Fluid Dynamics

March 14, 2023

Contents

Lectures

Start of lecture 1

0 Preliminary Introduction

Continuum Mechanics:

- Fluid dynamics (liquids, gases)
- Solid mechanics (solids, elasticity, fracture)
- Other (complex fluids, soft matter, biomechanics)

Average over irrelevant molecular detail to get a continuum description in terms of fields. For example

- *velocity* $\mathbf{u}(\mathbf{x}, t)$
- *pressure* $p(\mathbf{x}, t)$
- *density* $\rho(\mathbf{x}, t)$

$\mathbf{x} = \mathbf{U}(t)$ is uniform flow, $\mathbf{u} = \mathbf{U}(t) + \mathbf{R}(t) \times \mathbf{x}$ solid body motion. $\mathbf{u}(\mathbf{x})$ steady, $\mathbf{u}(\mathbf{x}, t)$ unsteady.

Simple physics - mass, momentum, Newton's Laws, vector calculus, maths methods. Combine these to find the velocity $\mathbf{u}$ and the accompanying forces.

Kinematics: velocities and trajectories

Dynamic: forces and equations of motion.

Forces: gravity, pressure, viscous stress (internal friction).

We will consider *inviscid flow* good approximation for water / air (except on very small scales). We will also consider simple viscous flow.

More to come in Part II Fluids & Waves and lots of courses in Part III.

Applications

Fluids are everywhere! Application for the environment: atmosphere / polar ice caps (climate change), pollution, ventilation, wind power. For biology: breathing, blood, flying, swimming, cells. Aerosols to astrophysics.

1 Kinematics

1.1 Pathlines & Streamlines

There are two natural perspectives on a flow $\mathbf{u}(\mathbf{x}, t)$:

(1) A stationary observer watching the flow go past (Eulerian picture)

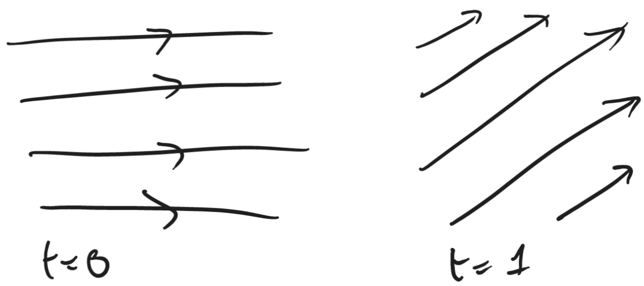
(2) A moving observer, travelling along with (some bit of) the flow.

Definition (Streamlines)**.** *Streamlines* are curves that are everywhere parallel to the flow at a given instant. Given parametrically as $\mathbf{x} = \mathbf{x}(s; \mathbf{x}_0, t_0)$ from

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \mathbf{u}(\mathbf{x}, t_0)$$

with $\mathbf{x} = \mathbf{x}_0$ at $s = 0$. (See later for stream functions). Similar to characteristics.

The set of streamlines shows the direction of flow at that instant – all particles at one time. For example $\mathbf{u} = (t, t)$



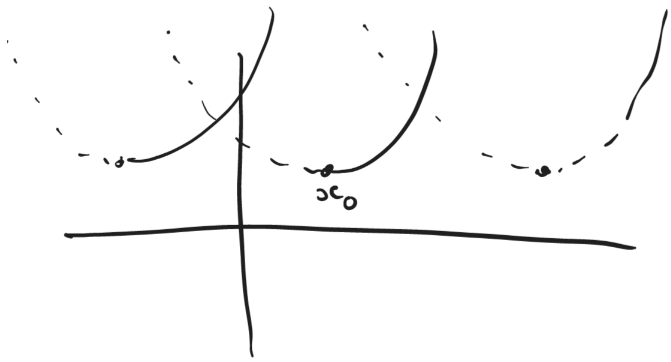
$$x = x_0 + s, y = y_0 + t_0 s \implies y = y_0 + t_0(x - x_0)$$

Definition (Pathline)**.** A *pathline* (particle path) is the trajectory of a fluid 'particle' ($\equiv$ a very small blob of fluid). The pathline $\mathbf{x} = \mathbf{x}(t; \mathbf{x}_0)$ of the particle at $\mathbf{x}_0$ at $t = 0$ found from

$$\boxed{\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}, t) \quad \text{with} \quad \mathbf{x} = \mathbf{x}_0 \quad \text{at } t = 0}$$

one particle at many times.

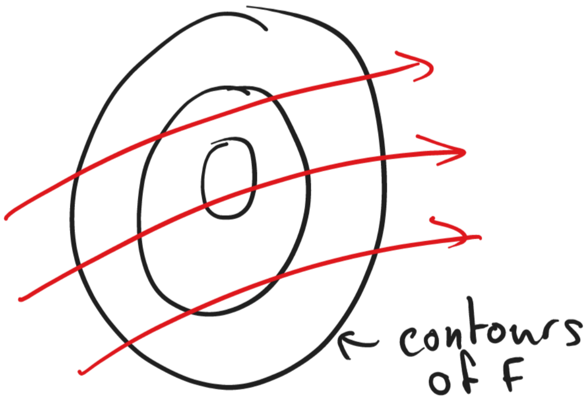
For example, $\mathbf{u} = (1, t) \implies x = x_0 + t, y = y_0 + \frac{1}{2}t^2 \implies y = y_0 + \frac{1}{2}(x - x_0)^2$



Can consider lots of particles, for example at $\mathbf{x}_0$ in a given region, to see how the shape and position of a dyed patch of fluid evolves – useful for thinking about transport (pollutant, aerosols) and mixing problems. For a *steady flow* (only), streamlines and pathlines are the same.

1.2 The Material Derivative

Rate of change moving along with the fluid (material). For example:



F varies along the flow.

For any quantity $F(\mathbf{x}, t)$, the rate of change (with time) seen by an observer moving with the fluid, $\frac{\mathrm{d}F}{\mathrm{d}t}$ is found from

$$\begin{aligned} \delta F &= F(x + \delta x, t + \delta t) - F(x, t) \\ &= \delta \mathbf{x} \cdot \nabla F + \delta t \frac{\partial F}{\partial t} + \text{higher order terms} \end{aligned}$$

Displacement of observer moving with the fluid $\delta \mathbf{x} = \mathbf{u}(\mathbf{x}, t)\delta t + \text{h.o.t.}$.

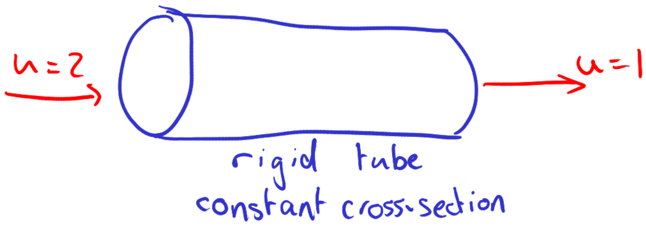
$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \mathbf{u} \cdot \nabla F$$

where $\frac{DF}{Dt}$ is the material / Lagrangian derivative, $\frac{\partial F}{\partial t}$ is the local / Eulerian derivative and $\mathbf{u} \cdot \nabla F$ is the convectional advective derivative.

Start of lecture 2

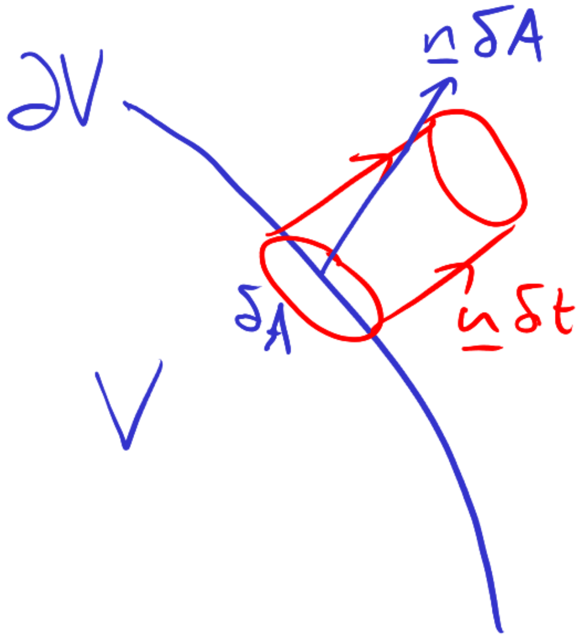
1.3 Conservation of Mass

Example



Experience says OK if fluid is air, because air is compressible, velocities make density go up (and pressure goes up). Problem if fluid is water because density of water is approximately constant, so velocities would imply mass destroyed.

Suggests there must be a relationship between $\rho(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}, t)$ as mass is not destroyed or created. Consider an arbitrary volume V, *fixed in space*, bounded by a surface ∂V with outward normal $\mathbf{n}$. Mass in V, $\int_V \rho \mathrm{d}V$, can only change due to flow of mass across the bounding surface ∂V.



Volume out over δA in time δt is $\mathbf{u}\delta t \cdot \mathbf{n}\delta A$. Mass out is $\rho\mathbf{u} \cdot \mathbf{n}\delta A\delta t$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \mathrm{d}V = -\int \rho\mathbf{u} \cdot \mathbf{n}\mathrm{d}A$$

$$\int_V \frac{\partial \rho}{\partial t}\mathrm{d}V = -\int_V \nabla \cdot (\rho\mathbf{u})\mathrm{d}V$$

The two left expressions are equal for fixed V, and the two right expressions are equal by Divergence Theorem. Since this is true for arbitrary V,

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0}$$

$\rho\mathbf{u}$ is the *mass flux*. Using $\nabla \cdot (\rho\mathbf{u}) = \mathbf{u} \cdot \nabla\rho + \rho\nabla \cdot \mathbf{u}$ can rewrite as

$$\boxed{\frac{D\rho}{Dt} = -\rho\nabla \cdot \mathbf{u}}$$

The density of blob decreases if the velocity is divergent (spreading out). If a fluid is *incompressible*, i.e. the density of a fluid particle can't change, $\frac{D\rho}{Dt} = 0$

$$\implies \boxed{\nabla \cdot \mathbf{u} = 0}$$

incompressible flow. This course considers flows where the density of a fluid is constant and uniform: $\rho = \text{const.}$

Note (Non-examinable)**.** Incompressibility is a very good approximation if

$$|\mathbf{u}| \ll \text{sound speed } c$$

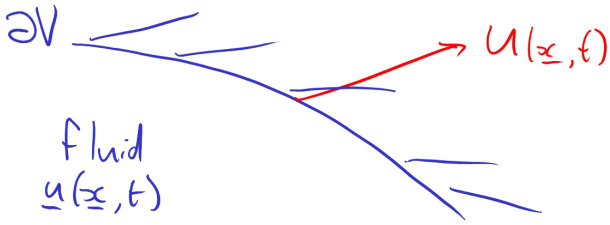
and

$$\text{timescale} \ll \frac{\text{lengthscale}}{c}$$

$c = 330\text{ms}^{-1}$ (air), $c = 1500\text{ms}^{-1}$ (water).

1.4 Kinematic Boundary Condition

Mass conservation at a boundary. Suppose the material boundary of a body of fluid has velocity $(\mathbf{x}, t)$.



At a point $\mathbf{x}$ on boundary, fluid velocity relative to the moving boundary is $\mathbf{u}(\mathbf{x}, t) - \mathbf{U}(\mathbf{x}, t)$. Condition that there is no mass flux across boundary $\rho(\mathbf{u} - \mathbf{U}) \cdot \mathbf{n} \delta D \delta t = 0$, i.e.

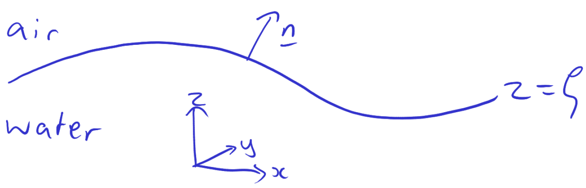
$$\mathbf{u} \cdot \mathbf{n} = \mathbf{U} \cdot \mathbf{n}$$

For example:

(1) At a stationary rigid boundary $\mathbf{U} = \mathbf{0} \implies \mathbf{u} \cdot \mathbf{n} - 0$. So a fixed boundary is a streamline.

(2) Water waves (Section 5) have an air-water interface $z = \zeta(x, y, t)$. Think of the water surface as a contour of the function

$$F(x, y, z, t) = z - \zeta(x, y, t)$$



Then $\mathbf{n} \parallel \nabla F = (-\zeta_x, -\zeta_y, 1)$. Can take $\mathbf{U} = (0, 0, \zeta_t)$, $\mathbf{u} = (u, v, w)$ hence

$$-u\zeta_x - v\zeta_y + w = \zeta_t$$

(at $z = \zeta$). Equivalent to $\frac{D}{Dt}(z - \zeta) = 0$, i.e. fluid particles on a surface remain on surface.

1.5 Stream Function for 2D Incompressible Flow

Incompressible $\equiv \nabla \cdot \mathbf{u} = 0 \iff \mathbf{u} = \nabla \times \mathbf{A}$ for some vector potential $\mathbf{A}$. For a 2D flow, $\mathbf{u} = (u(x, y), v(x, y), 0)$ can take

$$\mathbf{A} = (0, 0, \psi(x, y)) \implies \mathbf{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The scalar $\psi(x, y)$ is the *stream function.*

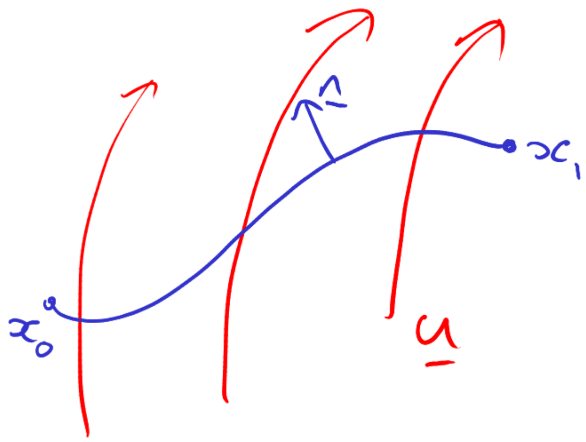
Properties of the stream function

Exercises on Question 4 of Example Sheet 1:

(i) The streamlines are given by $\psi = \text{const}$ ($\mathbf{u}$ is parallel to the contours of ψ).

(ii) $|\mathbf{u}| = |\nabla \psi|$ so flow faster when streamlines are closer.

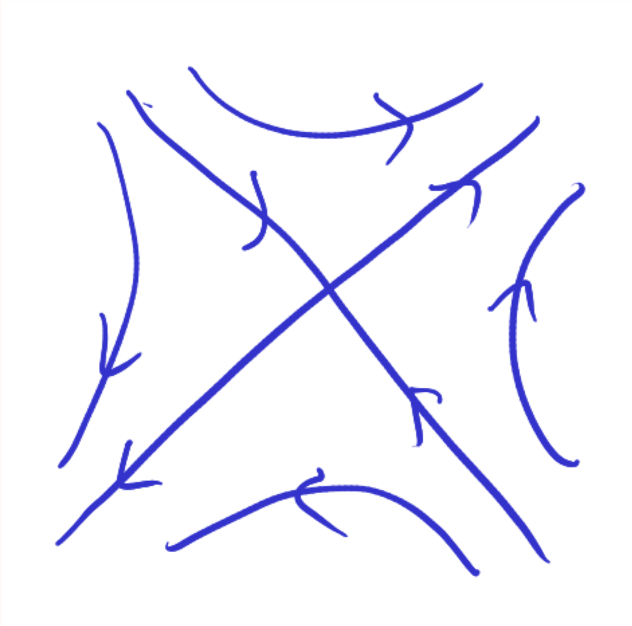
(iii) $\psi(\mathbf{x}_1) - \psi(\mathbf{x}_0) = \int_{x_0}^{x_1} \mathbf{u} \cdot \mathbf{n} \mathrm{d}l$ = volume flux crossing line from $\mathbf{x}_0$ to $\mathbf{x}_1$.



