Electromagnetism

March 14, 2023

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Lectures

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1 Introduction

1.1 Charges and currents

Electric charge is a physical property of elementary particles. It is

- Positive, negative or zero.
- Quantized (an integer multiple of the *elementary charge* e)
- Conserved (even if particles are created or destroyed).

By convention, the electron has charge $-e$, proton has charge $+e$ and neutron has charge 0.

On macroscopic scales, the number of particles is so large that charge can be considered to have a continuous *electric charge density* $\rho(\mathbf{x}, t)$. The total charge in a volume V is

$$Q = \int_V \rho \mathrm{d}V$$

The *electric current density* $\mathbf{J}(\mathbf{x}, t)$ is the flux of electric charge per unit area. The current flowing through a surface S is

$$I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}.$$

Consider a time-independent volume V with boundary S. Since charge is conserved,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -I$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \mathrm{d}V + \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = 0$$

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}\right) \mathrm{d}V = 0$$

Since this is true for any V,

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0} \tag{1}$$

This *equation of charge conservation* has the typical form of a conservation law. The discrete charge distribution of a single particle of charge q_i and position vector $\mathbf{x}_i(t)$ is

$$\rho = q_i \delta(\mathbf{x} - \mathbf{x}_i(t))$$

$$\mathbf{J} = q_i \dot{\mathbf{x}}_i \delta(\mathbf{x} - \mathbf{x}_i(t))$$

For N particles it is

$$\rho = \sum_{i=1}^{N} q_i \delta(\mathbf{x} - \mathbf{x}_i(t))$$

$$\mathbf{J} = \sum_{i=1}^{N} q_i \dot{\mathbf{x}}_i \delta(\mathbf{x} - \mathbf{x}_i(t))$$

Exercise: Verify that these satisfy equation (1).

1.2 Fields and Forces

Electromagnetism is a *field theory*. Charged particles interact not directly but by generating fields around them that are experienced by other charged particles. In general we have two time-dependent vector fields:

- electric field $\mathbf{E}(\mathbf{x}, t)$.
- magnetic field $\mathbf{B}(\mathbf{x}, t)$.

The *Lorentz force* on a particle of charge q and velocity $\mathbf{v}$ is

$$\boxed{\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

1.3 Maxwell's Equations

In this course we will explore some consequences of *Maxwell's equations*:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{M1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{M2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{M3}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \tag{M4}$$

Some properties:

- Coupled linear PDEs in space and time.
- Involve two positive constants:
  - ε_0 (vacuum permittivity)
  - μ_0 (vacuum permeability)
- Charges (ρ) and currents ($\mathbf{J}$) are the sources of the electromagnetic fields.
- Each equation has an equivalent integral form (see later) related via the divergence theorem or Stoke's theorem.

- These are the vacuum equations that apply on microscopic scales (or in a vacuum). A related macroscopic version applies in media (for example air) (see Part II Electrodynamics).
- The equations are consistent with each other and with charge conservation.
  - $\nabla \cdot$ (M3) agrees with $\frac{\partial}{\partial t}$(M2)
  -

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} &= \frac{\partial}{\partial t}(\varepsilon_0 \nabla \cdot \mathbf{E}) + \nabla \cdot \left(-\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0} \nabla \times \mathbf{B}\right) \\ &= 0\end{aligned}$$

1.4 Units

The SI unit of electric charge is the coulomb (C). The elementary charge is (exactly)

$$e = 1.602176634 \times 10^{-19}\mathsf{C}.$$

The SI unit of electric current is the ampere or amp (A), equal to $1\mathsf{Cs}^{-1}$. The SI base units needed in electromagnetism are

- second (s)
- metre (m)
- kilogram (kg)
- ampere (A)

From the Lorentz force law we see that the units of $\mathbf{E}$ and $\mathbf{B}$ must be $\mathsf{kgs}^{-3}\mathsf{A}^{-1}$ and $\mathsf{kgs}^{-2}\mathsf{A}^{-1}$ (also called tesla (T)). From Maxwell's equations we can work out the units of ε_0 and μ_0. Experimentally determined values are

$$\begin{aligned}\varepsilon_0 &= 8.854\ldots \times 10^{-12}\mathsf{kg}^{-1}\mathsf{m}^{-3}\mathsf{s}^4\mathsf{A}^2 \\ \mu_0 &= 1.256\ldots \times 10^{-6}\mathsf{kgms}^{-2}\mathsf{A}^{-2} \\ &\approx 4\pi \times 10^{-7}\end{aligned}$$

The speed of light is (exactly)

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 299792458\mathsf{ms}^{-1} \approx 3 \times 10^8\mathsf{ms}^{-1}$$

Start of lecture 2

2 Electrostatics

In a time-independent situation, Maxwell's equations reduce to

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \mathbf{E} = \mathbf{0}$$
$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Since $\mathbf{E}$ and $\mathbf{B}$ are decoupled, we can study them separately.

Electrostatics is the study of the electric field generated by a stationary charge distribution.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{M1}$$
$$\nabla \times \mathbf{E} = \mathbf{0} \tag{M3'}$$

2.1 Gauss's Law

Consider a surface S enclosing a volume V. Integrate (M1) over V and use the divergence theorem to obtain *Gauss's law*

$$\boxed{\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\varepsilon_0}}$$

where $Q = \int_V \rho \mathrm{d}V$ is the total charge in V. Gauss's law is the integral version of (M1) and is valid generally.

$$\text{electric flux} \propto \text{total charge enclosed}$$

In special situations we can use Gauss's law together with symmetry to deduce $\mathbf{E}$ from ρ, by choosing the *Gaussian surface* S appropriately.

