^a \frac{2\mathrm{d}s}{1-s^2} \\
&= 2\tanh^{-1}(a)
\end{aligned}$$

with equality if and only if $\dot{v} = 0$ and $\dot{u} \geq 0$, i.e. $v = 0$ and u monotonic. So the arc of diameter is globally length minimising and if parametrised by arc-length it becomes a geodesic with length

$$\boxed{d_{g_{\text{hyp}}}(0, a) = 2\tanh^{-1}(a)} \tag{†}$$

Now $0, a \in \mathbb{R}_+ \cap D$ are joined by a unique geodesic (any geodesic going through 0 must be a diameter). An element of $\text{Möb}(D)$ can be used to send any $p, q \in D$ to $0, a \in \mathbb{R}_+ \cap D$ (check, but see below). Since isometries map geodesics to geodesics, every $p, q \in D$ is joined by a unique geodesic. And Möbius maps send circles to circles and preserve angles, and hence orthogonality to ∂D. This implies our description of geodesics. □

Corollary 4.6. If $p, q \in D$, then

$$d_{\text{hyp}}(p, q) = 2\tanh^{-1}\left|\frac{p - q}{1 - \overline{p}q}\right|$$

Proof. $Q(z) = \frac{z-p}{1-\overline{p}z}$ maps p to 0. Pick θ such that $e^{i\theta}Q(q) \in \mathbb{R}_+ \cap D$. For $T = e^{i\theta}Q$,

$$\begin{aligned} d_{\text{hyp}}(p, q) &= d_{\text{hyp}}(T(p), T(q)) \\ &= d_{\text{hyp}}(0, T(q)) \\ &= 2\tanh^{-1}\left|\frac{p - q}{1 - \overline{p}q}\right| \qquad \text{(using (†))} \end{aligned}$$

□

Definition. The hyperbolic upper half plane (also called Poincaré upper half-plane) $(\mathcal{H}, g_{\text{hyp}})$ is the set

$$\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

with abstract Riemannian metric

$$\frac{\mathrm{d}z^2 + \mathrm{d}y^2}{y^2}$$

(or $\frac{|\mathrm{d}z|^2}{(\operatorname{Im}(z))^2}$).

Lemma 4.7. The Poincaré disc (D, g_{hyp}) and the Poincaré upper-half plane $(\mathcal{H}, g_{\text{hyp}})$ are *isometric.*

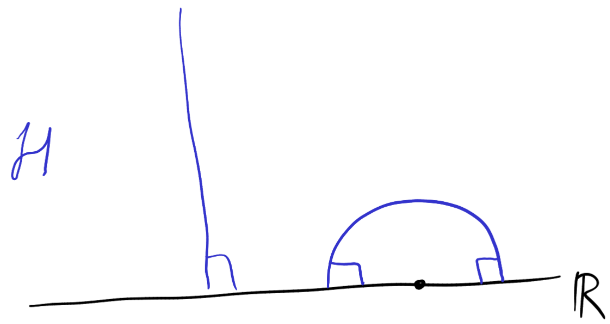
Proof. We have maps: $\mathcal{H} \to D$, $w \mapsto \frac{w-i}{w+i}$ and $D \to \mathcal{H}$, $z \mapsto i\frac{i+z}{i-z}$, which are inverse diffeomorphisms (easy to check). If $w \in \mathcal{H}$, $T(w) = \frac{w-i}{w+i} = z \in D$, then $T'(w) = \frac{2i}{(w+i)^2}$,

then

$$\begin{aligned}
\frac{|\mathrm{d}z|}{1-|z|^2} &= \frac{|\mathrm{d}(Tw)|}{1-|Tw|^2} \\
&= \frac{|T'(w)||\mathrm{d}w|}{1-|Tw|^2} \\
&= \frac{2|\mathrm{d}w|}{|w+i|^2\left(1-\left|\frac{w-i}{w+i}\right|^2\right)} \\
&= \frac{|\mathrm{d}w|}{w\,\mathrm{Im}(w)}
\end{aligned}$$

□

Corollary 4.8. In $(\mathcal{H}, g_{\mathrm{hyp}})$ every pair of points is joined by a unique geodesic and the geodesics are vertical straight lines and semicircles centred on $\mathbb{R}$.



Proof. Our isometry $\mathcal{H} \xrightarrow{T} D$ is a Möbius map sending $\mathbb{R} \cup \{\infty\}$ to ∂D and Möbius maps preserve circles and orthogonality. □

Remarks

(1) When we discussed surfaces in $\mathbb{R}^3$ with constant Gauss curvature, we saw that if something had $\kappa = -1$, its FFF in Fermi coordinates was

$$\mathrm{d}u^2 + \cosh^2 u \mathrm{d}v^2$$

and after a change of variables we got

$$\frac{\mathrm{d}v^2 + \mathrm{d}w^2}{w^2}$$

(Lecture 17). Thus $\mathcal{H}$ (and hence D) has Gauss curvature -1.

(2) Suppose we looked for a metric $d : D \times D \to \mathbb{R}_{\geq 0}$ on D with properties:

(a) Möb(D)-invariant:

$$d(Tx, Ty) = d(x, y) \qquad \forall x, y \in D, \forall T \in \text{Möb}(D)$$

(b) $\mathbb{R}_+ \cap D$ is "length-minimising".

Möb(D)-invarianve means that d is completely determined by $d(0, a)$ for $a \in \mathbb{R}_+ \cap D$. Let us call this $p(a)$. If $\mathbb{R}_+ \cap D$ is "length-minimising", then if $0 < a < b < 1$,

$$\underbrace{d(0, a)}_{p(a)} + \underbrace{d(a, b)}_{p\left(\frac{b-a}{1-ab}\right)} = \underbrace{d(0, b)}_{p(b)}$$