$\mathbf{n}\mathrm{d}l = (+\mathrm{d}y, -\mathrm{d}x)$.

(iv) $\psi = \text{const}$ on a stationary rigid boundary. (k.b.c and iii)

Example. $\mathbf{u} = (y, x)$, $\nabla \cdot \mathbf{u} = 0$ so a stream function exists. $\frac{\partial \psi}{\partial y} = y$ so $\psi = \frac{1}{2}y^2 + f(x)$. $-\frac{\partial \psi}{\partial x} = x$ so $f(x) = -\frac{1}{2}x^2$. Streamlines are $\frac{1}{2}(y^2 - x^2) = \text{const}$.



2D Incompressible Flow in Polar Coordinates

$$\mathbf{u} = (u_r(r,\theta), u_\theta(r,\theta), 0)$$

$$\mathbf{A} = (0, 0, \psi(r,\theta))$$

$$\implies \mathbf{u} = \nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}, -\frac{\partial \psi}{\partial r}, 0\right)$$

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0$$

Note (Non-examinable)**.** Axisymmetric Flow, in spherical polar coordinates (r, θ, ϕ), has

$$u_\phi = \frac{\partial}{\partial \phi} = 0$$

If $\nabla \cdot \mathbf{u} = 0$ and we take $\mathbf{A} = \left(0, 0, \frac{\partial \Psi(r,\theta}{\partial r \sin\theta}\right)$ then

$$\mathbf{u} = \left(\frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial \theta}, -\frac{1}{r \sin\theta}\frac{\partial \Psi}{\partial r}, 0\right)$$

is parallel to contours of Ψ. The Stokes stream function $\Psi(r, \theta)$ – stream tubes. (see textbooks for other properties).

Start of
lecture 3

2 Dynamics of Inviscid Flow

2.1 Surface and Volume Forces

Two types of force act on a fluid (liquid / gases)

(i) Those proportional to the volume, like gravity

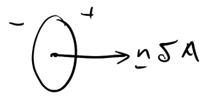
(ii) Those proportional to the area, like *pressure* or *viscous stress* (friction between moving fluid and neighbouring material, whether a boundary or other fluid).

(i) Volume / body forces

We denote the force on a small volume element $\mathbf{f}(\mathbf{x}, t)\delta V$. $\mathbf{f}$ is often *conservative* with potential energy / unit volume χ and $\mathbf{f} = -\nabla\chi$. The most common case for us $\mathbf{f}\delta V = \rho\mathbf{g}\delta V$ with $\chi = \rho g z$ if $\mathbf{g} = (0, 0, -g)$.

(ii) Surface Forces

Consider a small element of area $\mathbf{n}\delta A$



Denote the surface force exerted *by the + side on the − side* by $\boldsymbol{\tau}(\mathbf{x}, t; \mathbf{n})\delta A$ where $\boldsymbol{\tau}$ is the *stress* acting on the area element. (A stress is a force per unit area). Note, the stress $\boldsymbol{\tau}$ depends on the orientation $\mathbf{n}$. For example, by Newton III, the surface force exerted by the − side on the + side is

$$-\boldsymbol{\tau}(\mathbf{x}, t; \mathbf{n}) = \boldsymbol{\tau}(\mathbf{x}, t, -\mathbf{n})$$

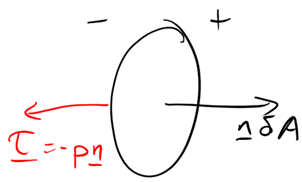
We defer any discussion of viscous stresses (friction) to Chapter 3.

In many phenomena, the viscous stresses are negligible and the fluid behaves as if it is *inviscid* (frictionless). For example it takes over an hour to slow down a 10cm layer of water, 10^{-3} effect when pouring tea.

For *inviscid (frictionless) fluids*, the stress $\boldsymbol{\tau}$ acting across $\mathbf{n}\delta A$ has no tangential component and has magnitude independent of the orientation:

$$\boldsymbol{\tau}(\mathbf{x}, t; \mathbf{n}) = -p(\mathbf{x}, t)\mathbf{n}$$

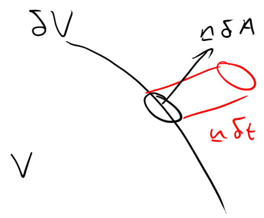
where p is the pressure. Note the − sign! The + side pushes on the − side in the direction $-\mathbf{n}$:



2.2 The Euler Momentum Equation

(Inviscid for the rest of the section) Consider an arbitrary volume V, fixed in space, bounded by surface δV with an outward normal $\mathbf{n}$. The momentum $\int_V p\mathbf{u}\mathrm{d}V$ inside V can change due to

1. Flow of momentum across the boundary ∂V.
2. Volume / body forces.
3. Surface forces.



Volume out across δA in time δt is $\mathbf{n}\delta A \cdot \mathbf{u}\delta t$, so momentum out across δA in time δt is $\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n}\delta A\delta t)$. Hence

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{u}\mathrm{d}V = -\int_{\partial V}\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\mathrm{d}A + \int_{\partial V}(-p\mathbf{n})\mathrm{d}A + \int_V \mathbf{f}\mathrm{d}V}$$

(Momentum integral equation). $\rho u_i u_j$ is the momentum flux (tensor). Or in components:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho u_i \mathrm{d}V = -\int_{\partial V}(\rho u_i u_j)n_j \mathrm{d}A - \int_{\partial A} p n_i \mathrm{d}A + \int f_i \mathrm{d}V$$

$$\int_V \frac{\partial}{\partial t}(\rho u_i)\mathrm{d}V = -\int_V \frac{\partial}{\partial x_j}(\rho u_i u_j)\mathrm{d}V - \int_V \frac{\partial p}{\partial x_i}\mathrm{d}V + \int_V f_i \mathrm{d}V$$

Since V is arbitrary,

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + f_i$$

$$LHS = u_i\left(\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right) + \rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j}$$

But $\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$ by mass conservation. So

$$LHS = \rho\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)u_i$$

Therefore

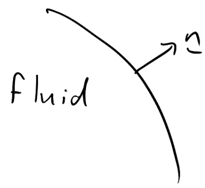
$$\boxed{\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{f}}$$

Euler momentum equation.

A fluid accelerates due to the difference in pressure on either side and to the volume force.

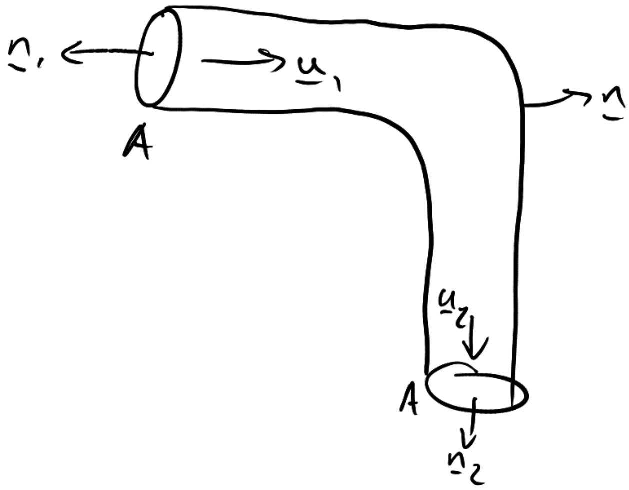
Dynamic boundary condition

The stress $\boldsymbol{\tau}$ exerted on the fluid by the boundary is $p\mathbf{n}$ (the stress exerted on the boundary by the fluid is $p\mathbf{n}$).



Application of momentum integral equation

Bent hose pipe:



Assume steady uniform flow U in and out with same cross-sectional area A. Neglect gravity. ($\mathbf{f} = \mathbf{0}$).

$$\int_{\text{walls}} + \int_{\text{ends}} [\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) + p\mathbf{n}] \mathrm{d}A = 0$$

by steady momentum integral equation.

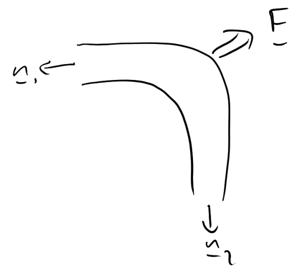
$$\int_{\text{walls}} [\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) + p\mathbf{n}] \mathrm{d}A = \int_{\text{walls}} p\mathbf{n} \mathrm{d}A$$ ($\mathbf{u} \cdot \mathbf{n} = 0$ by the kinematic boun

$$= \text{Force by the fluid on the pipe}$$

$$\int_{\text{ends}} [\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) + p\mathbf{n}] \mathrm{d}A = A(p_1 \mathbf{n}_1 + \rho(-U\mathbf{n}_1)(-U) + p_2 \mathbf{n}_2 + \rho(U\mathbf{n}_2)U)$$

$p_1 = p_2$ (see next section).

$$\text{Force on pipe} = -A(p + pU^2)(\mathbf{n}_1 + \mathbf{n}_2)$$



pU^2 contribution comes from charge in momentum flux: momentum in goes to momentum out due to force from wall.

Start of lecture 4

2.3 Bernoulli's Equation for Steady Flow with Potential Forces

Euler equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mathbf{f}$$

Steady $\equiv \frac{\partial}{\partial t} = 0$. Potential forces $\implies \mathbf{f} = -\nabla\chi$. Use vector identity

$$\boxed{\mathbf{u}\times(\nabla\times\mathbf{u}) = \nabla\left(\frac{1}{2}u^2\right) - (\mathbf{u}\cdot\nabla)\mathbf{u}}$$

(check) (where $u = |\mathbf{u}|$). Introduce the *vorticity* $\mathbf{w} = \nabla\times\mathbf{u}$.

Euler equation reduces to

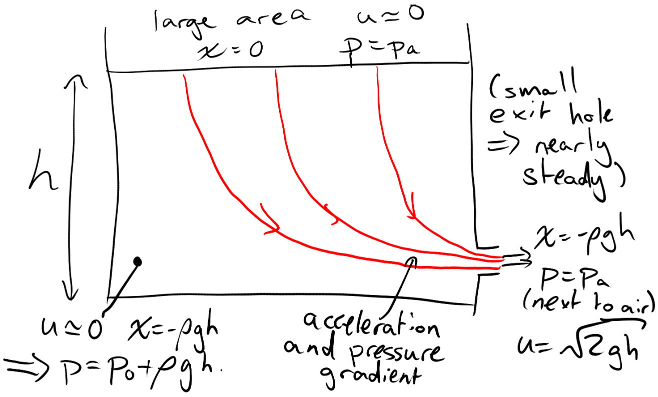
$$\begin{aligned}
\rho\left(0 + \nabla\left(\frac{1}{2}u^2\right) - \mathbf{u}\times\mathbf{w}\right) &= -\nabla\rho - \nabla\chi \\
\implies \nabla\left(\frac{1}{2}\rho u^2 + p + \chi\right) &= \rho\mathbf{u}\times\mathbf{w} \qquad\qquad (\rho = \text{const}) \\
\implies \mathbf{u}\cdot\nabla\left(\frac{1}{2}\rho u^2 + p + \chi\right) &= 0
\end{aligned}$$

$H = \frac{1}{2}\rho u^2 + p + \chi$ is constant along streamlines. Bernoulli (1738). So $H = \text{const}$ implies p low where u is high and vice versa.

See Question 9 on Sheet 1 for interpretation on conservation of energy.

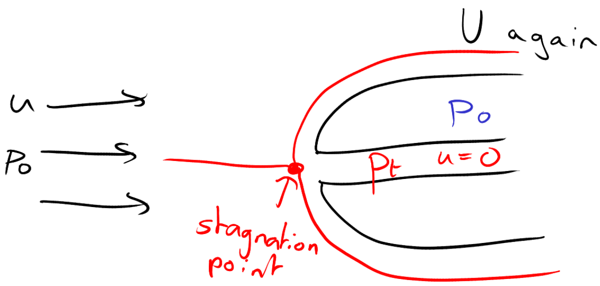
Applications of Steady Bernoulli

Emptying Tank

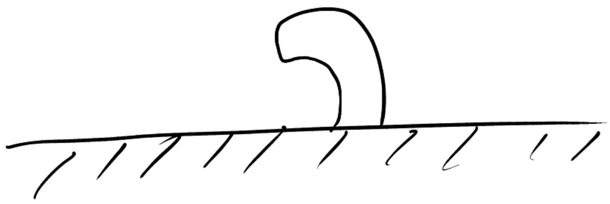


Pitot Tube

To measure air speed.



Mount facing into the flow

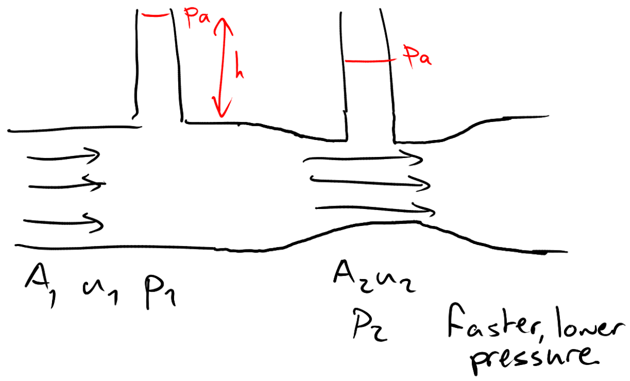


Bernoulli along the streamline

$$\frac{1}{2}\rho_{\text{air}}U^2 + p_0 = 0 + p_1 \implies U = \left[\frac{2(p_1 - p_0)}{\rho_{\text{air}}}\right]^{1/2}$$

Venturi Meter

To measure flow rate in pipe without any moving parts.



Assume steady flow, uniform across a cross-section – OK for gentle variation of A. Mass conservation

$$A_1u_1 = A_2u_2 = Q$$

Bernoulli:

$$\frac{1}{2}\rho u_1^2 + p_1 = \frac{1}{2}\rho u_2^2 + p_2$$

$$\implies p_1 - p_2 = \frac{1}{2}\rho u_1^2\left(\frac{A_1^2}{A_2^2} - 1\right) > 0$$

Measure h $\rightarrow$

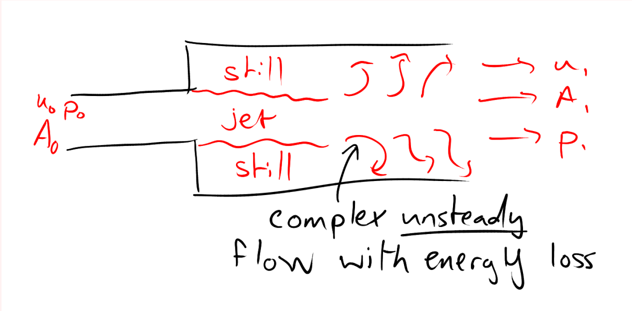
$$\rho g h = p_1 - p_2$$

$$\implies u_1$$

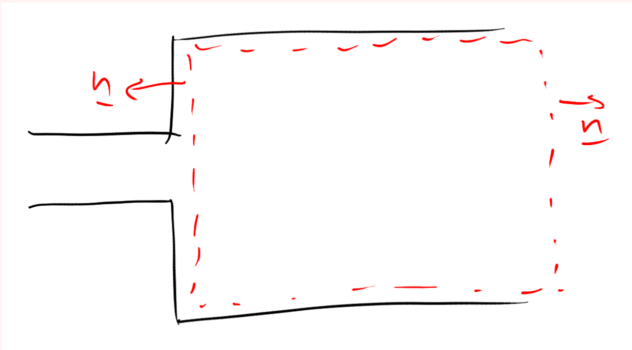
$$\implies Q = \sqrt{2gh}\frac{A_1A_2}{\sqrt{A_1^2 - A_2^2}}$$

(check)

Remark (Non-Examinable)**.** Cannot use Bernoulli for sudden enlargement of pipe. See



Exercise: Assume that $p_{\text{jet}} \simeq p_{\text{still}}$ (because no sideways acceleration) and $p_{\text{jet}} \simeq p_0$. Apply the momentum integral equation to

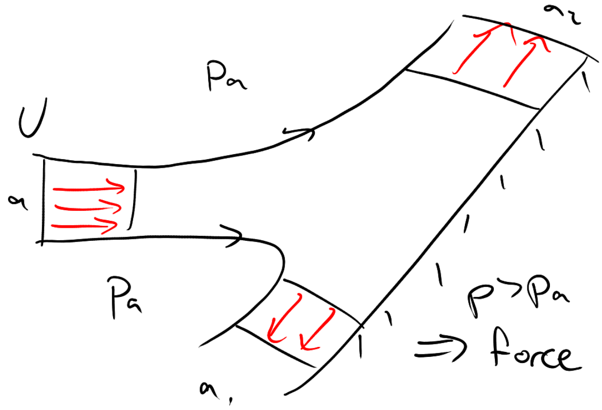


$\mathbf{f} = 0$, $\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \rho \mathbf{u} \mathrm{d}V\right) \simeq 0$ on average. Show

$$p_1 = p_0 + \rho u_1^2 \left(\frac{A_1}{A_0} - 1\right)\left(\frac{A_1}{A_0}\right)$$

Water jet hitting an oblique wall

Two-dimensional version (neglect gravity). Assume steady.