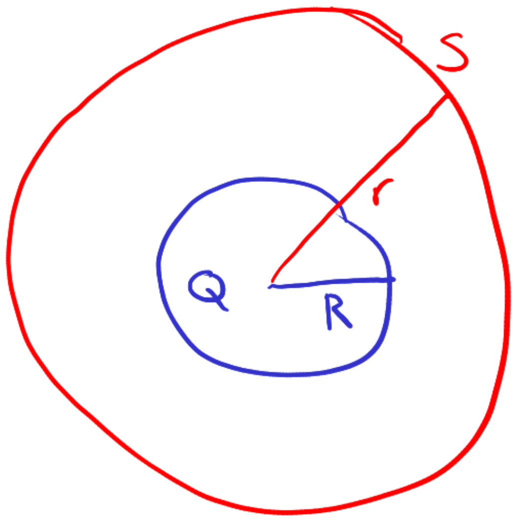
2.1.1 Spherical symmetry

Consider a spherically symmetric charge distribution, $\rho(r)$ in spherical coordinates, with total charge Q contained within an outer radius R. To have spherical symmetry, the electric field should have the form

$$\mathbf{E} = E(r)\mathbf{e}_r$$

This will satisfy (M3'), as required. To find $E(r)$, apply Gauss's law to a sphere of radius r. If $r > R$ then

$$\begin{aligned}\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} &= E(r) \int_S \mathbf{e}_r \cdot \mathrm{d}\mathbf{S} \\ &= E(r) \int_S \mathrm{d}S \\ &= E(r) 4\pi r^2 \\ &= \frac{Q}{\varepsilon_0}\end{aligned}$$



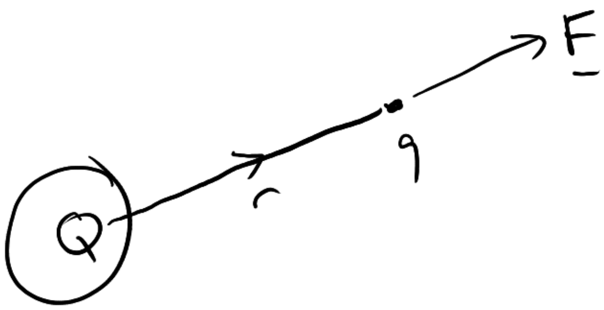
Thus

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{e}_r$$

So the external electric field of a spherically symmetric body depends only on the total charge. The Lorentz force on a particle of charge q in $r > R$ is

$$\mathbf{F} = q\mathbf{E} = \frac{Qq}{4\pi\varepsilon_0 r^2}\mathbf{e}_r$$

This is the *Coulomb force* between charged particles. The force is repulsive if the charges have the same sign ($Qq > 0$) and attractive if they have opposite signs ($Qq < 0$).



In the limit $R \to 0$, we obtain the electric field of a *point charge* Q, corresponding to $\rho = Q\delta(\mathbf{x})$.

Comparison with gravity

There is a close analogy between the Coulomb force and the gravitational force between massive particles,

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e}_r$$

Both involve an inverse-square law and the product of the charges / masses. However:

- While gravity is always attractive, electric forces can be repulsive or attractive.
- Gravity is very much weaker.

For example, for two protons the ratio of the electric to gravitational forces is

$$\frac{e^2}{4\pi\varepsilon_0 G m_\rho^2} \approx 10^{36}$$

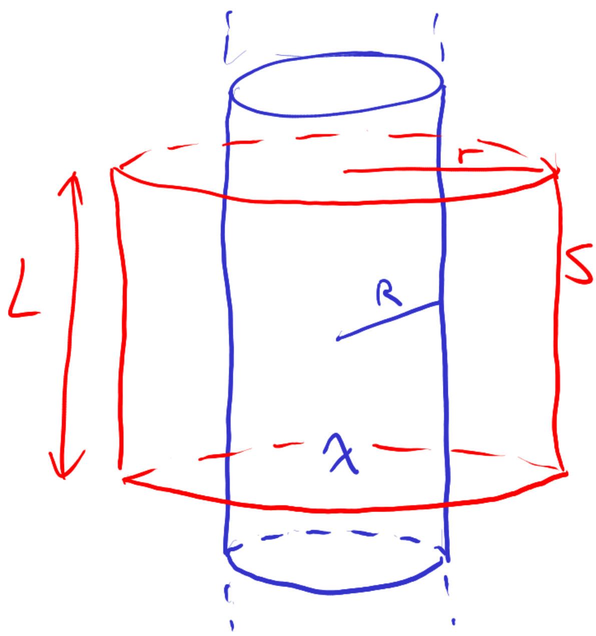
On the atomic scale, gravity is irrelevant. But + and − charges balance so accurately that on the planetary scale, gravity is dominant.

2.1.2 Cylindrical symmetry

Consider a cylindrically symmetric charge distribution, $\rho(r)$ in *cylindrical* polar coordinates, with total charge λ per unit length contained within an outer radius R. To have cylindrical symmetry,

$$bfE = E(r)\mathbf{e}_r$$

(Will satisfy (M3') as required.) To find $E(r)$, apply Gauss's law to a cylinder of radius r and arbitrary length L.