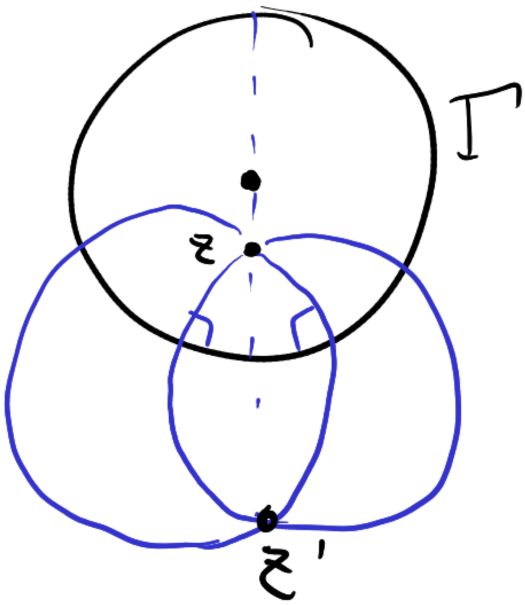
If we furthermore suppose p is differentiable and differentiate with respect to b and set $a = b$ we get

$$p'(a) = \frac{p'(0)}{1 - a^2}$$

hence $p(a) = C \tanh^{-1}(a)$ for some constant C. So up to scaling, the length metric associated with g_{hyp} is the only metric with these nice properties. The scale is chosen so that $\kappa \equiv -1$.

What about the *full* isometry group of $D(, g_{\text{hyp}})$ on $(\mathcal{H}, g_{\text{hyp}})$? The result is that we need to add "reflections" in hyperbolic lines. These are called *inversions*.

Definition. Let $\Gamma \subset \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be a circle or a line. We say that points $z, z' \in \hat{\mathbb{C}}$ are *inverse* for Γ if every circle through z and orthogonal to Γ also passes through z'.



Lemma 4.9. For every circle $\Gamma \subset \hat{\mathbb{C}}$ there is a unique inverse point with respect to Γ for z.

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Proof. Recall Möbius maps send circles (in $\hat{\mathbb{C}}$) to circles and preserve cycles. So if z and z' are inverse for Γ and $T \in$ Möb, then Tz and Tz' are inverse for $T(\Gamma)$. If $\Gamma = \mathbb{R} \cup \{\infty\}$, then $J(z) = \overline{z}$ gives inverse points; this map satisfies the requirements and is unique such.

Now if Γ is any circle, there exists $T \in$ Möb such that $T(\mathbb{R} \cup \{\infty\}) = \Gamma$. Define inversion in Γ by $J_\Gamma = TJT^{-1}$. This works (one should also check that this is unique, but this is left as an exercise). □

Definition. The map $z \to J_\Gamma(z)$ sending z to the unique inverse point z' for z with respect to Γ is called *inversion* in Γ. (This fixes all points in Γ and exchanges the two complementary regions).

Examples

(1) If Γ is a straight line (circle in $\hat{\mathbb{C}}$ through $\infty \in \hat{\mathbb{C}}$), then J_Γ is reflection in Γ.

(2) If $S^1 = \{z : |z| = 1\}$ then $J_{S^1}(z) = \frac{1}{\overline{z}}$ (see Example Sheet 4).

Remark. A composition of *two* inversions is a Möbius map. Let $C(z) = \bar{z}$ be inversion in $\mathbb{R} \cup \{\infty\}$. So if $\Gamma \subset \hat{\mathbb{C}}$, $J_\Gamma = TCT^{-1}$ where $T(\mathbb{R} \cup \{\infty\}) = \Gamma$. Now given circles Γ_1 and Γ_2 and $T_i(\mathbb{R} \cup \{\infty\}) = \Gamma_i$,

$$\begin{aligned} J_{\Gamma_1} \circ J_{\Gamma_2} &= (J_{\Gamma_1} \circ C) \circ (C \circ J_{\Gamma_2}) \\ &= (C \circ J_{\Gamma_1})^{-1} \circ (C \circ J_{\Gamma_2}) \end{aligned}$$

and also note

$$C \circ J_\Gamma = C \circ T \circ C \circ T^{-1}$$

We know $T^{-1} \in$ Möb so want to prove $C \circ T \circ C \in$ Möb. Let

$$T(z) = \frac{az + b}{cz + d}$$

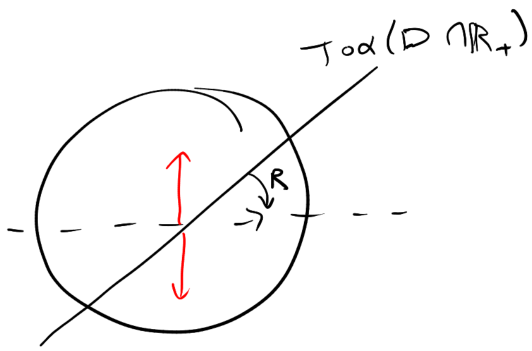
Then

$$C \circ T \circ C(z) = \frac{\bar{a}z + \bar{b}}{\bar{c}z + \bar{d}} \in \text{Möb}$$

Using this, we have constructed a Möbius map $J_{\Gamma_1} \circ J_{\Gamma_2}$ which maps Γ_1 to Γ_2.

Lemma 4.10. An orientation preserving isometry of $(\mathbb{H}, g_{\text{hyp}})$ is an element of Möb$(\mathbb{H})$, where $\mathbb{H} = D$ or $\mathbb{H}\mathcal{H}$. The *full* isometry group is generated by inversions in hyperbolic lines (circles perpendicular to $\partial\mathbb{H}$).

Proof. Suffices to prove this in either model. In D, inversion in $\mathbb{R} \cap D$, i.e. conjugation, preserves $g_{\text{hyp}} = \frac{4|\mathrm{d}z|}{(1-|z|^2)^2}$. Now Möb$(D)$ acts transitively on geodesics, and it acts by isometries, so all inversions in hyperbolic lines are isometries. Have we got them all? Now suppose $\alpha \in \text{Isom}(D, g_{\text{hyp}})$. Define $a = \alpha(0) \in D$ and using $T(z) = \frac{z-a}{1-\bar{a}z}$ we see that $T \circ \alpha(0) = 0$. Now $\exists$ a rotation $R \in$ Möb(D) such that $R \circ T \circ \alpha$ sends $D \cap \mathbb{R}_+$ to itself (fixed set wise).