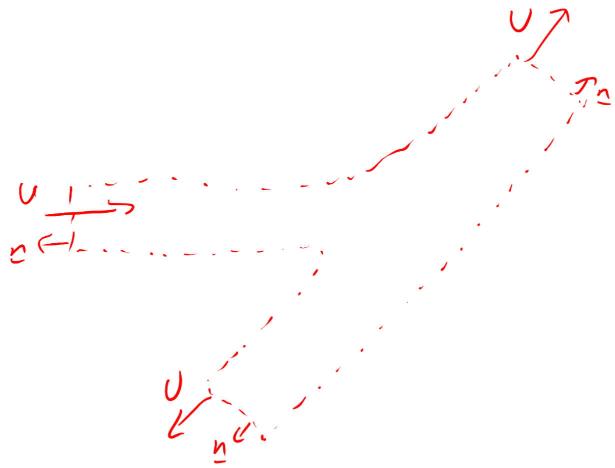
Suppose that the small angle between the two sections is β. Bernoulli on surface streamline where $p = p_a$, constant implies speed is constant $= U$ along this streamline.

So far from impact where flow is uniform, and $p = p_a$, (vorticity zero by section 2.5) have $u = U$ again. Mass conservation

$$\implies aU = a_1 + a_2 U \tag{1}$$

Start of lecture 5

Momentum equation applied to



$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \mathbf{u} \mathrm{d}V}_{0 \text{ because steady}} = -\int_{\partial V} \rho \mathbf{u}(\mathbf{u}\cdot\mathbf{n}) + p\mathbf{n} \mathrm{d}A + \underbrace{\int_V \mathbf{f} \mathrm{d}V}_{\text{neglect}}$$

$p = p_a$ except near the impact, $\mathbf{u}\cdot\mathbf{n} = 0$ except at the ends.

Component parallel to the diagonal plane imply

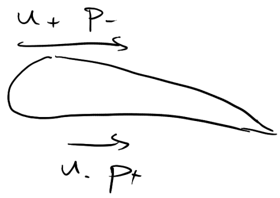
$$-\rho U^2 a\cos\beta + pU^a a_2 - pU^2 a_1 \tag{2}$$

(1) and (2) implies $a_2 = a\frac{1+\cos\beta}{2}$, $a_1 = a\frac{1-\cos\beta}{2}$.

Component perpendicular to plane

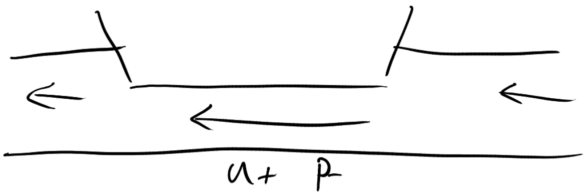
$$\frac{1}{2}\rho U^2 a\sin\beta = \int_{\partial V} p\mathbf{n}\mathrm{d}A = \int_{\partial V}(p-p_a)\mathbf{n}\mathrm{d}A = \text{Force on wall}$$

(because $p_a\int_{\partial V}\mathbf{n}\mathrm{d}A = 0$ by divergence theorem). Also: Aerofoil lift



so sucked up.

Barges in canals:



2.4 Hydrostatic Pressure and Archimedes Principle

If $\mathbf{u}\equiv\mathbf{0}$ then Euler gives $0 = -\nabla p + \rho\mathbf{g} = -\nabla(p+\chi)$

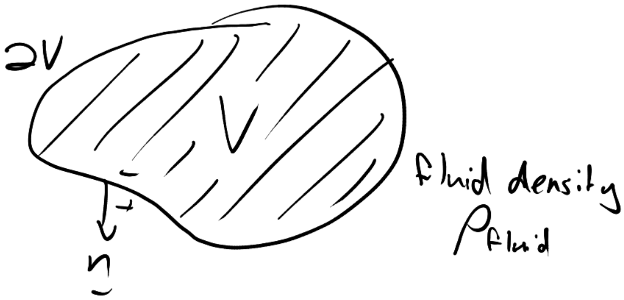
$$\implies p+\chi = \text{const} \implies p = p_0 - \rho g z$$

hydrostatic pressure. (here gravity is in the z direction).

Archimedes: Pressure force on a submerged body with $\mathbf{u}=\mathbf{0}$

$$\begin{aligned}\text{Force} &= -\int p\mathbf{n}\mathrm{d}A \qquad \text{(fluid on body)}\\ &= -\int (p_0-\rho g z)\mathbf{n}\mathrm{d}A\\ &= -\int \nabla(p_0-\rho g z)\mathrm{d}V\\ &= \rho_{\text{fluid}}\mathbf{g}\hat{\mathbf{z}}\end{aligned}$$

upthrust / buoyancy = weight of fluid displaced



For $\mathbf{u} \neq \mathbf{0}$ and constant ρ everywhere we can write $p = p_0 - \rho g z + p'$

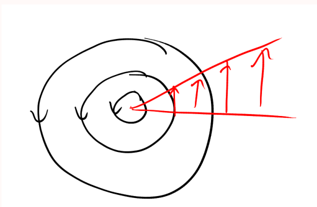
$$\implies \rho \frac{D\mathbf{u}}{Dt} = -\nabla p'$$

(no $\mathbf{f} = \rho \mathbf{g}$). p' is the *dynamic* (*modified*) pressure – variation due to the motion after subtracting hydrostatic balance. Can ignore gravity if there are no density variations and no free surfaces. Free surface between air and water, $\rho_{\text{water}} \neq \rho_{\text{air}}$ hence gravity clearly has an effect → waves (Section 5).

2.5 Vorticity

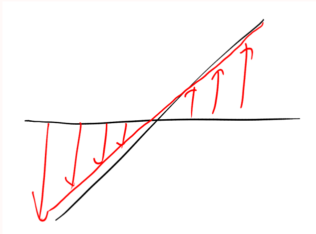
Vorticity defined by $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ (angular momentum by another name).

Example. $\mathbf{u} = \Omega \times \mathbf{x}$, solid body rotation



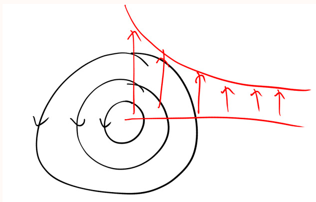
$\implies \boldsymbol{\omega} = 2\boldsymbol{\Omega}$ (check)

Example. $\mathbf{u} = (0, \gamma x, 0)$, simple shear



$\implies \boldsymbol{\omega} = (0, 0, \gamma)$

Example. $\mathbf{u} = \left(0, \frac{R}{2\pi r}, 0\right)$ in cylindrical polars, "line vortex"



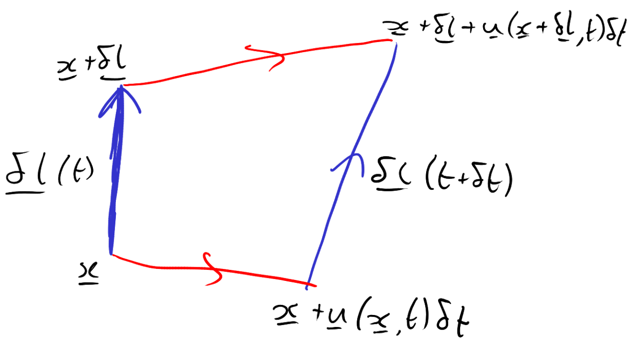
$\boldsymbol{\omega} = \mathbf{0}$ except at $r = 0$

$$\int_{r<a} \boldsymbol{\omega} \cdot \hat{\mathbf{z}} \mathrm{d}A = \oint_{r=a} \mathbf{u} \cdot \mathrm{d}l = \int_0^{2\pi} \frac{R}{2\pi r} r \mathrm{d}\theta = R$$

$\implies \boldsymbol{\omega} = (0, 0, R\delta(r))$.

2.5.1 Interpretation of ω as 2 $\times$ average location rate

Consider a material line element δl (i.e. one moving with the fluid).



i.e. $\boldsymbol{\delta}\mathbf{l} \to \boldsymbol{\delta}\mathbf{l} + (\boldsymbol{\delta}\mathbf{l} \cdot \nabla)\mathbf{u}\delta t$

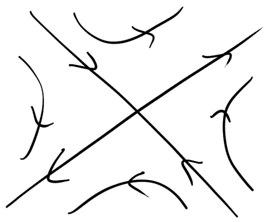
$$\implies \boxed{\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\delta}\mathbf{l} = (\boldsymbol{\delta}\mathbf{l} \cdot \nabla)\mathbf{u}}$$

Hence the tensor $\frac{\partial u_i}{\partial x_j}$ determines the rate of change of $\boldsymbol{\delta}\mathbf{l}$. We write

$$\begin{aligned}\frac{\partial u_i}{\partial x_j} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \\ &= e_{ij} + \frac{1}{2}\varepsilon_{ijk}\omega_k\end{aligned}$$

The local rotation of line elements due to the second term is $\frac{1}{2}\varepsilon_{jik}\omega_k\delta k_j = \frac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{\delta}\mathbf{l})_i =$ rotation at angular velocity $\frac{1}{2}\boldsymbol{\omega}$.

Local rotation due to first term, called the *strain rate*, gives same angular velocity averaged over all orientations $\boldsymbol{\delta}\mathbf{l}$. $\underline{\mathbf{e}}$ is symmetric (hence diagonalisable) and traceless. Therefore $\nabla \cdot \mathbf{u} = 0$.



Note $\frac{1}{2}\boldsymbol{\omega}$ gives the rate of rotation of blobs, not whether the blobs are going in circles!

Start of lecture 6

2.5.2 The vorticity equation

Start with

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) = -\nabla p + \mathbf{f}$$

Assume ρ is constant and $\mathbf{f}$ is conservative (i.e. $\mathbf{f} = -\nabla\chi$). Earlier vector identity $(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla\left(\frac{1}{2}u^2\right) - \mathbf{u} \times \mathbf{w}$. Take curl:

$$\rho\left(\frac{\partial \mathbf{w}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{w})\right) = 0$$

Second vector identity:

$$\begin{aligned}(\nabla \times (\mathbf{u} \times \mathbf{w}))_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\varepsilon_{klm}u_L w_m) \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\left(\frac{\partial u_l}{\partial x_j}w_m + u_l\frac{\partial w_m}{\partial x_j}\right) \\ &= (\mathbf{w} \cdot \nabla)\mathbf{u} - \mathbf{w}\underbrace{\cancel{(\nabla \cdot \mathbf{u})}}_{\text{incompressible}} + \mathbf{u}\underbrace{\cancel{(\nabla \cdot \mathbf{w})}}_{\nabla\cdot\nabla\times=0} - (\mathbf{w} \cdot \nabla)\mathbf{u}\end{aligned}$$

$$\implies \boxed{\frac{D\mathbf{w}}{Dt} \equiv \frac{\partial \mathbf{w}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{w} = (\mathbf{w}\cdot\nabla)\mathbf{u}}$$

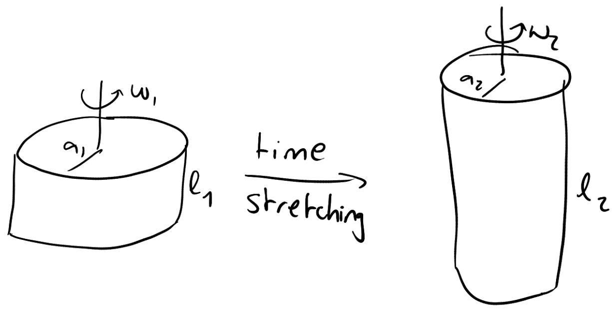
Vorticity equation. We call the $\frac{\partial \mathbf{w}}{\partial t}$ term the local derivative, the $(\mathbf{u}\cdot\nabla)\mathbf{w}$ term the advection of $\mathbf{w}$ and $\mathbf{w}\cdot\nabla)\mathbf{u}$ the "vortex stretching".
i.e. moving with the fluid, $\mathbf{w}$ changes if $\mathbf{u}$ changes in the direction of $\mathbf{w}$.

2.5.3 Vortex stretching and the "ballerina effect"

Compare

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\delta l} = (\boldsymbol{\delta l}\cdot\nabla)\mathbf{u} \qquad \text{and} \qquad \frac{D\mathbf{w}}{Dt} = (\mathbf{w}\cdot\nabla)\mathbf{u}$$

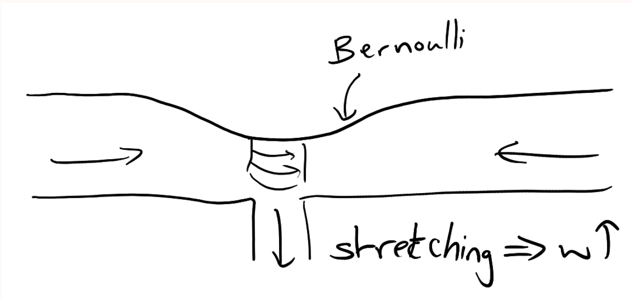
Moving with the fluid, $\mathbf{w}$ changes just like a material line element $\boldsymbol{\delta l}$ initially aligned with $\mathbf{w}$. In particular, if $\boldsymbol{\delta l}$ gets longer (stretching) then $\mathbf{w}$ gets bigger – this is just conservation of angular momentum! For example, consinder a uniformly rotating fluid cylinder:



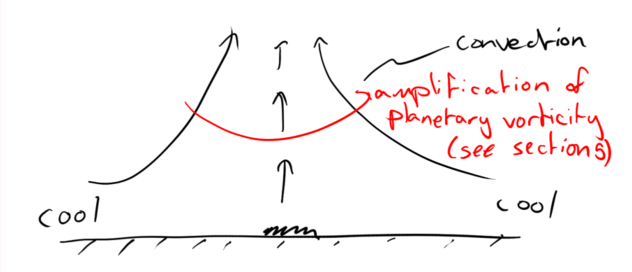
- Mass conservation: $a_1^2 l_1 = a_2^2 l_2$
- Angular momentum: $a_1^4 l_1 w_1 = a_2^4 l_2 w_2$

Hence $w_2 = w_1 \frac{l_2}{l_1}$, i.e. $\mathbf{w}$ increases as fluid stretches in the direction of existing $\mathbf{w}$ – called *vortex stretching* or the "ballerina effect".