If $r > R$ then

$$\begin{aligned}\int_S \mathbf{E}\cdot \mathrm{d}\mathbf{S} &= E(r)\int_S \mathbf{e}_r \cdot \mathrm{d}\mathbf{S} \\ &= E(r)\int_{\substack{S\\ \text{(curved part)}}} \mathrm{d}S \\ &= E(r) 2\pi r L \\ &= \frac{\lambda L}{\varepsilon_0}\end{aligned}$$

Thus

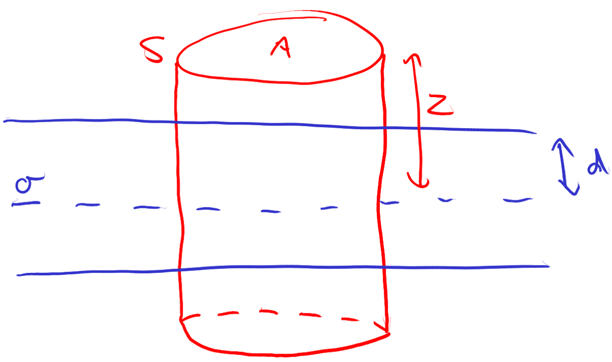
$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\mathbf{e}_r$$

In the limit $R \to 0$ we obtain the electric field of a *line charge* λ per unit length, corresponding to

$$\rho = \lambda \delta(x)\delta(y)$$

2.1.3 Planar symmetry

Planar charge distribution: $\rho(z)$ in Cartesian coordinates, with total charge σ per unit area contained within a region $-d < z < d$ of thickness $2d$. Assume reflectional symmetry, then $\rho(z)$ is even. To have planar symmetry, $\mathbf{E} = E(z)\mathbf{e}_z$. Will satisfy (M3') as required. Reflectional symmetry implies $E(-z) = -E(z)$. To find $E(z)$ for $z > 0$, apply Gauss's law to a "Gaussian pillbox" of hight $2z$ and arbitrary area A.



If $z > d$ then

$$\begin{aligned}\int_S \mathbf{E}\cdot \mathrm{d}\mathbf{S} &= E(z)A - E(-z)A \\ &= 2E(z)A \\ &+ \frac{\sigma A}{\varepsilon_0}\end{aligned}$$

Thus

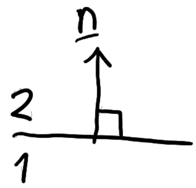
$$\mathbf{E} = \begin{cases} \frac{\sigma}{2\varepsilon_0}\mathbf{e}_z & z > d \\ -\frac{\sigma}{2\varepsilon_0}\mathbf{e}_z & z < -d \end{cases}$$

In the limit $d \to 0$ we obtained the electric field of a *surface charge* σ per unit area, corresponding to

$$\rho = \sigma\delta(z)$$

2.1.4 Surface charge and discontinuity

Let $\mathbf{n}$ be a unit vector normal to the charged surface, pointing from region 1 to region 2.



In our example, $\mathbf{n} = \mathbf{e}_z$. The discontinuity in $\mathbf{E}$ is given by

$$[\mathbf{n} \cdot \mathbf{E}] = \frac{\sigma}{\varepsilon_0} \tag{1}$$

where σ is the surface charge density and

$$[X] = X_2 - X_1$$

denotes a discontinuity. The tangential components are continuous:

$$[\mathbf{n} \times \mathbf{E}] = \mathbf{0} \tag{2}$$

Start of lecture 3

2.2 The electrostatic potential

For general $\rho(\mathbf{x})$ we cannot determine $\mathbf{E}(\mathbf{x})$ using Gauss's law alone.

(M3') implies that $\mathbf{E}$ can be written in terms of an *electrostatic* (or *electric*) *potential* $\Phi(\mathbf{x})$:

$$\boxed{\mathbf{E} = -\nabla\Phi}$$

The *potential difference* (or *voltage*) between two points $\mathbf{x}_1$ and $\mathbf{x}_2$ is

$$\begin{aligned}\Phi(\mathbf{x}_2) - \Phi(\mathbf{x}_1) &= \int \mathrm{d}\Phi \\ &= -\int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{E}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x}\end{aligned}$$

and is path-independent because $\nabla \times \mathbf{E} = \mathbf{0}$.

The electric force on a particle of charge q,

$$\mathbf{F} = q\mathbf{E} = -q\nabla\Phi$$

is a conservative force associated with the potential energy

$$\boxed{U(\mathbf{x}) = q\Phi(\mathbf{x})}$$

(M1) implies that Φ satisfies *Poisson's equation*

$$\boxed{-\nabla^2\Phi = \frac{\rho}{\varepsilon_0}}$$

The solution can be written as an integral (over all space, assuming decay at infinity)

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^2\mathbf{x}'$$

This is the convolution of $\rho(\mathbf{x})$ with the potential of a unit point charge, $\frac{1}{4\pi\varepsilon_0|\mathbf{x}|}$, which is the solution of

$$-\nabla^2\Phi = \frac{\delta(\mathbf{x})}{\varepsilon_0}$$

satisfying $\Phi \to 0$ as $|\mathbf{x}| \to \infty$.