$$D(R \circ J \circ \alpha)|_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Composing with conjugation C if necessary, there exists $A \in \text{Isom}(D, g_{\text{hyp}})$ of the required form such that $A \circ \alpha$ fixes $\mathbb{R} \cap D$ pointwise and fixes $i\mathbb{R} \cap D$ pointwise, so

$$D(A \circ \alpha)|_0 = \text{id}$$

Now implies $A \circ \alpha = \text{id}$. Hence α has the required form. In Example Sheet 4, you'll prove that every Möbius map is a product of inversions. If α preserved orientation and fixed $\mathbb{R} \cap D$, it necessarily fixes $i\mathbb{R} \cap D$ and so in fact $\alpha = (R \circ T)^{-1} \in \text{Möb}$. □

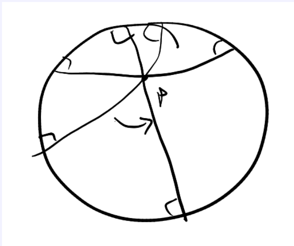
Remark. In the $\mathcal{H}$ model,

$$\begin{aligned} \text{Möb}(\mathcal{H}) &= \text{PSL}_2(\mathbb{R}) \\ &= \left\{ z \mapsto \frac{az+b}{cz+d} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}) \right\} \end{aligned}$$

(Note $\mathbb{R}$ rather than $\mathbb{C}$!) and $d_{\text{hyp}}(p, q) = 2 \tanh^{-1} \left| \frac{p-q}{p-\bar{q}} \right|$.

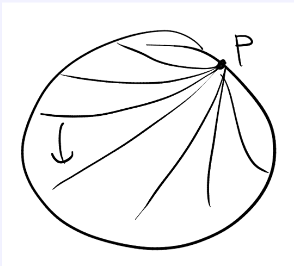
Definition. Let $\alpha \in \text{Isom}^+(\mathbb{H}) = \text{Möb}(\mathbb{H})$. Suppose $\alpha \neq \text{id}$. We say that α is:

- *elliptic* if α fixes $p \in \mathbb{H}$.



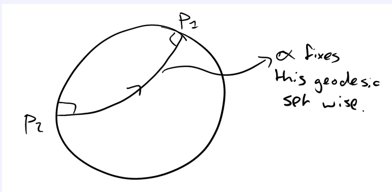
  If $p = 0 \in D$ this is a rotation.

- *parabolic* if α fixes a unique point of $\partial\mathbb{H}$



  If $p = \infty \in \partial\mathcal{H}$ then $\alpha(z) = \alpha(z + t)$.

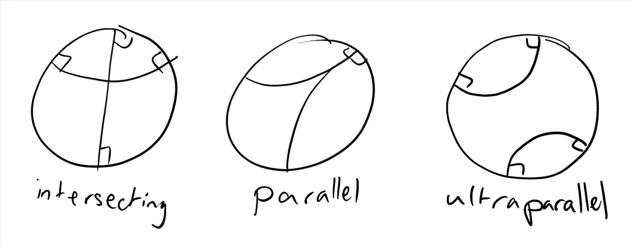
- *hyperbolic* if α fixes 2 points of $\partial\mathbb{H}$.



Exercise: All elements of Möb($\mathbb{H}$) fall into one of 3 cases.

Definition. Let l, l' be hyperbolic lines. We say that l, l' are

- *parallel* if they emet in $\partial\mathbb{H}$ but not in $\mathbb{H}$.
- *ultraparallel* if they don't meet in $\mathbb{H} \cup \partial\mathbb{H}$.
- *intersecting* if they meet in $\mathbb{H}$.

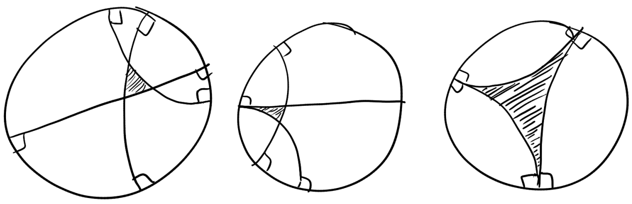


The parallel postulate of Euclid fails!

Hyperbolic Triangles

Definition. A *hyperbolic triangle* is the region bounded by 3 hyperbolic lines, not two of which are ultraparallel. Vertices lying at infinity ($\partial\mathbb{H}$) are called *ideal* vertices.

An *ideal triangle* has all its vertices ideal.



Next lecture we'll study triangles in more detail and compute their area.

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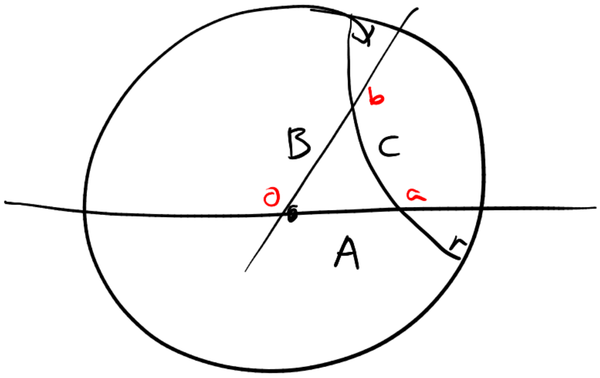
Note: $g_{\text{hyp}} = \frac{4(\mathrm{d}u^2+\mathrm{d}v^2)}{(1-u^2-v^2)^2}$, so $E = G$, $F = 0$ so this is *conformal* to the flat metric. Angles computed with respect to g_{hyp} agree with Euclidean angles.

Hyperbolic cosine formula

$\cosh C = \cosh A \cosh B - \sinh A \sinh B \cos\gamma$.

Proof. To simplify, by an isometry, put vertex with angle γ at $0 \in D$ and put vertex with angle β on $\mathbb{R}_+ \cap D$.