Example. Amplification of bathtub vortex:



Example. Hurricane / tornado:



Example.

$$\mathbf{u} = \left(-\frac{1}{2}\beta x, -\frac{1}{2}\beta y, \beta z\right) + \Omega(t)(y, -x, 0)$$

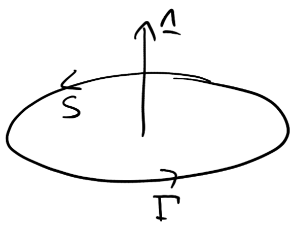
$$\implies \mathbf{w} = (0, 0, 2\Omega)$$

$$\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{w} = (\mathbf{w} \cdot \nabla)\mathbf{u}$$

z-component $2\dot{\Omega} + 0 = (2\Omega)\beta$ implies $\Omega \propto e^{\beta t}$.

Circulation

The *circulation around a closed curve* Γ is defined by $C(\Gamma) = \oint_\Gamma \mathbf{u} \cdot \mathrm{d}\mathbf{l} = \int_S \mathbf{w} \cdot \mathrm{d}\mathbf{S}$



For example a line vortex $\mathbf{u} = \left(0, \frac{\kappa}{2\pi r}, w\right)$ has circulation κ for any circle $r = a$.

Note (Non-examinable)**.** Kelvin's Circulation Theorem. If $\Gamma(t)$ is material curve, i.e. one moving with the fluid, then

$$\frac{\mathrm{d}}{\mathrm{d}t}[C(\Gamma)] = \oint_\Gamma \left(\frac{D\mathbf{u}}{Dt} \cdot \mathrm{d}\mathbf{l} + \mathbf{u} \cdot \underbrace{(\mathbf{u} \cdot \nabla)\boldsymbol{\delta l}}_{\dot{t}\boldsymbol{\delta l}} \right) \overset{\text{exercise}}{=} 0$$

For example $C = \pi a^2(2\Omega)$ above is constant for a material circle $a(t) = a(0)e^{-\frac{1}{2}\beta t}$.

3 Introduction to Viscous Flow

Section 2 was about *inviscid* flow.

- We neglected friction (viscous stresses) between layers of fluid or boundaries.
- Inviscid fluid exerts only a normal stress $-p\mathbf{n}$.
- Forces $-\nabla p + \mathbf{f}$ were balanced by inertia $\rho \frac{D\mathbf{u}}{Dt}$.
- If $\mathbf{f} = -\nabla \chi$ then energy and angular momentum are conserved.

With viscous flow, we will find:

- Velocity gradients give rise to visous stresses (friction)
- Fluids also exert tangential (shear) stresses on boundaries
- New term in equation of motion
- Dissipation of energy (non-examinabl) and diffusion of vorticity.

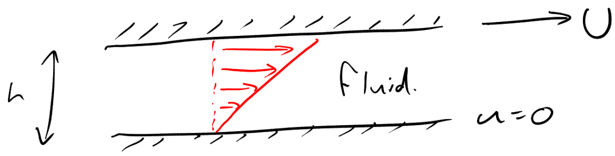
We will focus on simple case of *2D parallel viscous flow*

$$\mathbf{u} = (u(y,t), 0, 0)$$

– the full treatment of viscosity is in Part II.

Start of lecture 7

3.1 Plane Couette Flow and Viscosity



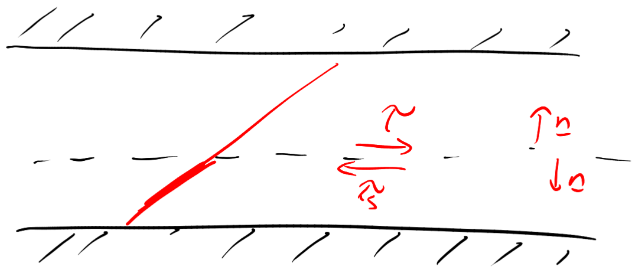
Steady flow between parallel plates driven only by the motion of the top plate. Find experimatelly for simple (Newtonian) fluids – air, water, oild, syrup, glycerol – that

(i) Fluid velocity is U on the top plate and 0 on the bottom plate.

(ii) The fluid velocity varies linearly between the top and bottom plates.

(iii) Tangential force per unit area τ_3 is…

TODO By considering a slab of fluid $a < y < b$ as a Couette Flow, for example



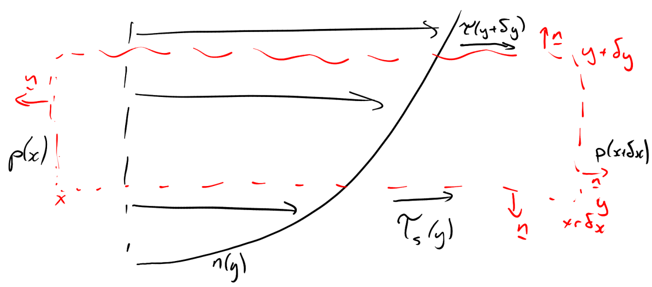
deduce: The tangential shear stress exerted by the + side on the − side of a surface $y = \text{const}$ is

$$\boxed{\tau_3 = \mu \frac{\partial u}{\partial n}}$$

For example, the fluid exerts a shear stress $\mu \frac{\partial u}{\partial y}$ on the bottom plate and $-\mu \frac{\partial u}{\partial y}$ on the top plate ($\mathbf{n}$ into fluid)

3.2 2D Parallel Viscous Flow

Steady case with $\mathbf{f} = \mathbf{0}$



Steady $\implies$ no acceleration $\implies$ forces exerted by the surrounding fluid on the dashed rectangle must balance.

In the x-direction

$$p(x)\delta y - p(x+\delta x)\delta y + \mu \left.\frac{\partial u}{\partial y}\right|_{y+\delta y} \delta x - \mu \left.\frac{\partial u}{\partial y}\right|_{y} \delta x = 0$$

Divide by $\delta x \delta y$ and take the limit:

$$-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} = 0$$

and similarly in the y-direction

$$-\frac{\partial p}{\partial y} = 0$$

General Case

For 2D unsteady parallel viscous flow $\mathbf{u} = (u(y,t), 0, 0)$ with body force $\mathbf{f} = (f_x, f_y, 0)$ (Sheet 2 Question 1)

$$\boxed{\begin{aligned} p\frac{\partial u}{\partial t} &= -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} + f_x \\ 0 &= -\frac{\partial p}{\partial y} + f_y \end{aligned}}$$

Note: $\mathbf{u}(u(y,t), 0, 0) \implies \nabla \cdot \mathbf{u} = 0$ and $(\mathbf{u} \cdot \nabla)\mathbf{u} = 0$.

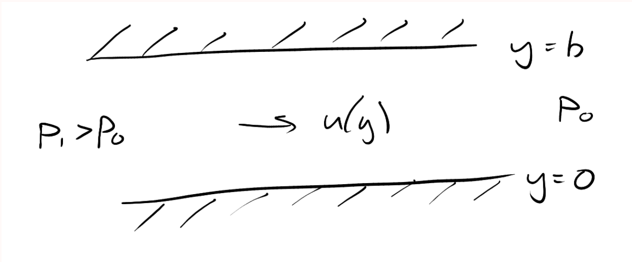
No-slip boundary condition

It has been verified experimentally (down to molecular scales) that at a rigid boundary viscous fluids satisfy a *no slip boundary condition*:

> the tangential component of the fluid must equal that of the boundary.

Combined with the mass conserving kinematic boundary condition $\mathbf{u} = \mathbf{U}$ (contrast with $\mathbf{u} \cdot \mathbf{n} = \mathbf{U} \cdot \mathbf{n}$ only for inviscid fluids).

Example (Poiseuille Flow in a Channel)**.** Steady flow in a channel driven by a pressure gradient.



$$\frac{\partial p}{\partial y} = 0 \implies p = p(x)$$

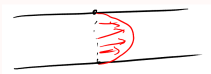
Steady, so

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\mathrm{d}p}{\mathrm{d}x} = -G \qquad \text{const}$$

$$u(0) = u(h) = 0$$

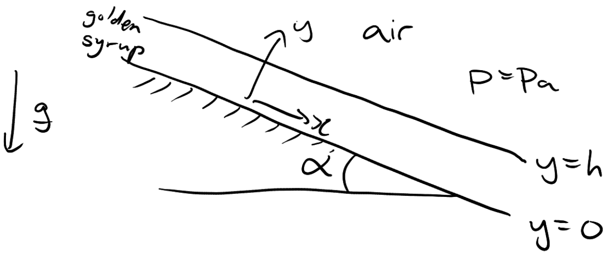
$$\implies u = \frac{G}{2\mu} y(h-y)$$

Flux (per unit width into page), $q = \int_0^h u \mathrm{d}y = \frac{Gh^3}{12\mu}$.



Overall force balance $Gh - \mu \left.\frac{\partial u}{\partial y}\right|_0 + \mu \left.\frac{\partial u}{\partial y}\right|_h = 0$. ($Gh$ is pressure gradient, and the next two terms are the shear stress $R \to L$).

Example Viscous Flow Down a Slope



Assume p_{air} is uniform (because $p_{\text{air}} \ll p_{\text{syrup}}$). Shear stress exerted by the air is negligible (because $\mu_{\text{air}} \ll \mu_{\text{syrup}}$). Take coordinates parallel and perpendicular to the plane:

$$\mathbf{f} = (\rho g \sin\alpha, -\rho g \cos\alpha)$$

perpendicular:

$$\frac{\partial p}{\partial y} = \rho g \cos\alpha, \quad p = p_a \text{ at y = h} \implies p = p_a + \rho g \cos\alpha (h-y) \implies \frac{\partial p}{\partial x} = 0$$

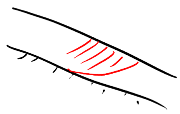
Parallel:

$$\mu \frac{\partial^2 u}{\partial y^2} = -\rho g \sin\alpha$$

$$u(0) = 0 \qquad \text{(no slip)}$$

$$\mu \left. \frac{\partial u}{\partial y} \right|_h = 0 \qquad \text{(no shear stress from / on air)}$$

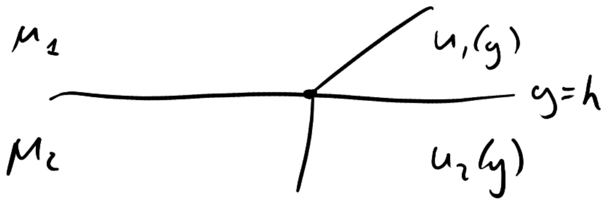
$$\implies u = \frac{\rho g \sin\alpha}{2\mu} y(2h-y)$$



Start of lecture 8

Boundary Conditions at an Interface

Consider two fluids in parallel viscous flow



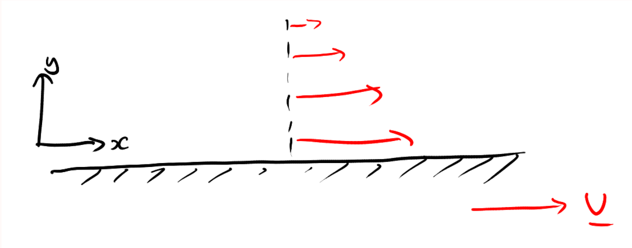
At $y = h$, $u_1 = u_2$ by no-slip. By continuity of stress,

$$\mu_1 \frac{\partial u_1}{\partial y} = \mu_2 \frac{\partial u_2}{\partial y}$$

$$p_1 = p_2$$

3.3 Unsteady Parallel Viscous Flow and Viscous Diffusion

Example (Impulsively started flat plate)**.** Consider a semi-infinite fluid domain $y > 0$, initially at rest, no applied pressure gradient and $\mathbf{f} = \mathbf{0}$. At $t = 0$, the boundary at $y = 0$ starts to move with velocity $(U, 0)$



$$\rho\frac{\partial u}{\partial t} = -\underbrace{\frac{\partial p}{\partial x}}_{0} + \mu\frac{\partial^2 u}{\partial y^2} + \underbrace{f_x}_{0}$$

$$u = 0 \qquad (t = 0, y \to \infty)$$

$$u = U \qquad \text{at } y = 0, t > 0 \text{(no-slip)}$$

Hence the velocity satisfies the *diffusion equation*

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}$$

where the kenimatic velocity $\nu = \frac{\mu}{\rho}$ can be thought of as a diffusivity for momentum (or vorticity – see later).

For other solutions of the diffusion equation see Methods (for example separation of variables, Fourier Transforms, Green's functions etc) and Sheet 2 Questions 4 and 5. Here there is no externally imposed length scale for y since $0 < y < \infty$. But we need a lengthscale! Can find a similarity solution using a scaling argument (see IA DEs, IA D&R).

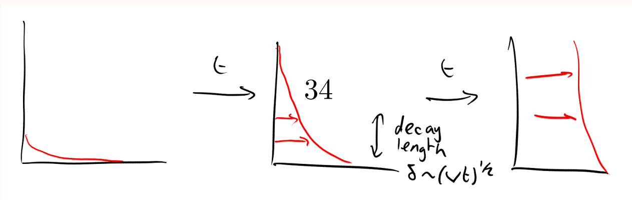
$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}$$

suggests $\frac{u}{t} \sim \nu\frac{u}{y^2}$. $u(0) = U$ suggests $u \sim U$. Hence $y \sim (\nu t)^{\frac{1}{2}}$. Try $u = Uf(\eta)$ where $\eta = \frac{y}{(\eta t)^{\frac{1}{2}}} \implies \frac{\partial \eta}{\partial t} = -\frac{1}{2}\frac{\eta}{t}$

$$\implies -\frac{1}{2}\eta f' = f'' \implies -\frac{1}{2}\eta = \frac{f''}{f'}$$

$$\implies -\frac{1}{4}\eta^2 = \ln f' + \text{constant} \implies f = A + B\int_\eta^\infty e^{-\frac{1}{4}\eta'^2}\mathrm{d}\eta'$$

$f(\infty) = 0$, $f(0) = 1$ implies $A =$), $B = \frac{1}{\sqrt{\pi}}$ hence $u = U\,\mathrm{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right)$.



same shape – self-similar – but with width ↑ as $(\nu t)^{\frac{1}{2}}$.

Comments and Variations

(1) Pure dimensional analysis would also allow $\frac{y}{ut}$ or $\frac{yU}{\nu}$ as dimensionless groups, but the equations show these are not relevant – the equations and scaling arguments are better!

(2) Kinematic and dynamic viscosity

	ρ	μ	ν
Water	10^3kgm^{-3}	$10^{-3}p_{as}$	10^{-6}m^{2}s^{-1}
Water	10^3kgm^{-3}	$10^{-3}p_{as}$	10^{-6}m^{2}s^{-1}
Water	10^3kgm^{-3}	$10^{-3}p_{as}$	10^{-6}m^{2}s^{-1}

(a) $\nu_{\text{air}} \approx 20\nu_{\text{water}}$ so motion spreads further / faster in air.

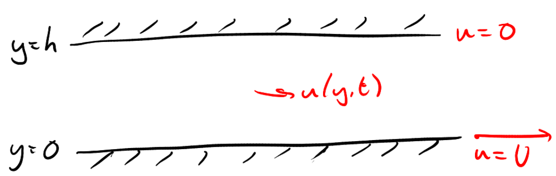
(b) Sheer stress exerted by fluid on the plate

$$\tau_s = \mu\frac{\partial u}{\partial y} = -\frac{\mu U}{(\nu t)^{\frac{1}{2}}}f'(0) = -\frac{\mu U j}{\sqrt{\pi\nu t}}$$

$$\left.\frac{\mu}{\sqrt{\nu}}\right|_{\text{water}} \approx 200\left.\frac{\mu}{\sqrt{\nu}}\right|_{\text{air}}$$

so water exerts a much bigger stress.