$\mathbf{E}$ is unaffected if we add an arbitrary constant to Φ. We usually choose this such that $\Phi \to 0$ as $|\mathbf{x}| \to \infty$. If $\rho(\mathbf{x})$ does not decay sufficiently rapidly, this may not be possible, for example a line charge $E_r \propto \frac{1}{r}$ and $\Phi \propto \ln r$.

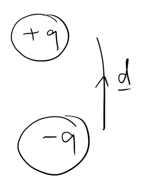
2.2.1 Point Charge

Potential due to a point charge q at the origin:

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0|\mathbf{x}|} = \frac{q}{4\pi\varepsilon_0 r}$$

2.2.2 Electric Dipole

Two equal and opposite charges at different positions. Without loss of generality, consider charges $-q$ at $\mathbf{x} = \mathbf{0}$ and $+q$ at $\mathbf{x} = \mathbf{d}$.



Potential due to the dipole:

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{|\mathbf{x}|} + \frac{1}{|\mathbf{x}-\mathbf{d}|}\right)$$

Apply Taylor's theorem for a scalar field,

$$f(\mathbf{x}) = f(\mathbf{x}) + (\mathbf{h}\cdot\nabla)f(\mathbf{x}) + \frac{1}{2}(\mathbf{h}\cdot\nabla)^2 f(\mathbf{x}) + O(|\mathbf{h}|^3)$$

to $f(\mathbf{x}) = \frac{1}{|\mathbf{x}|} = \frac{1}{r}$:

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{r} + \frac{1}{r} - (\mathbf{d}\cdot\nabla)\frac{1}{r} + O(|\mathbf{d}|^2)\right)\\ &= \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{d}\cdot\mathbf{x}}{|\mathbf{x}|^3} + O(|\mathbf{d}|^2)\end{aligned}$$

In the limit $|\mathbf{d}| \to 0$ with $q\mathbf{d}$ finite, we obtain a *point dipole* with *electric dipole moment*

$$\mathbf{p} = q\mathbf{d},$$

potential

$$\Phi(\mathbf{x}) = \frac{\mathbf{p}\cdot\mathbf{x}}{4\pi\varepsilon_0|\mathbf{x}|^3}$$

and electric field

$$\mathbf{E} = -\nabla\Phi = \frac{3(\mathbf{p}\cdot\mathbf{x})\mathbf{x} - |\mathbf{x}|^3\mathbf{p}}{4\pi\varepsilon_0|\mathbf{x}|^5}$$

In spherical polar coordinates aligned with $\mathbf{p} = p\mathbf{e}_z$,

$$\begin{aligned}\Phi &= \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}\\ E_r &= -\frac{\partial\Phi}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}\\ E_\theta &= -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}\\ (E_\phi &= 0)\end{aligned}$$

Note

- Φ and $\mathbf{E}$ are not spherically symmetric.
- They decrease more rapidly with r than for a point charge.

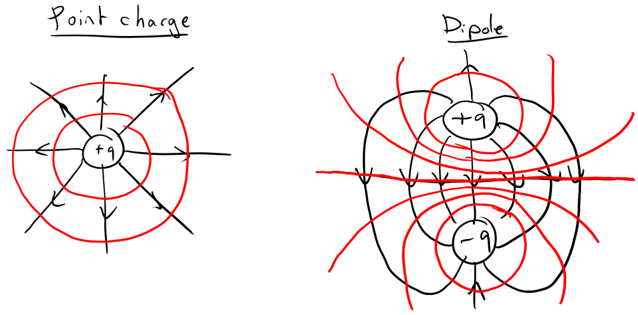
A point dipole $\mathbf{p}$ at the origin corresponds to

$$\begin{aligned}\rho(\mathbf{x}) &= -\mathbf{p}\cdot\nabla\delta(\mathbf{x})\\ \Phi(\mathbf{x}) &= \mathbf{p}\cdot\nabla\left(\frac{1}{4\pi\varepsilon_0|\mathbf{x}|}\right)\end{aligned}$$

2.2.3 Field lines and equipotentials

Electric field lines are the integral curves of $\mathbf{E}$, being tangent to $\mathbf{E}$ everywhere.

Since $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$, field lines begin on $+$ charges and end on $-$ charges. In electrostatics, $\mathbf{E} = -\nabla\Phi$, field lines are perpendicular to the equipotential surfaces $\Phi = \text{const.}$



2.2.4 Dipole in an external field

Consider a dipole $\mathbf{p}$ in an external electric field $\mathbf{E} = -\nabla\Phi$ generated by distinct charges. With $-q$ at $\mathbf{x}$ and $+q$ at $\mathbf{x} + \mathbf{d}$, the potential energy of the dipole due to the external field is

$$\begin{aligned} U &= -q\Phi(\mathbf{x}) + q\Phi(\mathbf{x} + \mathbf{d}) \\ &= q(\mathbf{d} \cdot \nabla)\Phi(\mathbf{x}) + O(|\mathbf{d}|^3) \end{aligned}$$

In the limit of a point dipole,

$$\begin{aligned} U &= \mathbf{p} \cdot \nabla\Phi \\ &= -\mathbf{p} \cdot \mathbf{E} \end{aligned}$$

and sis minimised when $\mathbf{p}$ is aligned with $\mathbf{E}$.