Then

$$A = d_{\text{hyp}}(0, a) = 2\tanh^{-1}(a)$$

$$a = \tanh\frac{A}{2}$$

and

$$b = e^{i\gamma}\tanh(B/2)$$

$$\left|\frac{b-a}{1-\overline{a}b}\right| = \tanh\left(\frac{C}{2}\right)$$

If $t = \tanh(\lambda/2)$, "recall"

$$\cosh(\lambda) = \frac{1+t^2}{1-t^2}$$

$$\sinh(\lambda) = \frac{2t}{1-t^2}$$

so

$$\begin{aligned}
\cosh(A) &= \frac{1+|a|^2}{1-|a|^2} \\
\cosh(B) &= \frac{1+|b|^2}{1-|b|^2} \\
\cosh C &= \frac{|1-\overline{a}b|^2 + |b-a|^2}{|1-\overline{a}b|^2 - |b-a|^2} \\
&= \frac{(1+|a|^2)(1+|b|^2) - 2(\overline{a}b + a\overline{b})}{(1-|a|^2)(1-|b|^2)}
\end{aligned}$$

But $a \in \mathbb{R}$ and $b + \overline{b} = 2\operatorname{Re}(b) = 2b\cos\gamma$. Using the above one can check that

$$\cosh C = \cosh A \cosh B - \sinh A \sinh B \cos\gamma$$

as desired. □

Remarks

(1) If A, B and C small, then $\sinh \approx A$, $\cosh \approx 1 + \frac{A^2}{2}$ then the formula reduces to

$$C^2 = A^2 + B^2 - 2AB\cos\gamma$$

(up to higher order terms), which is just the usual Euclidean cosine formula. Recall: dilating a surface in $\mathbb{R}^3$ rescaled its curvature. Similarly, "zooming in" to any point on any abstract surface with a Riemannian metric, the surface looks closer and closer to being flat.

(2) $\cos\gamma \geq -1$, so formula gives

$$\cosh C \leq \cosh A \cosh B + \sinh A \sinh B = \cosh(A+B)$$

cosh is increasing, so this implies $C \leq A + B$, which is the triangle inequality! So hyperbolic cosine formula is a refinement of the triangle inequality.

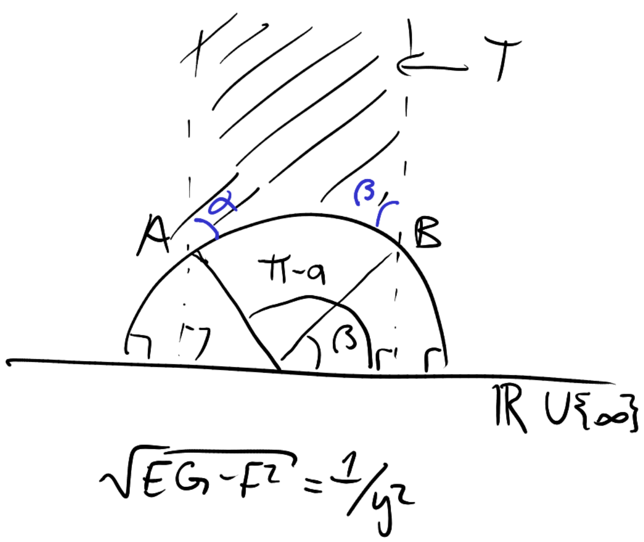
Areas of triangles

Let $T \subset \mathbb{H}$ be a hyperbolic triangle, with internal angles α, β, γ.

$$\boxed{\text{Area}_{\text{hyp}}(T) = \pi - \alpha - \beta - \gamma}$$

This is a version of the *Gauss-Bonnet* theorem. We allow T to have *ideal vertices.*

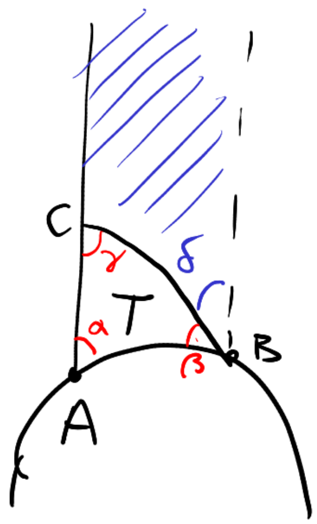
Proof. We can work in any model. We take $\mathbb{H} = \mathcal{H}$. First we compute the area of the traignel assuming $\gamma = 0$ (so one vertex is at ∞). Using an isometry we can assume that two vertices are on the unit circle as in the figure:



Then

$$\begin{aligned}\text{Area}(T) &= \int_{\cos(\pi-\alpha)}^{\cos\beta} \mathrm{d}x \int_{\sqrt{1-x^2}}^{\infty} \frac{\mathrm{d}y}{y^2} \\ &= \int_{\cos(\pi-\alpha}^{\cos\beta} \frac{\mathrm{d}x}{\sqrt{1-x^2}} \\ &= \int_{\pi-\alpha}^{\beta} -\mathrm{d}\theta \\ &= \pi-\alpha-\beta\end{aligned}$$

For the general case, we can always arrange for one of the sides of a triangle to be a vertical line:



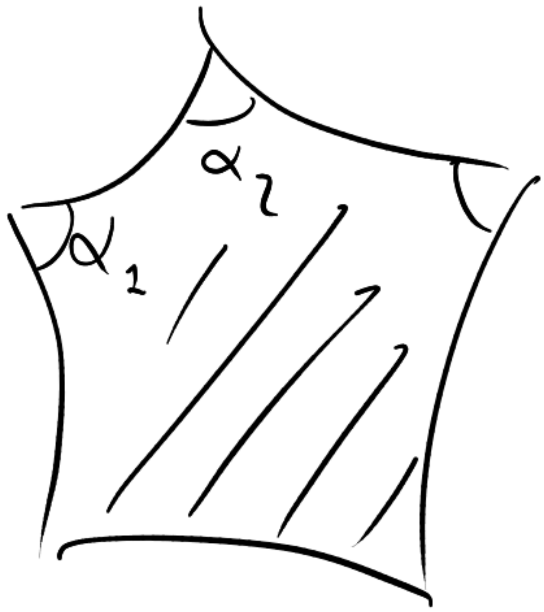
Then

$$\begin{aligned}\text{Area}(T) &= \pi-\alpha-(\beta+\delta)-(\pi-\delta-\pi+\gamma) \\ &= \pi-\alpha-\beta-\gamma\end{aligned}$$ □

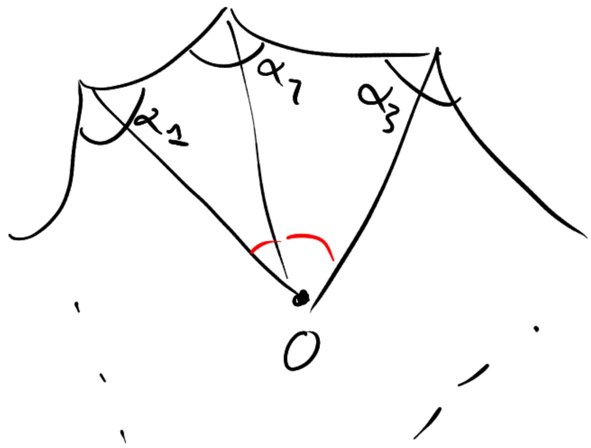
Corollary 4.11. The area of hyperbolic n-gon (with sides being arcs of hyperbolic lines) is given by the formula

$$(n-2)\pi-(\alpha_1+\cdots+\alpha_n)$$

where $\alpha_1,\ldots,\alpha_n$ are the internal angles.