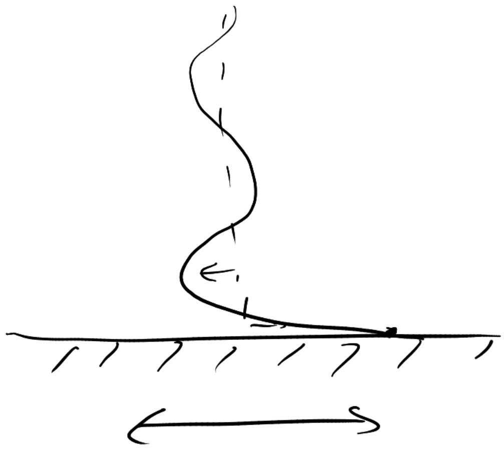
(3) Now add a stationary boundary at $y = h$



Have 2 relevant length scales h and $(\nu t)^{\frac{1}{2}}$. If $h \gg (\nu t)^{\frac{1}{2}}$ expect boundary to have little (exponentially small) effect on the previous solution. As $t \to \infty$ expect to approach steady Couette flow (linear profile). Deviations from this steady state decay like e^{-kt} with $k \propto \frac{\nu}{h^2}$ and are small for $\nu t \gg h^2$ (Example Sheet 2, Question 5) $\implies$ Characteristic timescale $T = \frac{h^2}{\nu}$ for diffusion accross the cell. For example $h = 10$cm, $T = 0.07$s for golden syrup, but 3hrs for water. $t \ll T$ effects of viscosity ar econfined to a "boundary layers" $\delta \sim (\nu t)^{\frac{1}{2}}$. $t \gg T$ almost steady, viscosity dominant everywhere.

(4) Simple oscillating boundary (Example Sheet 2 Question 4)

Imposed timescale $\frac{1}{\omega}$ during which velocity variation can diffuse $\left(\frac{\nu}{m}\right)^{\frac{1}{2}}$



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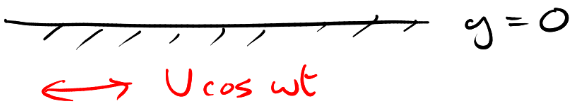
(5) Add a stationary boundary



Compare either timescales $\frac{\nu}{h^2}$ and $\frac{1}{\omega}$ or lengthscales $\left(\frac{\nu}{\omega}\right)^{\frac{1}{2}}$ and h. Then dimensionless parameter $S = \frac{\omega h^2}{\nu}$ (Stokes number). $s \gg 1$ looks like (4) with $\delta \sim \left(\frac{\nu}{\omega}\right)^{\frac{1}{2}} \ll h$, $s \gg 1$ looks like Couette flow (linear profile) with amplitude $U(t)$.

These examples illustrate that the inviscid solution ($\nu \equiv 0, \implies u = 0$ and slip at boundary) is *not* uniformly the same as the limit of "small" viscosity (smallness means $\nu \ll \frac{h^2}{t}$ or $\ll \omega h^2$) but the viscous effects only felt in thin boundary layers $\delta \ll h$.

3.4 The Navier-Stokes Equation

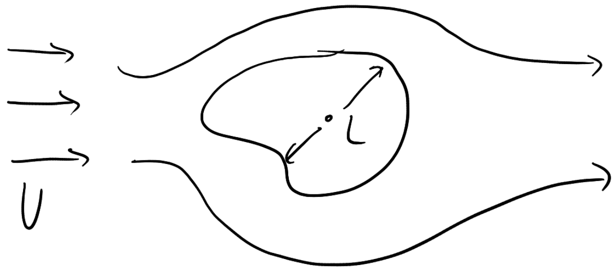
In Part II it is shown that $\boldsymbol{\tau}(\mathbf{x}, t; \mathbf{n}) = -p\mathbf{n} + \mu[(\mathbf{n} \cdot \nabla)\mathbf{u} + (\nabla \mathbf{u}) \cdot \mathbf{n})]$ and hence the flow satisfies the Navier-Stokes equation

$$\boxed{\begin{aligned} &\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f} \\ &\nabla \cdot \mathbf{u} = 0 \end{aligned}}$$

This reduces to the Euler equation if $\mu = 0$ and to the parallel viscous flow equations if $\mathbf{u} = (u(y,t), 0, 0)$. When is viscosity important?

The Reynolds Number

Suppose a flow has a characteristic lengthscale L and velocity scale U. For example



Assume the characteristic timescale T is $\frac{L}{U}$ (or steady) and let the characteristic scale of the pressure difference be denoted by P (to be found). Estimate the scale of the terms in the Navier-Stokes Equation:

$$\begin{array}{ccccccc} \frac{\partial \mathbf{u}}{\partial t} & + & \mathbf{u}\cdot\nabla\mathbf{u} & = & -\frac{1}{\rho}\nabla p & + & \nu\nabla^2\mathbf{u} \\ \sim \frac{U}{L/U} & & \sim \frac{U^2}{L} & & \sim \frac{p}{\rho L} & & \sim \frac{\nu}{U}L^2 \\ 1 & : & 1 & : & \frac{p}{\rho U^2} & : & \frac{\nu}{LU} \equiv \frac{1}{R_e} \end{array}$$

The Reynolds number $\boxed{R_e \equiv \frac{UL}{\nu}}$ is a dimensionless parameter describing the relative importance of inertia and viscosity:

$$\frac{\rho\frac{D\mathbf{u}}{Dt}}{\mu\nabla^2\mathbf{u}} \sim R_e$$

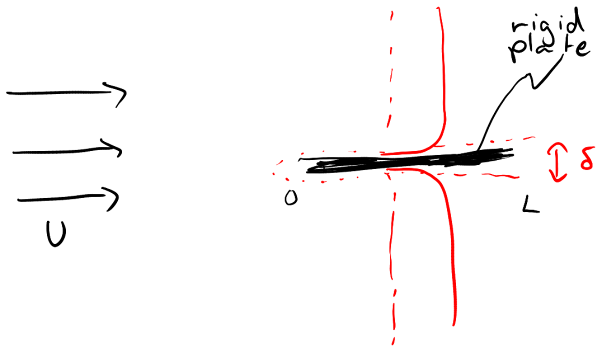
If $R_e \ll 1$ then expect $\rho\frac{D\mathbf{u}}{Dt} \ll \mu\nabla^2\mathbf{u}$, inertia negligible, and can approximate the N-S equation by the Stokes equations

$$\boxed{\begin{array}{l} \quad 0 = -\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \end{array}}$$

Scaling for pressure $p \sim \frac{\mu U}{L}$ like viscous stress.

If $R_e \gg 1$ expect $\mu\nabla^2\mathbf{u} \ll \rho\frac{D\mathbf{u}}{Dt}$ and viscosity negligible (except perhaps in thin boundary layers at rigit boundaries). Approximate the N-S equation by the Euler equation $\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{f}$ ouside the boundary layer. Pressure scaling $P \sim pU^2$ like Bernoulli.

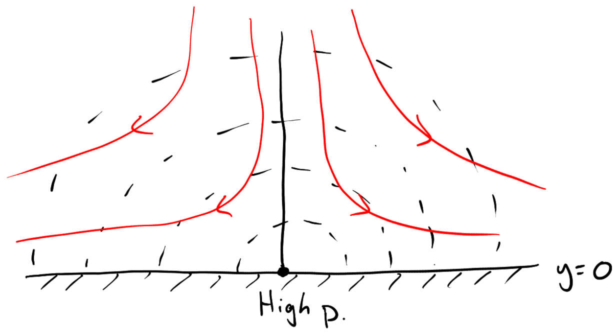
How thin are the boundary layers? For example



On timescale $\frac{L}{U}$ viscous diffusion affects the velocity over a distance $\delta \sim \left(\nu \frac{L}{U}\right)^{\frac{1}{2}}$ or $\frac{\delta}{L} \sim \left(\frac{\nu}{LU}\right)^{\frac{1}{2}} = \frac{1}{R_e^{\frac{1}{2}}}$.

3.5 Stagnation Point Flow: An Illustrative Example

Consider the 2D inviscid incompressible flow $\mathbf{u} = (Ex, -Ey)$ in $y > 0$, $\psi = Exy$ and $p = p_0 - \frac{1}{2}\rho E^2(x^2 + y^2)$ (Bernoulli)



Solves $\rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p$, $v = 0$ on $y = 0$. Also satisfies the N-S equation, but not the viscous no-slip boundary condition $u = 0$ if $y = 0$ is a rigid boundary. Fortunately, and unusually, there is an exact solution for this case!

Try / guess $\psi = Exf(y) \implies \mathbf{u} = (Exf', -Ef)$. Want $f(0) = f'(0) = 0$ and $f(y) \approx y$ as $y \to \infty$. Substitute into $(\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}$, x-component:

$$\left(Exf'\frac{\partial}{\partial x} - Ef\frac{\partial}{\partial y}\right)Exf' = -\frac{1}{\rho}\frac{\partial \phi}{\partial x} + \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(Exf')$$

$$\implies E^2x(f'^2 - ff'') = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu Exf''' \qquad (1)$$

Similarly, y-component:

$$E^2 f' f = -\frac{1}{\rho}\frac{\partial \phi}{\partial y} - \nu E f'' \qquad (2)$$

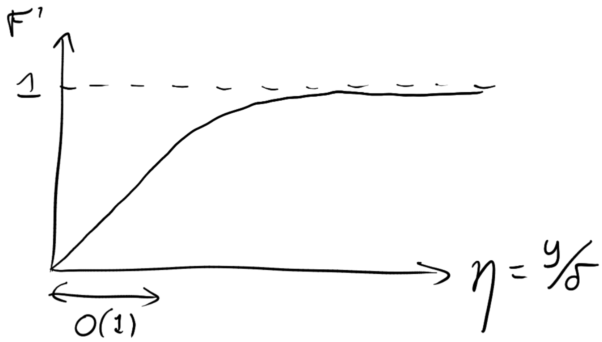
$$\frac{\partial}{\partial x}(2) \implies \frac{\partial^2 p}{\partial x \partial y} = 0 \implies p = X_{(x)} + Y_{(y)}$$

$$(1) \implies \frac{\partial p}{\partial x} \propto x$$

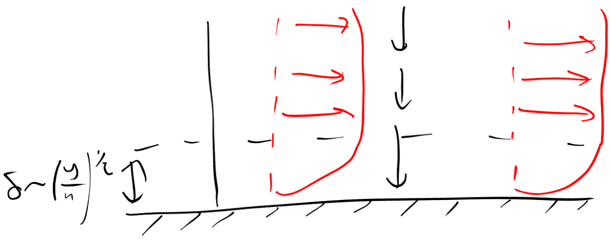
$f' \to 1$ as $y \to \infty \implies X(x) = p_0 - \frac{1}{2}\rho E^2 x^2$

$$\implies f'^2 - ff'' = 1 + \frac{\nu}{E} f'''$$

Rescale $f(y) = \delta F(\eta)$ where $\eta = \frac{y}{\delta}$ and $\delta = \left(\frac{\nu}{E}\right)^{\frac{1}{2}}$. So $F''' = F'^2 - FF'' - 1$, $F(0) = F'(0) = 0$, $F' \to 1$ as $\eta \to \infty$. Solve numerically:



For $\eta \gtrsim 2$, $F' \approx 1$, $F \approx \eta - 0.6$. $y \gtrsim w\sqrt{\frac{\nu}{E}}$, $u \approx Ex$, $v \approx -E(y - 0.65)$.



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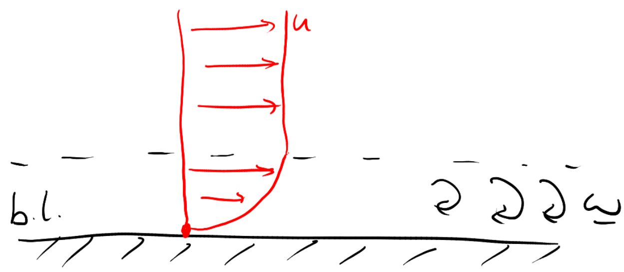
3.6 The Vorticity Equation for Viscous Flow

$\nabla \times (\nu \nabla^2 \mathbf{u}) = \nu \nabla^2 \mathbf{w}$! Take curl of Navier-Stokes, and use the previous derivation from Euler (Section 2.5.2) to obtain

$$\boxed{\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{w} = (\mathbf{w} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \mathbf{w}}$$

($(\mathbf{u} \cdot \nabla)\mathbf{w}$ is advection, $(\mathbf{w} \cdot \nabla)\mathbf{u}$ is vortex stretching, $\nu \nabla^2 \mathbf{u}$ is diffusion).

For $R_e \gg 1$: Away from rigid boundaries, vorticity is dominantly advected and amplified by vortex stretching, and the final term is small. At rigid boundaries vorticity is generated by the no-slip boundary condition and diffuses a short distance to form a boundary layer.



For parallel flow:

$$\mathbf{u} = (u(y,t), 0, 0) \implies \boldsymbol{\omega} = (0, 0, \omega(y,t))$$

where $\omega = -\frac{\partial \mathbf{u}}{\partial t}$

$$\frac{\partial \omega}{\partial t} = \nu \frac{\partial^2 \omega}{\partial t^2}$$

diffusion equation.

4 Inviscid Irrotational Flow

AKA Potential Flow.

4.1 The Velocity Potential

In inviscid flow, if $\boldsymbol{\omega} = \nabla \times \mathbf{u} = \mathbf{0}$ at $t = 0$ (irrotational flow) then

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} \implies \nabla \times \mathbf{u} = 0 \quad \forall t \geq 0$$

"irrotational flow remains irrotational" in inviscid flow.

$\nabla \times \mathbf{u} = \mathbf{0}$ implies there exists a *velocity potential* $\phi(\mathbf{x}, t)$

$$\boxed{\mathbf{u} = \nabla\phi}$$

(for example $\phi = \int^{\mathbf{x}} \mathbf{u} \cdot d\mathbf{x}$). Note:

(1) The + sign (cf $\mathbf{f} = -\nabla\chi$)

(2) Can add any $f(t)$ without changing $\mathbf{u}$.

Incompressibility $\nabla \cdot \mathbf{u} = 0$ implies

$$\boxed{\nabla^2\phi = 0}$$

Kinematic boundary condition $\mathbf{u} \cdot \mathbf{n} = \mathbf{U} \cdot \mathbf{n}$ implies

$$\boxed{\mathbf{n} \cdot \nabla\phi = \mathbf{U} \cdot \mathbf{n}}$$

So solving the Euler equation for an irrotational flow reduces to solving the more familiar, and linear, Laplace's equation with Neumann boundary conditions (or $\frac{\partial\phi}{\partial n}$) – the solution is non-zero because of the boundary conditions. Important examples of irrotational flow are flows starting from rest or with uniform flow upstream.

4.2 Examples

For simple boundaries (spheres, cylinders, rectangular channel, half-space). We can build solutions using separable solutions to $\nabla^2\phi = 0$ in suitable coordinate systems cf Methods.

- Cartesians: for example uniform flow, $\mathbf{u} = \mathbf{U} \implies \phi = \mathbf{U} \cdot \mathbf{x} = U_1 x_1 + U_2 y + U_3 z$
  For example
  $$\phi = Z(z)X(x)$$

with

$$Z(z) = \begin{cases} e^{\pm kz} \\ \cosh kz \\ \sinh kz \end{cases}$$

$$X(x) = \begin{cases} e^{\pm ikx} \\ \cos kx \\ \sin kx \end{cases}$$

so that $\nabla^2\phi = 0$. Corresponding flows are periodic in the x-direction, for example waves in section 5.