2.2.5 Multipole Expansion

For a general charge distribution $\rho(\mathbf{x})$ confined to a ball $\{V : |\mathbf{x}| < R\}$,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \mathrm{d}^3\mathbf{x}'$$

External potential ($|\mathbf{x}| > R$) – expand

$$\begin{aligned} \frac{1}{|\mathbf{x} - \mathbf{x}'|} &= \frac{1}{r} - (\mathbf{x}' \cdot \nabla)\frac{1}{r} + \frac{1}{2}(\mathbf{x}' \cdot \nabla)^2 \frac{1}{r} + O(|\mathbf{x}'|^3) \\ &= \frac{1}{r}\left[1 + \frac{\mathbf{x}' \cdot \mathbf{x}}{r^2} + \frac{3(\mathbf{x}' \cdot \mathbf{x})^2 - |\mathbf{x}'|^2|\mathbf{x}|^2}{2r^4} + O\left(\frac{R^3}{r^3}\right)\right] \end{aligned}$$

Leads to the *multipole expansion* of the potential (see Example Sheet 1, Question 11)

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} + \frac{1}{2}\frac{Q_{ij}x_ix_j}{r^5} + \cdots\right)$$

First three multipole moments:

- total charge (monopole - scalar) $Q = \int_V \rho(\mathbf{x})\mathrm{d}^3\mathbf{x}$
- electric dipole moment (vector) $\mathbf{p} = \int_V \mathbf{x}\rho(\mathbf{x})\mathrm{d}^3\mathbf{x}$.
- electric quadripole moment (traceless symmetric 2nd order tensor)

$$Q_{ij} = \int_V (3x_ix_j - |\mathbf{x}|^2\delta_{ij})\rho(\mathbf{x})\mathrm{d}^3\mathbf{x}$$

For $r \gg R$, Φ and $\mathbf{E}$ look increasingly like those of a point charge Q, unless $Q = 0$, in which case they look like those of a point dipole, unless $\mathbf{p} = \mathbf{0}$, etc (See Example Sheet 1, Question 11).

Start of lecture 4

2.3 Electrostatic Energy

The work done against the electric force $\mathbf{F} = q\mathbf{E}$ in bringing a particle of charge q from infinity (where we assume $\Phi = 0$) to $\mathbf{x}$ is

$$\int_\infty^{\mathbf{x}} \mathbf{F}\cdot\mathrm{d}\mathbf{x} = +q\int_\infty^{\mathbf{x}} \nabla\Phi\cdot\mathrm{d}\mathbf{x} = q\Phi(\mathbf{x}).$$

Consider assembling a configuration of N point charges one by one. Particle i of charge q_i is brought from ∞ to $\mathbf{x}_i$ while the previous particles remain fixed.

- Particle 1: No work is involved: $W_1 = 0$.
- Particle 2:

$$W_2 = q_2\left(\frac{q_1}{4\pi\varepsilon_0|\mathbf{x}_1 - \mathbf{x}_2|}\right)$$

- Particle 3:

$$W_3 = q_3\left(\frac{q_1}{4\pi\varepsilon_0|\mathbf{x}_3 - \mathbf{x}_1|} + \frac{q_2}{4\pi\varepsilon_0|\mathbf{x}_3 - \mathbf{x}_2|}\right)$$

  etc.

Total work done:

$$U = \sum_{i=1}^N W_i = \sum_{i=2}^N\sum_{j=1}^{i-1}\frac{q_iq_j}{4\pi\varepsilon_0|\mathbf{x}_i - \mathbf{x}_j|}$$

Can be rewritten as

$$U = \frac{1}{2}\sum_{i=1}^N\sum_{\substack{j=1\\ j\neq i}}\frac{q_iq_j}{4\pi\varepsilon_0|\mathbf{x}_i - \mathbf{x}_j|}$$

or alternatively

$$\boxed{U = \frac{1}{2}\sum_{i=1}^{N} q_i \Phi(\mathbf{x}_i)}$$

Generalise to a continuous charge distribution $\rho(\mathbf{x})$ occupying a finite volume V:

$$U = \frac{1}{2}\int_V \rho(\mathbf{x}\Phi(\mathbf{x}\mathrm{d}^3\mathbf{x}$$

$$\boxed{U = \frac{1}{2}\int_V \rho\Phi\mathrm{d}V}$$

Using (M1) we have

$$\begin{aligned} U &= \frac{1}{2}\int_V (\varepsilon_0 \nabla \cdot \mathbf{E})\Phi\mathrm{d}V \\ &= \frac{\varepsilon_0}{2}\int_V (\nabla \cdot (\Phi\mathbf{E}) - \mathbf{E}\cdot\nabla\Phi)\mathrm{d}V \\ &= \frac{\varepsilon_0}{2}\int_S \Phi\mathbf{E}\cdot\mathrm{d}\mathbf{S} + \int_V \frac{\varepsilon_0|\mathbf{E}|^2}{2}\mathrm{d}V \end{aligned}$$

Let $S = \partial V$ be a sphere of radius $R \to \infty$. Then $\Phi = O(R^{-1})$ and $\mathbf{E} = O(R^{-2})$ on S, while the area of S is $O(R^2)$, so $\int_S$ is $O(R^{-1})$ and $\to 0$ as $R \to \infty$. Thus

$$\boxed{U = \int \frac{\varepsilon_0|\mathbf{E}|^2}{2}\mathrm{d}V}$$

integrated over all space.

This implies that *energy is stored in the electric field even in a vaccum.*

Any of the expressions for U suggests that the self-energy of a point charge is infinite. We can discuss this as it is unchanging and causes no force.