Proof. Divide into n triangles

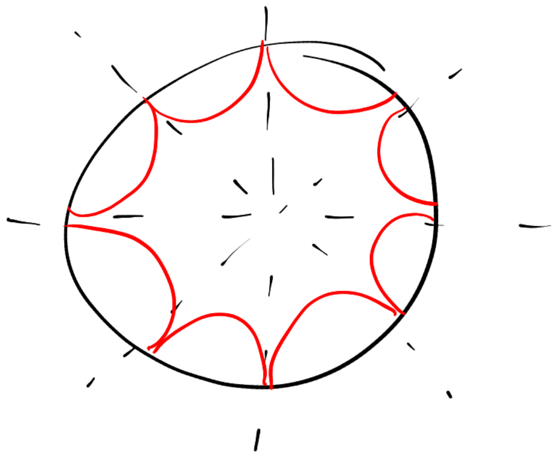


The n triangles have interior angles adding up to $2\pi + \sum_{i=1}^n \alpha_i$.

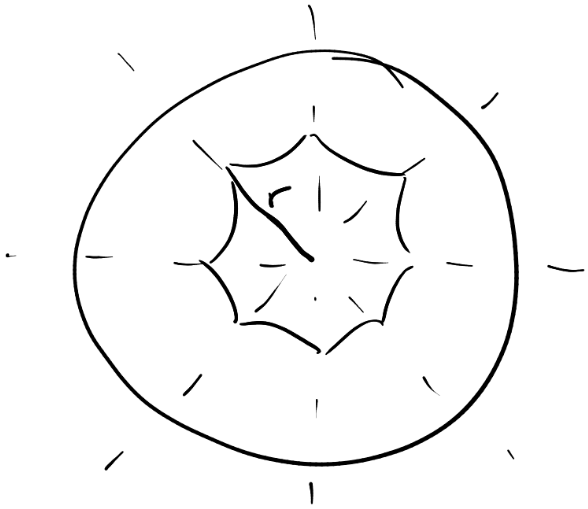
$$\implies \text{Area}(n\text{-gon}) = n\pi - \left(2\pi + \sum_{i=1}^n \alpha_i\right)$$
$$= (n-2)\pi - \sum_{i=1}^n \alpha_i \qquad \square$$

Lemma 4.12. If $g \geq 2$, there is a regular $4g$-gon in $\mathbb{H}$ with internal angle $\frac{\pi}{2g}$.

Proof. Take an *ideal* $4g$-gon in D with all vertices at ∂D being the $4g$-roots of unity



"slide" all vertices radially inwards:



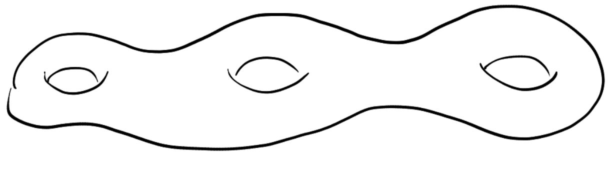
Then $A(r) = (4g-2)\pi - 4g\alpha(r)$. $A(r) \to 0$ as $r \to 0$, and $A(1) = (4g-2)\pi$. Hence, by the intermediate value theorem there exists r_0 for which $4\pi g - 4\pi = A(r_0) = (4g-2)\pi - 4g\alpha(r_0)$, which gives $\alpha(r_0) = \frac{\pi}{2g}$. ☐

We'll use this lemma to construct a Riemannian metric with $\kappa = -1$ a surface of genus ≥ 2.

Start of lecture 23

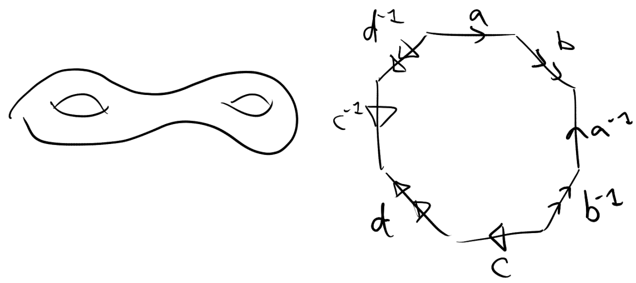
Theorem 4.13. For each $g \geq 2$, there exists an abstract Riemannian metric on the compact orientable surface of genus g with curvature $\kappa \equiv -1$ (locally isometric to $\mathbb{H}$).

Recall



g is the number of holes, $\chi = 2 - 2g$. $g = 0$, round sphere, $\kappa = 1$. $g = 1$, torus, $\mathbb{R}^2/\mathbb{Z}^2$ curvature $\kappa \equiv 0$.

Sketch proof. This will be an outline of the key points. Recall that:



Analogously, a $4g$-gon with side labels $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$ would give (after gluing) an orientable surface Σ_g of genus g.

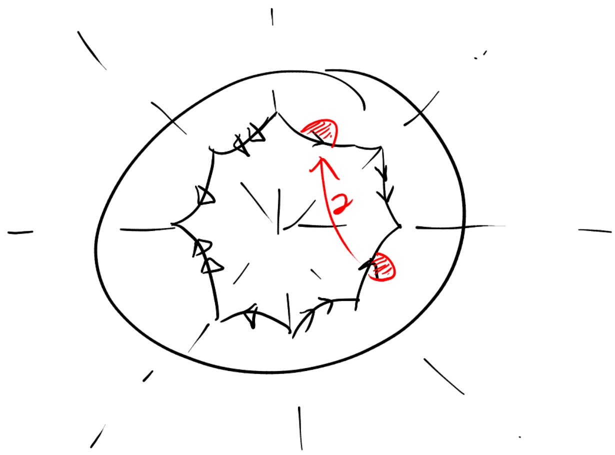
Observation: Let's say that a *flag* comprises:

(i) An oriented hyperbolic line.