- Spherical coordinates: General axisymmetric solution to $\nabla^2\phi = 0$ in spherical coordinates is

$$\phi = \sum_{n\geq 0}(A_n r^n + B_n r^{n-1})P_n(\cos\theta)$$

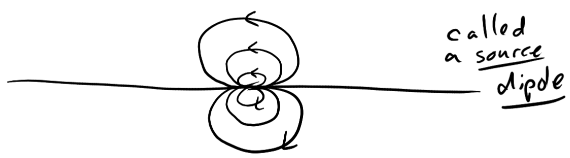
where P_n is a Legendre polynomial. We will only need the first few simple modes. (for example $\phi = A_0 \implies \mathbf{u} = \mathbf{0}$) For example $\phi = \frac{B}{r} \implies \mathbf{u} = \nabla\phi = -\frac{B}{r^2}\mathbf{e}_r$ radial flow $\propto \frac{1}{r^2}$. Volume fluw across any sphere $r = a$ is

$$q = \int_{r=a} \mathbf{u}\cdot\mathbf{n}\mathrm{d}A = 4\pi A^2\left(-\frac{B}{a^2}\right) = -4\pi B$$

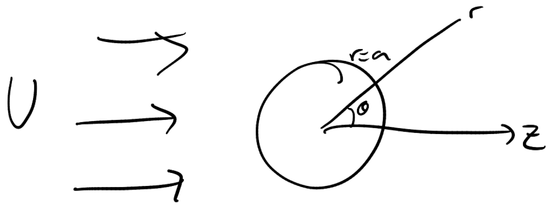
independent of a (by mass conservation).

$$\boxed{\phi = -\frac{q}{4\pi r}}$$

gives a *point source of strength (volume flux)* q. ($q < 0$ gives a *point sink*). For example $\phi = Ur\cos\theta = U_2$ implies $\mathbf{u} = \nabla\phi = U\mathbf{e}_z$ *uniform flow again.* Another example $\phi = \frac{U\cos\theta}{r^2} \implies \mathbf{u} = \nabla\phi = \cdots \implies$



Another example: Uniform flow past a stationary sphere:



$$\begin{cases} \nabla^2\phi = 0 \text{ in } r > a & \text{irrotational / incompressible} \\ \phi \to Ur\cos\theta \text{ as } r \to \infty & \text{far-field} \\ \mathbf{u}\cdot\mathbf{n} = \frac{\partial\phi}{\partial r} = 0 \text{ on } r = a & \text{kinematic bc} \end{cases}$$

Linear problem, forcing $\alpha \cos\theta = P_1(\cos\theta)$. Try $\phi = U\cos\theta\left(r + \frac{B}{r^2}\right)$:

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=a} = U\cos\theta\left.\left(1 - \frac{2B}{r^2}\right)\right|_{r=a}$$

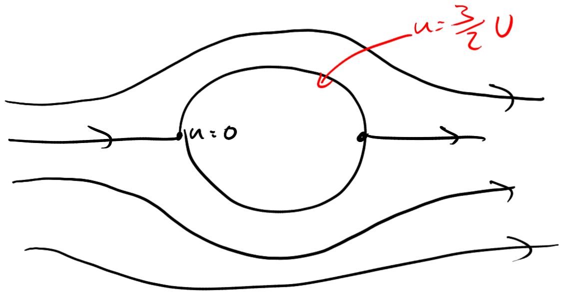
$$\implies B = \frac{a^3}{2}$$

$$\implies \phi = U\cos\theta\left(r + \frac{a^3}{2r^2}\right)$$

$$\implies \mathbf{u} = \nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, 0\right)$$

in sphericals

$$= \left(U\cos\theta\left(1 - \frac{a^3}{r^3}\right), -U\sin\theta\left(1 + \frac{a^3}{2r^3}\right), 0\right)$$



- Cylindrical Geometry / 2D Flow: General solution to $\nabla^2\phi = 0$ in plane polars is

$$\phi = C_0 + A_0 \ln r + B_0\theta + \sum_{n\geq 1}(A_n r^n + B_n r^{-n})(C_n\cos n\theta + D_n \sin n\theta)$$

– again we will only need a few modes.

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For example $\phi = \frac{q}{2\pi}\ln r$ implies $\mathbf{u} = \frac{q}{2\pi r}\mathbf{e}_r$, radial flow $u_r \propto \frac{1}{r}$. Flux across any circle $r = a$ is $2\pi a u_r =$ q (indepdendent of a). Called a *2D point source* of strength q (line source in 3D). For example $\phi = \frac{\kappa}{2\pi}\theta$ implies $\mathbf{u} = \nabla\phi = \frac{\kappa}{2\pi r}\mathbf{e}_\theta$ circular flow $\propto \frac{1}{r}$. Circulation $\oint \mathbf{u} \mathrm{d}\boldsymbol{l}$ around a circle $r = a$ is $2\pi a \frac{\kappa}{2\pi a} = \kappa$ independent of a – see section 2.5. A *point vortex* of circulation (strength) κ. (line vortex in 3D).
eg $\phi = Ur\cos\theta$ uniform flow again
eg $\phi = \frac{U\cos\theta}{r}$ 2D dipole
eg Uniform flow past a cylinder with circulation κ:

$$\nabla^2\phi = 0 \text{ in } r > a \qquad \phi \to Ur\cos\theta \text{ as } r \to \infty$$

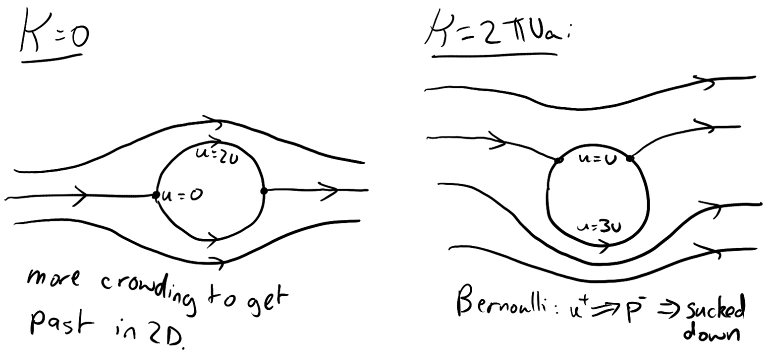
$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = 0 \text{ on } r = 0 \qquad \oint \mathbf{u}\cdot\boldsymbol{dl} = [\phi]_{r=a} = \kappa$$

– condition needed to get a unique solution.

$$\implies \phi = U\cos\theta\left(r + \frac{a^2}{r}\right) + \frac{\kappa}{2\pi}\theta$$

$$\implies \mathbf{u} = \nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

$$= \left(U\cos\theta\left(1 - \frac{a^2}{r^2}\right), -U\sin\theta\left(1 + \frac{a^2}{r^2}\right) + \frac{\kappa}{2\pi r}\right)$$



For the right diagram: this is like a tennis ball with top spin (consider change of reference frame).

4.3 Pressure in Potential Flow with Potential Forces

Non-linear equation of motion has been reduced to linear Laplace for the kinematics. But still have non-linearity in the dynamic boundary condition (pressure). Momentum equation:

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mathbf{f}$$

Potential forces $\mathbf{f} = -\nabla\chi$. Previous identity:

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\left(\frac{1}{2}u^2\right) - \mathbf{u}\times\boldsymbol{\omega}$$

($\mathbf{u}\times\boldsymbol{\omega}$ is zero because irrotational flow). Potential flow

$$\frac{\partial\mathbf{u}}{\partial t} = \nabla\frac{\partial\phi}{\partial t}$$

Thus the momentum equation reduces to

$$\nabla\left(\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho u^2 + p + \chi\right) = 0$$

$$\Longrightarrow \boxed{\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho u^2 + p + \chi = f(t), \qquad \text{independent of } \mathbf{x}}$$

"unsteady Bernoulli". Notes:

(1) $f(t)$ is actually irrelevant, since we can add arbitrary $g(t)$ to ϕ. – its the independence of $\mathbf{x}$ that matters.

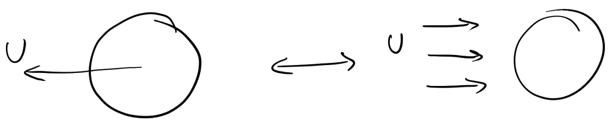
(2) Not the same as steady Bernoulli! – see handout

(3) For steady, irrotational flow, H is constant everywhere.

4.4 Force on Translating Sphere and Cylinder

4.4.1 Steadym Translating Sphere

For steady motion, it is most convenient to use the frame of reference moving with the sphere



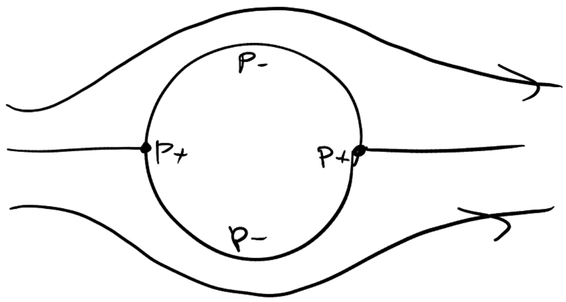
From section 4.2, uniform flow past stationary sphere is given by

$$\phi = U\cos\theta\left(r + \frac{a^3}{2r^2}\right)$$

and on $r = a$, $u_r = 0$, $u_\theta = -\frac{3}{2}U\sin\theta$. Apply either form of Bernoulli: ($\frac{\partial\phi}{\partial t} = 0$, $\chi = 0$) to obtain

$$\frac{1}{2}\rho\left(-\frac{3}{2}U\sin\theta\right)^2 + p(a,\theta) = \frac{1}{2}\rho U^2 + p_\infty$$

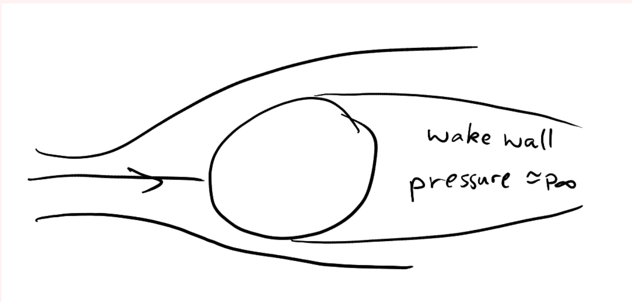
i.e. $p(a,\theta) = p_\infty + \frac{1}{2}\rho U^2\left(1 - \frac{a}{4}\sin^2\theta\right)$.



Pressure distribution is symmetric force-aft and round equator. $\implies$ The net force on the sphere is zero!

This surprising result is *d'Alembert's paradox*. In fact, there is no drag ($\equiv$ the net force parallel to the motion) on any steadily moving body in unbounded potential flow: KE is constant, we've neglected viscosity / friction. So conservation of energy gives no work done.

Note (Non-examinable)**.** effect of viscosity on translating sphere. See experimentally:



and find $F = 0.4\frac{1}{2}\rho U^2(\pi a^2)$ – see handout.

However, potential flow is good slippery bubbles, for rapid acceleration of a rigid particle, or small amplitude oscillations.

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4.4.2 Steady Translation of a Cylinder with Circulation

Work in frame moving steadily with the cylinder. From section 4.2, uniform flow past a cylinder with circulation κ given by

$$\phi = U\cos\theta\left(r + \frac{a^2}{r}\right) + \frac{\kappa}{2\pi}\theta, \qquad \text{and on } r = a, u_r = 0, u_\theta = -2U\sin\theta + \frac{\kappa}{2\pi a}$$

(For $\kappa > 4\pi Ua$, there is no streamline connecting the cylinder to ∞.) Use expression for pressure in potential flow (section 4.3) to deduce

$$-\frac{1}{2}\rho\left(-2U\sin\theta + \frac{\kappa}{2\pi a}\right)^2 + p(a,\theta) = \frac{1}{2}\rho U^2 + p_\infty$$

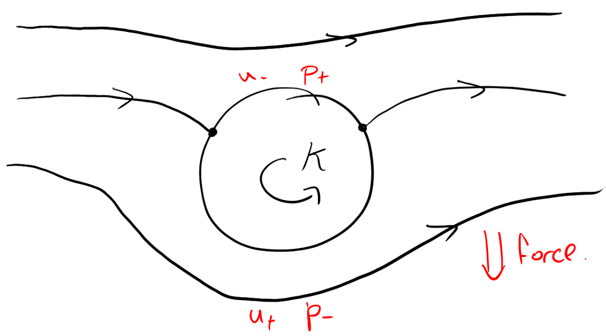
Fluid force on the cylinder (per unit length) is $-\int_{r=a} p(a,\theta)\mathbf{n}\mathrm{d}A$.

$$= -\int\left[p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left(4U^2\sin^2\theta - \frac{2U\kappa}{\pi a}\sin\theta + \frac{\kappa^2}{4\pi^2 a^2}\right)\right](\cos\theta,\sin\theta)a\mathrm{d}\theta$$

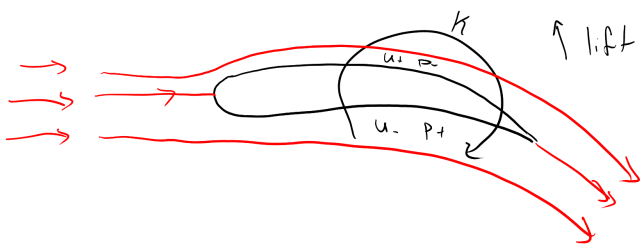
The only term that doesn't integrate to 0 is the $-\frac{2U\kappa}{\pi a}\sin\theta \times (0,\sin\theta)$ term. So it equals

$$= \left(0, -\frac{1}{2}\rho\cdot\frac{2U\kappa}{\pi a}\frac{1}{2}2\pi a\right) = (0, -\rho U\kappa)$$

i.e. we get a *lift* force (perpendicular to $\mathbf{U}$) – down for $\kappa > 0$, up for $\kappa < 0$. Recall $\kappa = 2\pi Ua$:



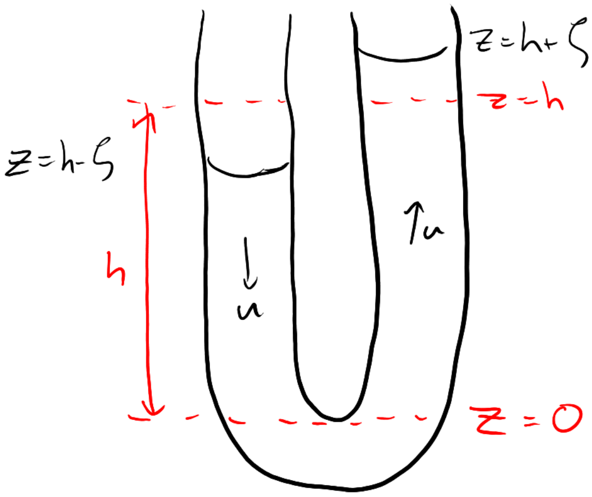
Can show (Sheet 3 Question 1) that uniform potential flow past *any* 2D body with circulation κ produces lift $-\rho U\kappa$. For example Aerofoils – wings, propellors, wind turbines.



Circulation is generated by the condition that the flow separates smoothly at the back (see handout).

4.5 Some Unsteady Potential Flows

4.5.1 Free Oscillations in a U-tube Manometer



Start from rest, displacement $\zeta(0) \neq 0$. $\implies$ irrotational $\implies$ potential. Assume long tubes of equal areas, and short, wide junction. At the bottom,

$$\Delta\phi = \int_L^R \mathbf{u} \cdot \mathrm{d}\mathbf{x} \simeq 0$$

because $\mathrm{d}\mathbf{x}$ is small (short) and $\mathbf{u}$ is small (wide). WLOG can take $\phi = 0$ at the base of both tubes. By mass conservation, u is uniform and equals $\dot{\zeta}$. RHS

$$\phi = uz, \quad \frac{\partial\phi}{\partial t} = \dot{u}z \quad \left.\frac{\partial\phi}{\partial t}\right|_{h+\zeta} = \ddot{\zeta}(h+\zeta)$$

(Note, this is not $\frac{\mathrm{d}}{\mathrm{d}t}(\phi(h+\zeta))$.) LHS

$$\phi = -uz \quad \frac{\partial\phi}{\partial t} = -\dot{u}z \quad \left.\frac{\partial\phi}{\partial t}\right|_{h-\zeta} = -\ddot{\zeta}(h-\zeta)$$

Apply $\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho u^2 + p + \chi$ independent of position.