2.4 Conductors

In a *conductor* such as a metal, some charges (usually electrons) can move freely. In electrostatics we require

$$\boxed{\mathbf{E} = \mathbf{0}, \qquad \Phi = \text{constant}}$$

inside a conductor, hence $\rho = 0$. Otherwise free charges would move in response to the electric force and a current would flow.

A surface charge density σ can exist on the surface of a conductor, which is an equipotential.

Taking **n** to point out of the conductor, the condition

$$[\mathbf{n} \cdot \mathbf{E}] = \frac{\sigma}{\varepsilon_0}$$

implies

$$\boxed{\mathbf{n} \cdot \mathbf{E} = \frac{\sigma}{\varepsilon_0}}$$

immediately outside the conductor.

The constant potential of a conductor can be set by connecting it to a battery or another conductor. An *earthed* (or *grounded*) conductor is connected to the ground, usually taken as $\Phi = 0$.

To find $\Phi(\mathbf{x})$ and $\mathbf{E}(\mathbf{x})$ due to a charge distribution $\rho(\mathbf{x})$ in the presence of conductors with surfaces S_i and potentials Φ_i, we solve Poisson's equation

$$-\nabla^2 \Phi = \frac{\rho}{\varepsilon_0}$$

with Dirichlet boundary conditions

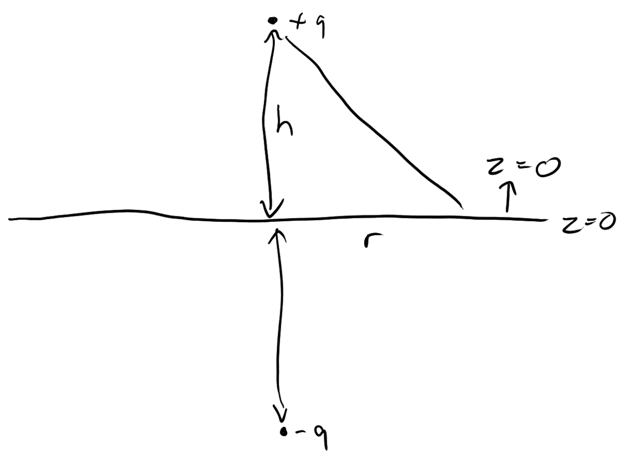
$$\Phi = \Phi_i \qquad \text{on} \qquad S_i.$$

The solution depends linearly on ρ and $\{\Phi_i\}$.

Example

A point charge q at position $(0, 0, h)$ in a half-space ($z > 0$) bounded by an earthed conducting wall ($\Phi = 0$ on $z = 0$).

By the method of images, the solution in $z > 0$ is identical to that of a dipole, with image charge $-q$ at $(0, 0, -h)$.



The wall coincides with an equipotential of the dipole. The induced surface charge density on the wall can be worked out from

$$\frac{\sigma}{\varepsilon_0} = \mathbf{n} \cdot \mathbf{E} = E_z = -\frac{2qh}{4\pi\varepsilon_0(r^2 + h^2)^{3/2}}$$

($r = \sqrt{x^2 + y^2}$). The total induced surface charge is

$$\begin{aligned} \int_0^\infty \sigma 2\pi r \mathrm{d}r &= -qh \int_0^\infty \frac{r\mathrm{d}r}{(r^2 + h^2)^{3/2}} \\ &= -q \end{aligned}$$

(equal to the image charge).

Capacitors

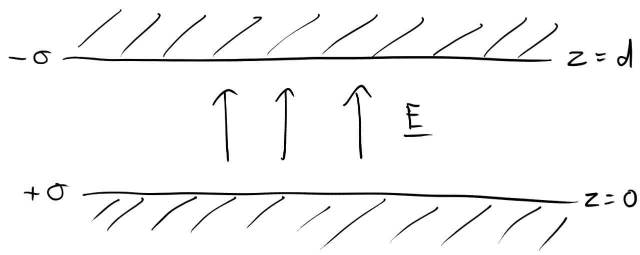
A simple *capacitor* consists of two separated conductors carrying charges $\pm Q$.

If the potential difference (voltage) between them is V, then the capacitance is defined by

$$\boxed{C = \frac{Q}{V}}$$

and depends only on the geometry, because Φ depends on linearly on Q.

For example, two infinite parallel plates separated by distance d. Let the plate surfaces at $z = 0, d$ have surface charge densities $\pm\sigma$. Then $\mathbf{E} = E\mathbf{e}_z$ with $E = \sigma/\varepsilon_0 = \text{const}$ for $0 < z < d$ (and $\mathbf{E} = \mathbf{0}$ elsewhere). $\Phi = -Ez + \text{const}$ and $V = Ed$.