(ii) A point on that line.

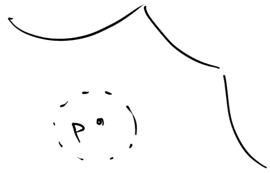
(iii) A choice of side to the line.

Given two such flags, there is a hyperbolic isometry taking one to the other: we can swap sides using inversions if *needed.* Now consider the regular $4g$-gon from Lemma 4.11 with internal angle $\frac{\pi}{2g}$.

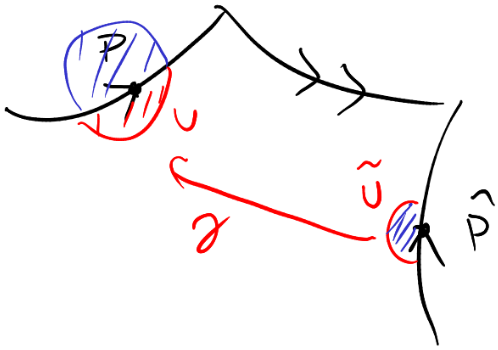


For each paired set of 2 edges, there is a hyperbolic isometry taking one to the other (respecting orientation) and taking "inside" of the polygon at e_1 to the "outside" of its paired edge e_2. Now we'll give an atlas for Σ_g as follows:

- If $p \in \text{Interior}(P)$, just take a small disc contained in interir of P and include it into $D \subset \mathbb{R}^2$.



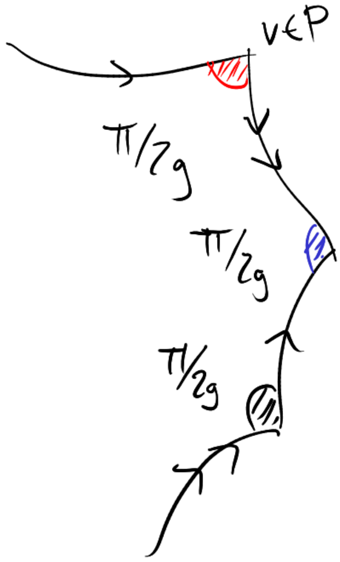
- If $p \in$ edge(P) (not a vertex), say e_1 and $\hat{p} \in e_2$ and $\gamma_{\hat{p}} = p$, we know that γ exchanges sides.

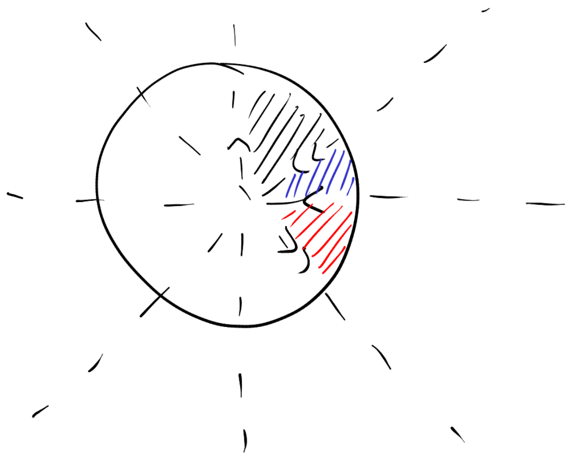


$$[p] = [\hat{p}] \in \Sigma_g = \text{Polygon} / \sim$$

Define $U \cup \tilde{U} \to D$ to be the inclusion in D and γ on $\tilde{U}$. These descend to maps on $[U] \subset \Sigma$, $[\tilde{U}] \subset \Sigma$ which agree on $[U \cap \tilde{U}]$ (projection to Σ_g).

- In our gluing pattern all $4g$ vertices are identified to one point on Σ_g and we'd like a chart there.





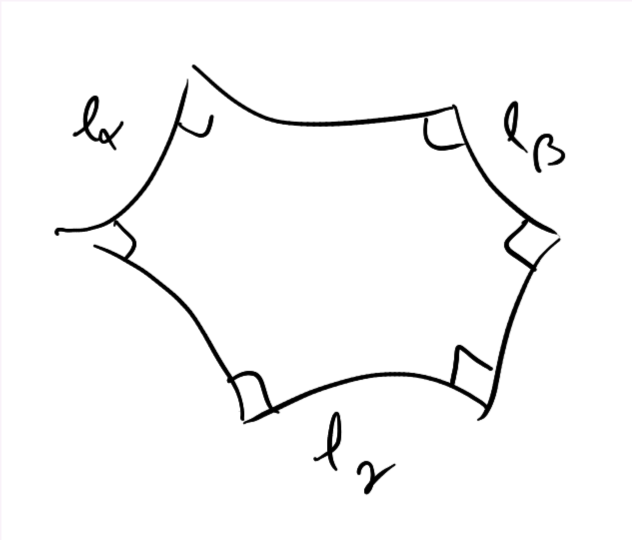
Condition that internal angles sum to 2π means that we have a neighbourhood of $[v] \in \Sigma_g$ that defines a chart at $[v] \in \Sigma_g$.

All chrarts are obtained either from inclusion into D or the composite of inclusion and some hyperbolic isometry (so smooth) and we got an atlas that defines a Riemannian metric on Σ_g that is locally isometric to $\mathbb{H}$. $\square$

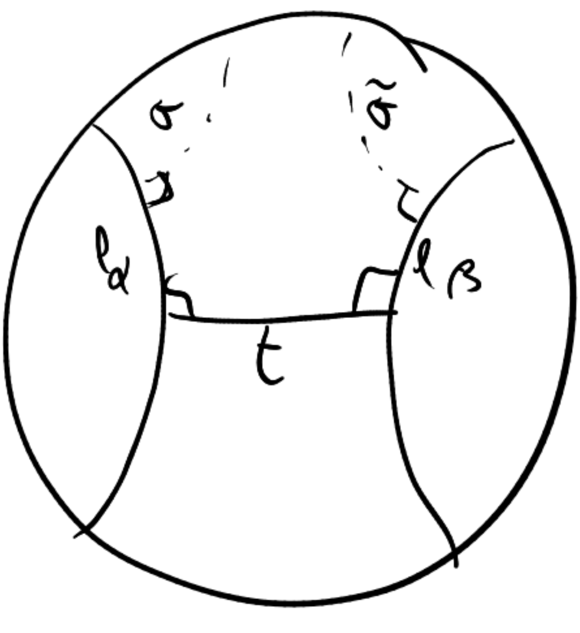
Remark. Really what is going on is that $\Sigma_g = \mathbb{H}/\Gamma$ where $\Gamma = \langle \gamma_1, \ldots, \gamma_{2g} \rangle$ Fuchsian group. ($T^2 = \mathbb{R}/\mathbb{Z}^2$). Algebraic topology: $\mathbb{H} \to \mathbb{H}/\Gamma$.