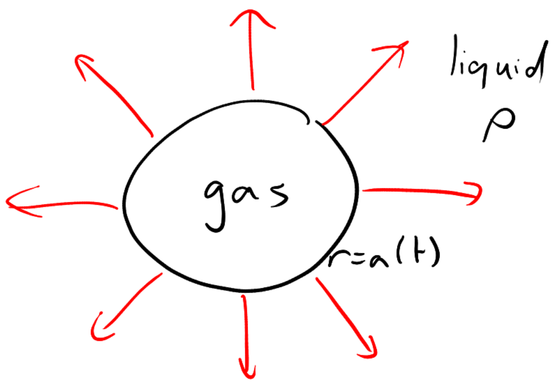
$$p\ddot{z}(h+\cancel{\zeta}) + \cancel{\frac{1}{2}\rho\dot{\zeta}^2} + \cancel{p_a} + \rho g(\cancel{h}+\zeta) = -\rho\ddot{\zeta}(h-\cancel{\zeta}) + \cancel{\frac{1}{2}\rho\dot{\zeta}^2} + \cancel{p_a} + \rho g(\cancel{h}-\zeta)$$

$$\rho\ddot{\zeta}(2h) = -\rho g(2\zeta)$$

implies SHM with frequency $\sqrt{\frac{g}{h}} = \omega$. But nonlinear if tubes have different areas.

4.5.2 Oscillation / Expansion / Collapse of a Bubble

Air is 15,000 times more compressible than water (compare $\frac{1}{v}\frac{\partial V}{\partial \rho}$). Consider incompressible flow of fluid outside a bubble of radius $a(t)$.



Neglect gravity. Spherical symmetry implies radial flow. $u \propto \frac{1}{r^2}$ by mass conservation. (implies $\boldsymbol{\omega} = \mathbf{0}$). $u = \dot{a}$ on $r = a$ implies $\mathbf{u} = \frac{a^2\dot{a}}{r^2}\mathbf{e}_r = \nabla\phi$ with $\phi = -\frac{a^2\dot{a}}{r}$ (source flow).

$$\frac{\partial \phi}{\partial t} = -\frac{a^2\ddot{a} + 2a\dot{a}^2}{r} \qquad \left.\frac{\partial \phi}{\partial t}\right|_{r=a} = -a\ddot{a} - 2\dot{a}^2$$

(Not $\frac{\mathrm{d}}{\mathrm{d}t}\phi(a(t),t)$). So using $\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho u^2 + p = f(t)$ $(\chi = 0)$ at $r = a$, $r = \infty$ we obtain

$$\boxed{-\rho a\ddot{a} - \frac{3}{2}\rho\dot{a}^2 = p_\infty - p(a(t),t)}$$

Multiply by $a^2\dot{a}$ implies $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\rho a^3\dot{a}^2\right) = a^2\dot{a}(p(a) - p_\infty)$. Interpret as / motivated by $\dot{KE}$ equals rate of working by the pressures at a and ∞. (Note $4\pi a^2\dot{a} = 4\pi r^2 ur$). Can integrate if $p(a,t)$ given as $F(a)$ ad $F(a)$ known from interior of bubble. For example under water explosion $p_a \gg p_\infty$, collapse of a void $p_a \ll p_\infty$.

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eg Small oscillations of a gass bubble $a = a_0 + \eta(t)$, $|\eta| \ll a_0$. Linearise (1) $\implies$ $\rho a_0\ddot{\eta} = p(a) - p_\infty$. Assume p_∞ is constant. Gas in bubble obeys $pV^\gamma = \text{const}$ (** adiabatic variation in an ideal gas $\gamma = \frac{C_\rho}{C_V} = 1$ for air **)

Then $\frac{\delta p}{p} = -\delta\frac{\delta V}{V} = -3\gamma\frac{\delta a}{a}$ because $V \propto a^3$.

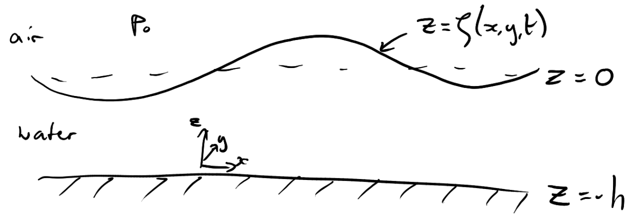
$$\implies \rho a_0\ddot{\eta} = p_0(-3\gamma)\frac{\eta}{a_0}$$

$\implies$ SHM with $\omega = \left(\frac{3\gamma p_0}{\rho a_0^2}\right)^{\frac{1}{2}} \approx 2 \times 10^4\mathrm{s}^{-1}$ for a 1mm bubble.

5 Geophysical Flows

5.1 Water Waves

Particularly successful application of potential flow. Consider



5.1.1 Governing Equations

Assume water is inviscid, and motion starts from rest. Hence, flow is and remains irrotational.

$$\implies \mathbf{u} = \nabla\phi \quad \text{and} \quad \boxed{\nabla^2\phi = 0 \text{ in } -h < z < \zeta(x,y,t)} \tag{1}$$

Kinematic boundary conditions

- At the rigid bottom

$$\boxed{\frac{\partial \phi}{\partial z} = 0 \text{ at } z = -h} \tag{2}$$

- At the air-water interface, often called the free surface as it is free to move,

$$\boxed{\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = w \text{ at } z = \zeta} \tag{3}$$

(See section 1.4)

Dynamic boundary condition

$$\boxed{p_{\text{water}} = p_{\text{air}} \text{ at } z = \zeta(x,y,t)}$$

As $p_{\text{air}} \ll p_{\text{water}}$, assume that pressure variations in the air are $\ll$ those in the water. Hence take $p_{\text{air}} = \text{const} = p_0$. Bernoulli for potential flow

$$\rho\frac{\partial \phi}{\partial t} + \frac{1}{2}\rho u^2 + p + \chi = f(t)$$

independent of $\mathbf{x}$ ($\chi = \rho g z$). Apply to the surface and use $p = p_0 = \text{const}$.

$$\implies \boxed{\rho\frac{\partial \phi}{\partial t} + \frac{1}{2}\rho|\nabla\phi|^2 + \rho g\zeta = f(t) \quad \text{at} \quad z = \zeta(x,y,t)} \tag{4}$$

Equations (1) - (4) give the full nonlinear problem. (1) is linear but on an unknown domain; (3) and (4) are complicated nonlinear boundary conditions to be applied at an unknown position ζ.

5.1.2 Linear Water Waves

For small-amplitudes ($\zeta \ll h$ and $\frac{\partial \zeta}{\partial x}, \frac{\partial \zeta}{\partial y} \ll 1$, waveheight $\ll$ fluid depth and wavelenth) can linearise the prolem (about a state of rest):

- Ignore the terms quadratic in disturbance quantities, for example $u\frac{\partial \zeta}{\partial x}$, ρu^2.
- Use Taylor series to expand boundary conditions at $z = \zeta$ in terms of information at $z = 0$. For example

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=\zeta} = \left.\frac{\partial \phi}{\partial z}\right|_{z=0} + \zeta \left.\frac{\partial^2 \phi}{\partial z^2}\right|_{z=0} + \cdots$$

  (on the RHS we ignore everything other than the $\left.\frac{\partial \phi}{\partial z}\right|_{z=0}$ term, as the others are quadratic or higher).

The linearise problem is

$$\begin{aligned}
\nabla^2 &= 0 && \text{in } -h < z < 0 && (1)\\
\frac{\partial \phi}{\partial z} &= 0 && \text{at } z = -h && (2)\\
\frac{\partial \zeta}{\partial t} &= \frac{\partial \phi}{\partial z} && \text{at } z = 0 && (3)\\
\rho\frac{\partial \phi}{\partial t} + \rho g \zeta &= f(t) && \text{independent of } x, y \text{ at } z = 0 && (4)
\end{aligned}$$

Using IB Michaelmas Methods, we can

- either look for separable solutions $\phi(x, y, z, t) = \Phi(z)X(x)Y(y)T(t)$ and find

$$X'' = -k^2 X \quad Y'' = -l^2 T \quad T'' = -\omega^2 T$$

  or, on an infinite domain, take Fourier transforms with respect to x, y, t.

In 2D, end up looking for solution $\phi = e^{i(kx-\omega t)}\Phi(z)$, $\zeta = \zeta_0 e^{i(kx-\omega t)}$ (with real part understood). (1) $\nabla^2\phi = 0$ implies $\Phi'' - k^2\Phi = 0$ $(*)$. (Could write down $\Phi = e^{\pm kz}$, but bottom boundary condition $\Phi'(-h) = 0$ suggests cosh and sinh better with origin shifted to $z = -h$, ie:)

$$(*) \implies \phi = A\cosh(k(z+h)) + B\sinh(k(z+h))$$

and boundary condition (2) implies $B = 0$. In (4), LHS $\propto e^{ikx} \implies f(t) = 0$ and LHS $= 0$. So (3) and (4)

$$\begin{cases} -i\omega\zeta_0 = Ak\sinh kh \\ -i\omega A\cosh kh + g\zeta_0 = 0 \end{cases}$$

Linear, homogeneous equations, with non-zero solution iff

$$\begin{vmatrix} -k\sinh kh & -i\omega \\ -i\omega\cosh kh & g \end{vmatrix} = 0$$

$$\implies \omega^2 \cosh kh = gk \sinh kh$$

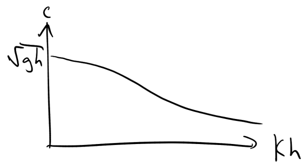
Solutions must satisfy the *dispersion relationship*

$$\boxed{\omega^2 = gk \tanh kh}$$

Wave crest moves with the *phase speed*

$$\boxed{c = \frac{\omega}{k}}$$

Long waves propagate faster.



Unlike light and sound, water waves are *dispersion*: waves of different frequencies at different speeds and thus disperse as they propagate away from a local disturbance (splash, storm in the Atlantic).

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Note (Non-examinable)**.** Can show (Part II Waves) that a wave packet (a group of nearly monochromatic waves) moves with the *group velocity* $c_g = \frac{\mathrm{d}q}{\mathrm{d}k}$.

Limiting Cases

(1) Deep-water limit. $kh \gg 1$, $\tanh kh \simeq 1$. So $w = \sqrt{gk}$, $c = \sqrt{g/k}$ and $c_g = \frac{1}{2}c$ independent of h. For example Atlantic storm produces swell waves with period 15s, $\omega = \frac{2\pi}{15} = 0.4\text{s}^{-1}$, $k^{-1} = \frac{5}{\omega^2} = 60\text{m} \ll$ ocean depth $= 4\text{km}$ (average). $c = \frac{\omega}{k} = 25\text{ms}^{-1} = 2000\text{km/day}$ so arrive before the storm.

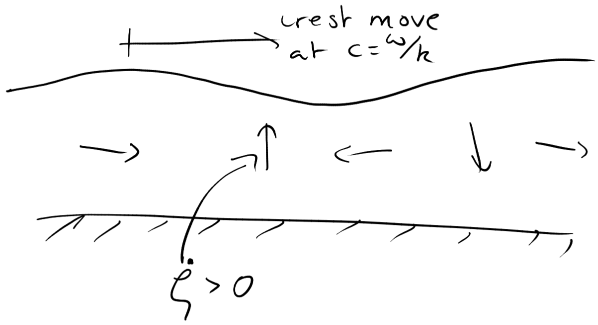
(2) Shallow water limit: $kh \ll 1$, $\tanh kh \simeq kh$. So $\omega = \sqrt{gh}k$, $c = \sqrt{gh}$, $c_y = c$ independent of k. For example flood waves on a river, $h = 2\text{m} \implies c = \sqrt{gh} = 4\text{ms}^{-1} = 16\text{kmh}^{-1}$. Or tsunami, $h = 4\text{km}$ (average) $\implies c = \sqrt{gh} = 200\text{ms}^{-1}$ roughly the speed of a commercial plane. Wavelength $\lambda \simeq 500\text{km}$. (Boxing Day 2004).

Velocities

$\zeta = \zeta_0 e^{i(kx-\omega t)}$, $\phi = -\frac{i\omega\zeta_0}{k}\frac{\cosh k(z+h)}{\sinh kh}e^{i(kx-\omega t)}$.

$$\mathbf{u} = \nabla\phi = (\cosh k(z+h), -i\sinh k(z+h))\frac{\zeta_0 w}{\sinh kh}e^{i(kx-\omega t)}$$

(real part understood). u_x and ζ are in phase.



Deep water limit ($kh \gg 1$)

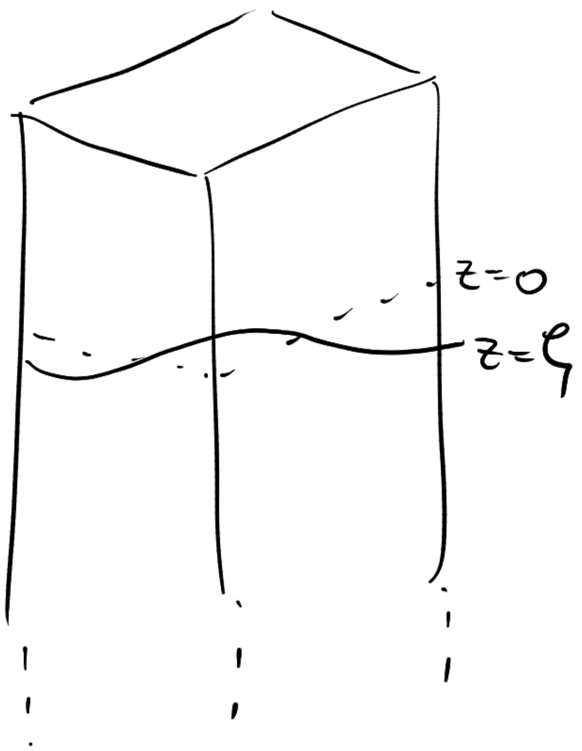
$\cosh, \sinh \simeq e^{+}$. Hence $\mathbf{u} = (1, -i)e^{k\zeta}\zeta_0 \omega e^{i(kx-\omega t)}$ circular motion, smaller at depth.

Shallow water limit ($kh \ll 1$)

$\mathbf{u} \simeq \left(\frac{1}{kh}, -i\frac{z+h}{h}\right)\zeta_0 \omega e^{i(kx-\omega t)}$ ($\frac{1}{kh}$ large, so horizontal velocity $\gg$ vertical velocity).

5.1.3 Standing Waves in a Container

Consider water in a deep rectangular box $0 < x < a$, $0 < y < b$, $z < 0$. Look for linearised waves with surface at $z = \zeta(x, y, t)$, $|z| \ll a, b$, $|\nabla\zeta| \ll 1$.



Want to solve $\nabla^2\phi = 0$ in $z < 0$, $\frac{\partial \phi}{\partial n} = 0$ on side boundaries. $\nabla\phi \to 0$ as $z \to -\infty$. Also want on $z = 0$, $\frac{\partial \zeta}{\partial t} = \frac{\partial \phi}{\partial z}$ and $\rho\frac{\partial \phi}{\partial t} + \rho g \zeta$ independent of x, y. Try separation of variables $\phi(x, y, z, t) = A\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{kx}e^{-i\omega t}$.