The same solution holds approximately for parallel plates of area $A \gg d^2$ if end-effects are neglected. So

$$C = \frac{Q}{V} \approx \frac{\sigma A}{Ed} \approx \frac{\varepsilon_0 A}{d}$$

Electrostatic energy stored in the capacitor:

$$U = \int \frac{\varepsilon_0 |\mathbf{E}|^2}{2} \mathrm{d}V \approx \frac{\varepsilon_0 E^2}{2} Ad \approx \frac{1}{2} CV^2$$

In general

$$\boxed{U = \frac{1}{2}CV^2 = \frac{Q^2}{2C}}$$

The work done in moving an element of charge δQ from one plate to another is $\delta W = V\delta Q$, so the total work done is

$$\int_0^Q \frac{Q'}{C}\mathrm{d}Q' = \frac{Q^2}{2C}$$

Or use

$$\begin{aligned} U &= \frac{1}{2}\int \rho\Phi \mathrm{d}V \\ &= \frac{1}{2}Q\Phi_+ - \frac{1}{2}Q\Phi_- \\ &= \frac{1}{2}QV \end{aligned}$$

Start of lecture 5

3 Magnetostatics

Magnetostatics is the study of the magnetic field generated by a stationary current distribution.

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{M4'}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{M2}$$

(M4') $\implies \nabla \cdot \mathbf{J} = 0$, the time-independent equation of charge conservation.

3.1 Ampère's Law

Consider a closed curve C that is the boundary of an open surface S. Integrate (M4') over S and apply Stoke's theorem to obtain Ampère's law:

$$\boxed{\int_C \mathbf{B} \cdot \mathrm{d}\mathbf{x} = \mu_0 I}$$

where $I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}$ is the total current through S.

Since $\nabla \cdot \mathbf{J} = 0$, the same current I flows through *any* open surface S such that $\partial S = C$. Ampère's law is the integral version of (M4') and is valid provided that $\frac{\partial \mathbf{E}}{\partial t} = \mathbf{0}$.

circulation of magnetic field around loop $\propto$ total current through loop

In special situations we can use Ampère's law together with symmetry to deduce $\mathbf{B}$ from $\mathbf{J}$. A cylindrically symmetric situation could involve

- An axial current distribution

$$J_z(r)\mathbf{e}_z \qquad (r, \phi, z)$$

- an azimuthal current distribution

$$J_\phi(r)\mathbf{e}_\phi$$

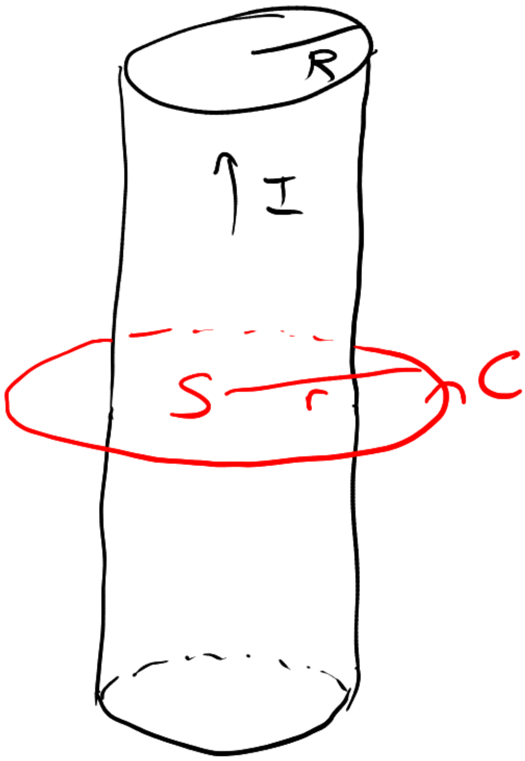
or a combination. ($\nabla \cdot \mathbf{J} = 0$ excludes a radial current).

The same applies to $\mathbf{B}$. The curl in (M4') implies:

- B_ϕ is linearly related to J_z
- B_z is linearly related to J_ϕ

Long straight wire

A cylindrical wire of radius R carries a total current I parallel to its axis. To find $B_\phi(r)$ generated by $J_z(r)$, apply Ampère's law to a circle C of radius r (S is a disc).



If $r > R$ then

$$\begin{aligned}\int_C \mathbf{B}\cdot \mathrm{d}\mathbf{x} &= B_\phi(r)\int_C \mathbf{e}_\phi \cdot \mathrm{d}\mathbf{x}\\ &= B_\phi(r)\int_C \mathrm{d}l\\ &= B_\phi(r)2\pi r\\ &= \mu_0 I\end{aligned}$$

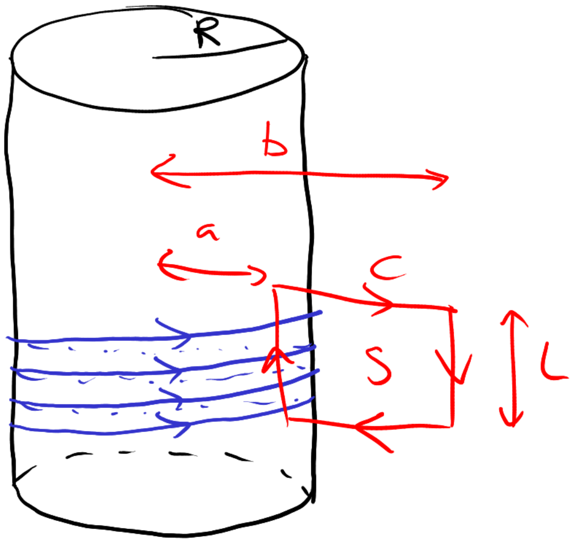
Thus, outside the wire,

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\mathbf{e}_\phi$$

3.2 Solenoid

A thin wire is coiled around a cylindrical tube of radius R. An *ideal solenoid* is infinitely long and tightly wound, having cylindrical symmetry and purely azimuthal current.