Hyperbolic *right-angled hexagons* are also very useful for similar constructions.

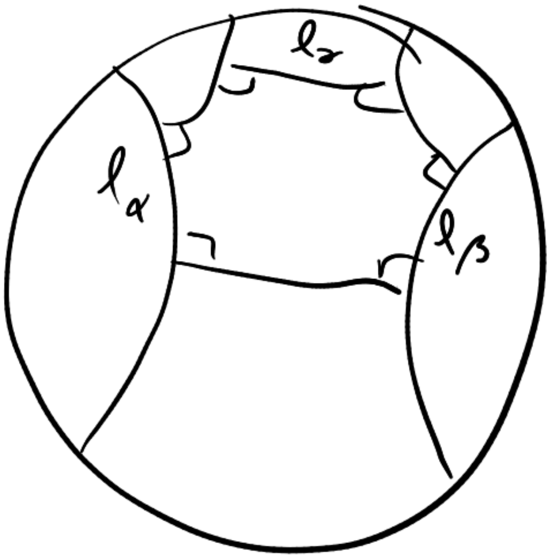
Lemma 4.14. For each $l_\alpha, l_\beta, l_\gamma$ positive numbers, there exists a right-angle hexagon with side lengths $l_\alpha, ?, l_\beta, ?, l_\gamma, ?$ in cyclic order:



Proof. Take a pair of ultraparallel lines. Example sheet 4 implies there exists a unique common perpendicular geodesic

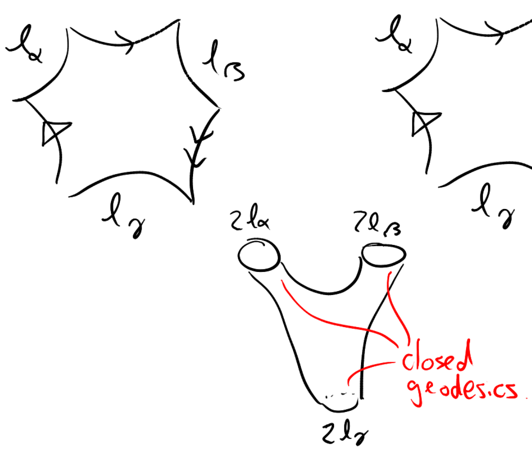


Given l_α and $l_\beta > 0$ we shoot off new geodesics σ and $\tilde{\sigma}$ orthogonal to the originals and having travelled l_α and l_β from common perpendicular. In fact given $t > 0$ there exists an original ultraparallel pair distance exactly t apart. There exists a threshold value t_0, by continuity when the new common perpendicular has length l_γ.



This is our right-angled hexagon. □

Gluing two identical right-angled hexagons gives rise to a *pair of pants*:



g, $\kappa \equiv -1$.

Interesting things to look up: Teichmüller space, Moduli space.

Start of lecture 24

5 Further Topics

Gauss Bonnet theorem revisited

Recall:

(i In a spherical triangle T with internal angles α, β, γ we saw in Example Sheet 2 that

$$\text{Area}_{S^2}(T) = \alpha + \beta + \gamma - \pi$$

while a hyperbolic triangle has area

$$\text{Area}_{\mathbb{H}}(T) = \pi - \alpha - \beta - \gamma$$

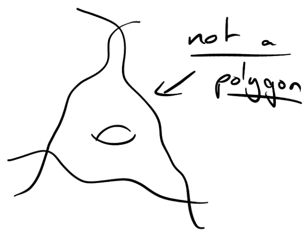
(ii We also saw that for compact surfaces $\Sigma \subset \mathbb{R}^3$,

$$\int_\Sigma \kappa \mathrm{d}A = 2\pi\chi(\Sigma)$$

Theorem 5.1 (Local Gauss-Bonnet)**.** Let Σ be an abstract smooth surface with Riemannian metric g. Take a *geodesic polygon* R on Σ, i.e. it is homeomorphic to a disc and its boundary is decomposed into finitely many geodesic arcs. Then

$$\int_R \kappa \mathrm{d}A = \sum_{i=1}^{n} \alpha_i - (n-2)\pi$$

where $n = \#$ of arcs, and α_i are the internal angles of the polygon.



Theorem 5.2 (Global Gauss-Bonnet)**.** If Σ is a compact smooth surface with Riemannian metric g, then

$$\int_\Sigma \kappa \mathrm{d}A = 2\pi\chi(\Sigma)$$

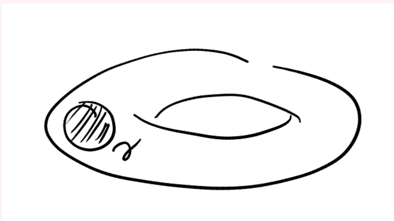
Remark. (i) κ and $\mathrm{d}A$ can be defined just using g.

(ii) For our hyperbolic surfaces observed by identifying the edges of a regular $4g$-gon with angles $\frac{\pi}{2g}$

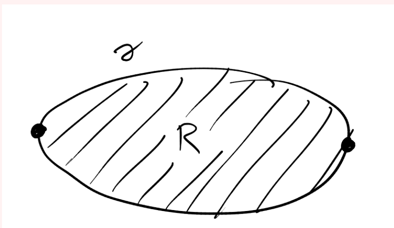
$$\begin{aligned}\int_\Sigma 1\mathrm{d}A &= \text{Area(Polygon)} \\ &= (4g-2)\pi - \sum_{n=1}^{4g} \frac{\pi}{2g} \\ &= (4g-4)\pi \\ &= 2(2g-2)\pi\end{aligned}$$

and we had $\chi(\Sigma_g) = 2 - 2g$ and $\kappa \equiv -1$, so this agrees with the Global Gauss-Bonnet.