Where Laplace's equation $-\frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2} + k^2 = 0$. Kinematic boundary condition

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \phi}{\partial z} \implies \zeta = \frac{iAk}{\omega}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{-i\omega t}$$

dynamic boundary condition

$$\frac{\partial \phi}{\partial t} + g\zeta \implies -i\omega A + g\frac{iAk}{\omega} = 0 \implies \omega^2 = gk$$

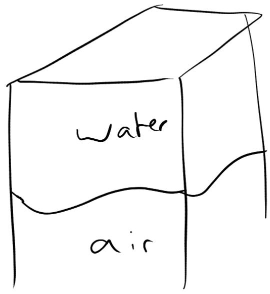
as before for deep-water. Only difference is that k is quantised by the side walls:

$$k_{mn} = \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)^{\frac{1}{2}}$$

These are standing waves / normal modes. $\zeta = f_{mn}(x, y)\cos(\omega_{mn}t)$.

The Rayleigh Instability

Consider an upside down container



Obvious instability! Just let $g \to -g \implies \omega^2 = -gk \implies \omega = \pm i\sqrt{gk}$, $e^{-i\omega t} = e^{\pm\sqrt{gk}t}$.

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5.2 Rotating Fluid Dynamics

Introduction: We live on a rotating Earth ($\Omega = \frac{2\pi}{86400\mathsf{s}} \simeq 10^{-4}\mathsf{s}^{-1}$) and the rotation has a huge effect on the large-scale motion of the oceans and atmosphere, hence climate. Moreover, the ocean and atmosphere are thin ($\simeq 4\mathsf{km}$, $\simeq 10\mathsf{km}$) compared to the Earth's radius ($R \simeq 6000\mathsf{km}$). Rotation of the Earth $\implies$ speed $R\Omega$ at the equation $\simeq 500\mathsf{ms}^{-1}$ $\gg$ winds relative to Earth. So sensible to work in a rotating frame.

5.2.1 Euler Equations in a Rotating Frame

Assume ρ uniform and $\nabla \cdot \mathbf{u} = 0$ – good for ocean, can make so for atmosphere by a fix. Acceleration of a fluid particle in a rotating frame is

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x})$$

($2\boldsymbol{\Omega} \times \mathbf{u}$ is Coriolis force and $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x})$ is centrifugal force). Now $\rho\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) = \nabla\left(\frac{1}{2}\rho|\boldsymbol{\Omega} \times \mathbf{x}|^2\right)$ so can combine with $\rho\mathbf{g} = -\nabla\chi$ and treat them as an "effective" $\mathbf{g}$ with an effective definition "vertical", which is normal to the surface of the spheroidal Earth (Or note $\frac{|\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x})|}{g} \simeq \frac{1}{300} \ll 1$). Hence

$$\boxed{\rho\frac{Dy}{Dt} + 2\rho\boldsymbol{\Omega} \times \mathbf{u} = -\nabla p + \rho\mathbf{g}}$$

$$\nabla \times (2\boldsymbol{\Omega} \times \mathbf{u}) = 2\boldsymbol{\Omega}\cancel{\nabla \cdot \mathbf{u}} - 2\boldsymbol{\Omega} \cdot \nabla\mathbf{u}$$

Hence the voriticty equation becomes

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla\mathbf{u}$$

The vortex stretching term now involves the *total / absolute vorticity*

$$\nabla \times (\mathbf{u} + \mathbf{\Omega} \times \mathbf{x}) = \mathbf{w} + 2\mathbf{\Omega}$$

($\mathbf{w}$ is relative vorticity and $2\mathbf{\Omega}$ is planetary voriticity).

The Rossby Number

Suppose we have a flow with a characteristic lengthscale L and velocity scale U and timescale $T = \frac{L}{U}$:

$$\rho(\mathbf{u} \cdot \nabla)\mathbf{u} \sim \frac{\rho U^2}{L}$$
$$2\rho\mathbf{\Omega} \times \mathbf{u} \sim \rho \Omega U$$

The ratio

$$\frac{\rho U^2/L}{\rho U \Omega} = \boxed{\frac{U}{\Omega L} \equiv R_0}$$

is the *Rossby number* describes the relative importance of inertia and Coriolis effects.

- For example Weather system $L \sim 1000\mathsf{km}$, $U \sim 10\mathsf{ms}^{-1}$, $T\ sim 10^5\mathsf{s} = 1\mathsf{day}$.

  $$\implies R_0 \sim 10^{-1}$$

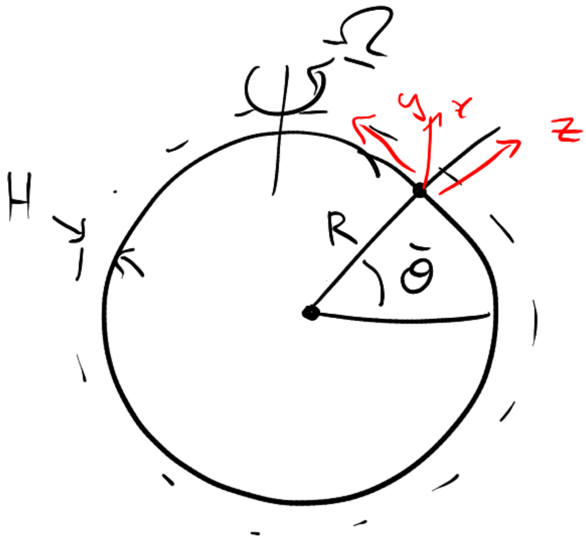
  so Coriolis terms dominate.

- Another example would be emptying bathtub. Near the plughole $L \sim 30\mathsf{cm}$, $U \sim 3\mathsf{cms}^{-1}$, $T \sim 10\mathsf{s}$

  $$\implies R_0 \sim 10^3$$

  so Coriolis effects are a tiny perturbation. $R_0 \ll 1$ also implies $\boldsymbol{\omega} \lesssim \frac{U}{L} \ll \mathbf{\Omega}$, so relative vorticity $\ll$ planetary.

Rotating Flow in a Shallow Layer



Consider flows with horizontal lengthscale L such that $H \ll L \ll R$. For example 100km to 1000km. Take local cartesian coordinates x eastwards, y northwards and z upwards, and $\mathbf{u} = (u, v, w)$. If $\overline{\theta}$ is the latitude ($\subset [-\pi/2, \pi/2]$) then $\mathbf{\Omega} = (0, \cos\overline{\theta}\Omega, \sin\overline{\theta}\Omega)$ and $2\mathbf{\Omega} \times \mathbf{u} = 2\Omega(\cos\overline{\theta}w - \sin\overline{\theta}v, \sin\overline{\theta}u, -\cos\overline{\theta}u)$. We make 3 good approximations:

(1) For $|\mathbf{u}| \sim 10\text{ms}^{-1}$, $\frac{\Omega u}{g} \sim \frac{10^{-4}10}{10} = 10^{-4} \ll 1$ so we can neglect $-\cos\overline{\theta}u$ in the vertical equation.

(2) For $L \gg H$,

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$$

suggests $\frac{w}{H} \sim \frac{u,v}{L} \implies w \sim \frac{H}{L}(u, v) \ll u, v$ so the flow is dominantly horizontal. (cf velocities in shallow water waves). Hence:

(i) Neglect $\cos\overline{\theta}w$ in the x-direction (except near the equator)

(ii) Neglect the vertical acceleration $\rho\frac{Dw}{Dt}$ so the vertical force balance is hydrostatic.

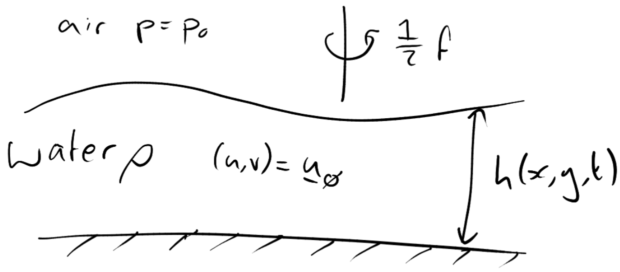
Hence

$$\boxed{\begin{aligned} \frac{Du}{Dt} - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (1) \\ \frac{Dv}{Dt} + fu &= -\frac{1}{\rho}\frac{Dp}{Dy} \qquad (2) \\ 0 &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \end{aligned}}$$

where $f = 2\Omega \sin \overline{\theta}$ is the *Coriolis parameter* (only the vertical component of $\boldsymbol{\Omega}$ matters). Also, $\frac{\partial w}{\partial z} \sim \frac{w}{H} \gg \frac{\partial w}{\partial x}, \frac{\partial w}{\partial x} \sim \frac{w}{L}$ so the vertical component of the vorticity equation becomes

$$\boxed{\begin{aligned} \frac{D\zeta}{Dt} &= (\zeta + f)\frac{\partial w}{\partial z} \\ &= -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \end{aligned}}$$

where $\boxed{\zeta = \omega_2}$. Consider a flow in a shallow layer of thickness $h(x, y, t)$:



$$\begin{aligned} \frac{\partial p}{\partial z} = \rho g &\implies p = p_0 + \rho g(h - z) \\ &\implies \nabla_H p = \rho g \nabla_H h \end{aligned}$$

is independent of z $\left(\nabla_H \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\right)$

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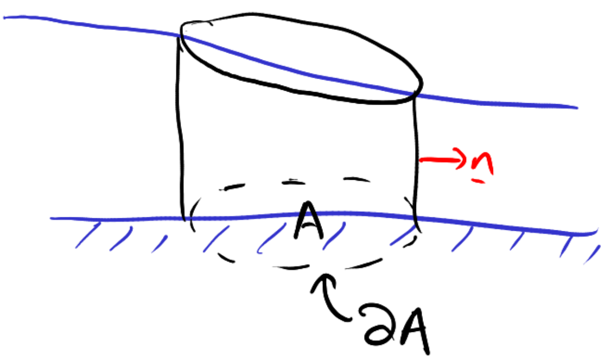
Assume the initial conditions are such that u, v independent of z $\implies$ remain so.

Mass conservation

Consider a cylindrical volume



$$\frac{\mathrm{d}}{\mathrm{d}t}\int_A \rho h \mathrm{d}A + \int_{\partial A} \rho h \mathbf{u}_H \cdot \mathbf{n} \mathrm{d}s = 0$$

$$\implies \int_A \frac{\partial h}{\partial t} + \nabla_H \cdot (h\mathbf{u}_H) = \mathrm{d}A = 0$$

$$\implies \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$$

or

$$\boxed{\frac{Dh}{Dt} = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}$$

Geostrophic balance

If $R_0 \equiv \frac{U}{\Omega L} \ll 1$ ($\implies$ $\mathbf{u} \cdot \nabla \mathbf{u} \ll 2\mathbf{\Omega} \times \mathbf{u}$) and steady ($\frac{\partial}{\partial t} = 0$) then (1) and (2)

$$\implies -fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial h}{\partial x}$$
$$+fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial h}{\partial y}$$

$\implies$ flow given by 2D stream-function $\psi = -\frac{p}{\rho f} = -\frac{gh}{f}$, i.e. pressure / height contours are streamlines. Hence winds blow parallel to isobars not perpendicular! $\rho(f\mathbf{e}_z) \times \mathbf{u} = -\nabla_H p$ is called *geostrophic balance.* Note, if given $h(x, y)$ can calculate steady flow, but no equation for h or $\frac{\partial h}{\partial t}$.

Potential Vorticity

Recall

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{v}$$

$$\frac{D}{Dt}h = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{m}$$

$$\implies \boxed{\frac{D}{Dt}\left(\frac{\zeta + f}{h} = 0\right)} \tag{ρv}$$

$\boxed{\frac{\zeta + f}{h}}$ is called the *potential vorticity.* PV is conserved for a material cylinder; the total vorticity $\zeta + f$ is $\propto$ the height of cylinder because vertical stretching $\rightarrow$ smaller radius $\rightarrow$ spins faster. PV is useful because $\nabla \times$ (momentum) eliminates the largest terms ∇p and $2\rho\mathbf{\Omega} \times \mathbf{u}$ of nearly geostrophic balance and reveals how the smaller terms give the evolution of h with time.

Linearised Evolution

Suppose $h = h_0 + \eta(x, y, t)$, $|\eta| \ll h_0$, $|\nabla\eta| \ll \frac{H}{L} \ll 1$. Linearised equations are

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{1}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \tag{2}$$

$$\frac{\partial \eta}{\partial t} + h_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{m}$$

and

$$\frac{\partial}{\partial t}\left(\frac{f}{h_0}\frac{1(+\zeta/f)}{(1+\eta/h_0)}\right) = 0 \implies \frac{\zeta}{f} - \frac{\eta}{h_0}(+O(z))$$

is constant, i.e. $\zeta h_0 - \eta f =$ initial value $\zeta_0 h_0 - \eta_0 f$. (pv). Now $h_0\left(\frac{\partial(1)}{\partial x} + \frac{\partial(2)}{\partial y}\right) - \frac{\partial(m)}{\partial t}$

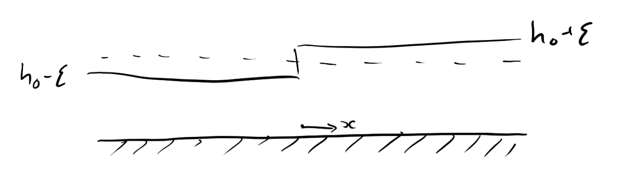
$$\implies -\frac{\partial^2 \eta}{\partial t^2} + gh_0\nabla^2\eta - h_0 f\zeta = 0.$$

Comine with (pv) $\implies$

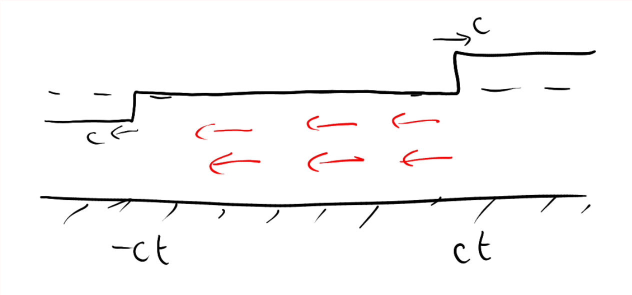
$$\boxed{\frac{\partial^2 \eta}{\partial t^2} - c^2\nabla^2\eta + f^2\eta = f^2\eta_0 - h_0 f\zeta_0}$$

$c = \sqrt{gh_0} =$ shallow-water wave speed.

Example. $\eta_0 = \varepsilon\,\mathrm{sgn}(x)$, $\mathbf{u}_0 = \mathbf{0} \implies \zeta_0 = 0$.



Without rotation ($f =$) just have $\frac{\partial^2 \eta}{\partial t^2} = c^2 \frac{\partial^2 \eta}{\partial x^2}$ with d'Alembert solution



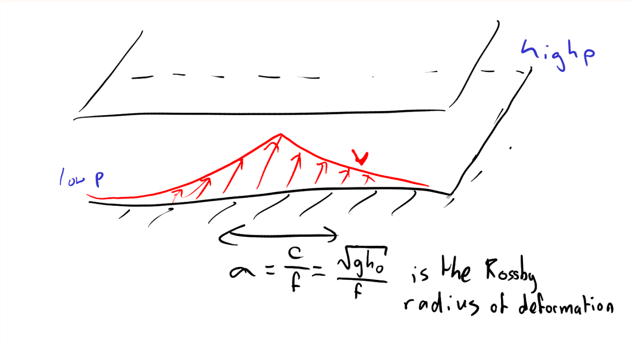
With rotation, the final steady state solves

$$\frac{\partial^2 \eta_\infty}{\partial x^2} - \frac{f^2}{c^2}\eta_\infty = -\frac{f^2}{c^2}\varepsilon\,\mathrm{sgn}(x)$$

$$\implies \eta_\infty = \varepsilon\,\mathrm{sgn}(x)(1 - e^{-|x|f/c})$$

$$\implies u_\infty = -\frac{g}{f}\frac{\partial \eta_\infty}{\partial y} = 0$$

$$v_\infty = \frac{g}{f}\frac{\partial \eta_\infty}{\partial x} = \varepsilon\frac{g}{c}e^{-|x|f/c}$$



Completely different!