(iii) If Σ is a flat surface so $\kappa = 0$ and γ is a closed geodesic, i.e. $\gamma : \mathbb{R} \to \Sigma$ and $\exists T > 0$ such that $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$, then γ cannot bound a disc:



Indeed, if we had such a γ, then



is a geodesic polygon with internal angles π and local Gauss-Bonnet gives:

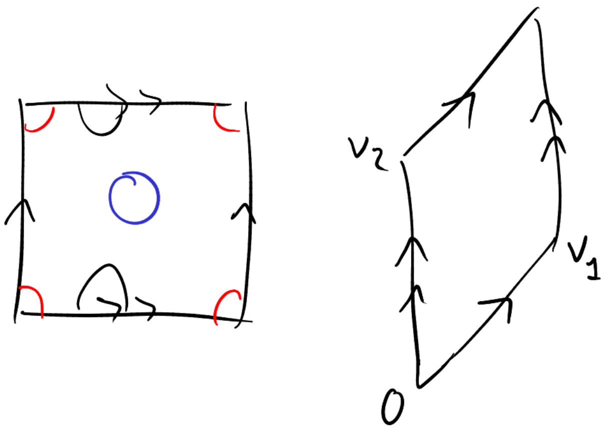
$$0 = \int_R \kappa \mathrm{d}A = \sum_{i=1}^n \alpha_i - (n-2)\pi = 2\pi$$

(since $n = 2$), so such a polygon would violate Local Gauss-Bonnet. However, in a non-flat metric, we clearly *can* have such a geodesic:

Flat metrics

Question: Can we understand *all* flat metrics on T^2?

The key to get a flat metric on T^2 was an atlas where transition functions were *isometries*.



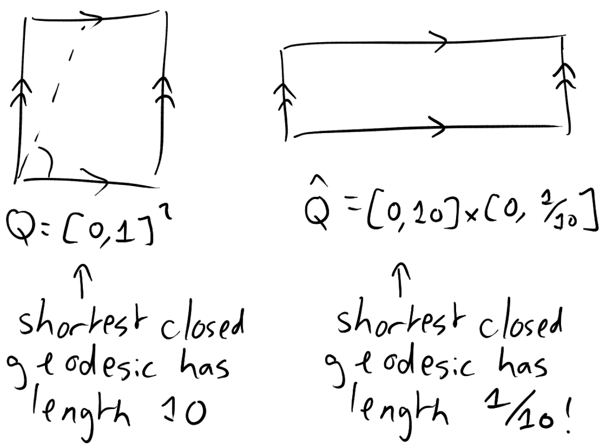
So *any* parallelogram $Q \subset \mathbb{R}^2$ delivers a flat metric g_Q on T^2.

$$(T^2, g_Q) = \mathbb{R}^2/\mathbb{Z}v_1 \oplus \mathbb{Z}v_2$$

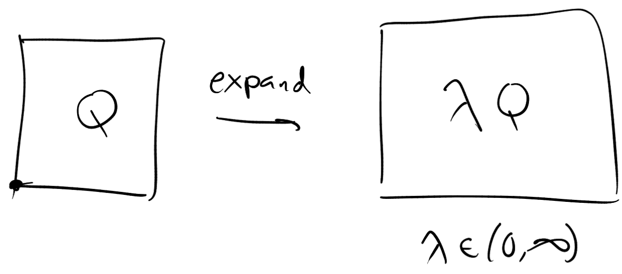
Observation:

$$\mathrm{Area}_{g_Q}(T^2) = \mathrm{Area}_{\mathrm{Euclidean}}(Q)$$

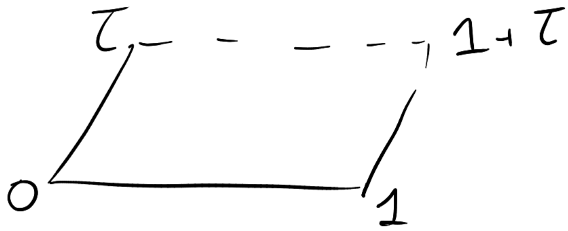
Since Q_1 and Q_2 could have different ares, the metrics g_{Q_1} and g_{Q_2} are not *isometric*.



We'll consider flat metrics up to dilations:



Under this assumption, given Q, we can put vertices at $0 \in \mathbb{R}^2$, $1 \in \mathbb{R}^2$ and $\tau \in \mathcal{H}$.



This defines a map

$$\mathcal{H} \to \{\text{flat metrics on } T^2\}/\text{Dilations}$$

But *diffeomorphisms* acct on the set of *flat metrics*. Given g and $g : T^2 \to T^2$ diffeomorphism, we can "pull-back" g by f:

$$Df|_p : T_p\Sigma \to T_{f(p)}\Sigma$$

f^*g is given by

$$\langle u, w\rangle_{f^*g} = \langle Df|_p(v), Df|_p(w)\rangle_g$$

$v, w \in T_p\Sigma$. $(\Sigma, f^*g) \xrightarrow{f} (\Sigma, g)$ isometry?

Now $\mathrm{SL}(2,\mathbb{Z})$ acts on T^2 by diffeomorphisms: it acts on $\mathbb{R}^2$ and preserves the lattice $\mathbb{Z}^2$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$$

$\mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$. $\mathrm{SL}(2,\mathbb{Z})$ also acts on $\mathcal{H}$ by Möbius maps (as isometries of g_{hyp}!)

Theorem 5.3. The map

$$\mathcal{H} \to \{\text{flat metrics on } T^2\}/\text{Dilations}$$

descends to a map

$$\mathcal{H}/\operatorname{SL}(2,\mathbb{Z}) \to \{\text{flat metrics on } T^2\}/\text{Dilations and Diffomorphisms}^+$$

which is a $\boxed{\text{bijection}}$. We say that $\mathcal{H}/\operatorname{SL}(2,\mathbb{Z})$ is the *moduli space of flat metrics of* T^2.