The wire carries current I and has N turns per unit length of the tube.



To find $B_z(r)$ generated by $J_\phi(r)$, apply Ampère's law to a rectangular loop C. Taking $a < b < R$ or $R < a < b$ gives

$$L(B_z(a) - B_z(b)) = 0$$

Taking $a < R < b$ gives

$$L(B_z(a) - B_z(b)) = \mu_0 NLI$$

Assuming that $B_z(r) \to 0$ as $r \to \infty$, we deduce that

$$B_z(r) \begin{cases} \mu_0 NLI & r < R \\ 0 & r > R \end{cases}$$

Surface current

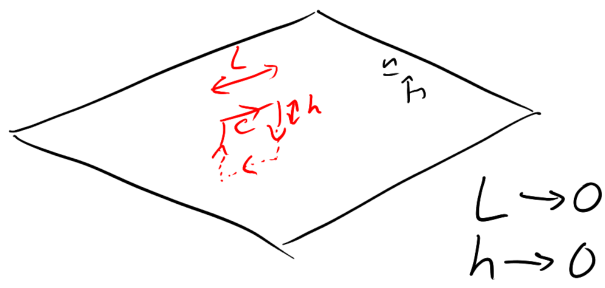
The ideal solenoid is an example of a *surface current*, here is of the form

$$J_\phi(r) = K_\phi \delta(r - R)$$

with $K_\phi = NI$. Generally, a *surface current density* $\mathbf{K}$ produces a discontinuity in the tangential magnetic field:

$$[\mathbf{n} \times \mathbf{B}] = \mu_0 \mathbf{K}$$

Follows from Ampère's law applied to



(M2) implies normal component is continuous

$$[\mathbf{n} \cdot \mathbf{B}] = 0.$$

3.3 The magnetic vector potential

(M2) implies that $\mathbf{B}$ can be written in terms of a *magnetic vector potential* $\mathbf{A}(\mathbf{X})$:

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$\mathbf{A}$ is not unique. If we make a *gauge transformation*, replacing $\mathbf{A}$ with

$$\mathbf{A}' = \mathbf{A} + \nabla\chi$$

where $\chi(\mathbf{x})$ is an arbitrary scalar field, then $\mathbf{B}$ is unchanged:

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \mathbf{A}'.$$

A convenient gauge for many calculations is the *Coulomb gauge* in which $\nabla \cdot \mathbf{A} = 0$. We can assume this condition without loss of generality. If $\nabla \cdot \mathbf{A} \neq 0$ then we can make a gauge transformation such that $\nabla \cdot \mathbf{A}' = 0$ by choosing χ to be the solution of Poisson's equation

$$-\nabla^2\chi = \nabla \cdot \mathbf{A}$$

In terms of $\mathbf{A}$, (M4') becomes

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}$$

Using the identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

and assuming Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$) we obtain Poisson's equation in vector form:

$$\boxed{-\nabla^2\mathbf{A} = \mu_0 \mathbf{J}}$$

(compare with $-\nabla^2\Phi = \frac{\rho}{\varepsilon_0}$)

3.4 The Biot-Savat Law

The solution of Poisson's equation is (integrating over al space, assuming decay at infinity)

$$\boxed{\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \mathrm{d}^3\mathbf{x}'}$$

We should check that the solution satisfies the assumed Coulomb gauge condition:

$$\begin{aligned}
\nabla \cdot \mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi} \int_V \nabla \cdot \left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \right) \mathrm{d}^3\mathbf{x}' \\
&= \frac{\mu_0}{4\pi} \int_V \mathbf{J}(\mathbf{x}') \cdot \nabla \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right) \mathrm{d}^3\mathbf{x}' \\
&= -\frac{\mu_0}{4\pi} \int_V \mathbf{J}(\mathbf{x}') \cdot \nabla' \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right) \mathrm{d}^3\mathbf{x}' \\
&= -\frac{\mu_0}{4\pi} \int_V \nabla' \cdot \left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \right) \mathrm{d}^3\mathbf{x}' \qquad \text{using } \nabla \cdot \mathbf{J} = 0 \\
&= -\frac{\mu_0}{4\pi} \int_{\partial V} \frac{\mathbf{J}(\mathbf{x}') \cdot \mathrm{d}\mathbf{S}'}{|\mathbf{x}-\mathbf{x}'|}
\end{aligned}$$

Thus $\nabla \cdot \mathbf{A} = 0$, as assumed, if the current is contained in some finite volume and we take V to be at least as large, or if $\mathbf{J}$ decays sufficiently as $|\mathbf{x}| \to \infty$.

Start of lecture 6

TO find the magnetic field, derive $\mathbf{b} = \nabla \times \mathbf{A}$ from (1):

$$\boxed{\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \mathrm{d}^3\mathbf{x}'} \qquad (2)$$

This is the *Biot-Savat law*, giving the magnetic field generated by a stationary current distribution.

A special case is when the current is restricted to a thin wire in the form of a curve C. Then the current element $\mathbf{J}\mathrm{d}^3\mathbf{x}$ can be replaced by $I\mathrm{d}\mathbf{x}$ (I is the current in the wire) ($\mathrm{d}\mathbf{x}$ is the vector line element parallel to wire). Charge conservation means that $I = \text{const}$ along the wire. The result is

$$\boxed{\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int_C \frac{\mathrm{d}\mathbf{x}' \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}} \qquad (3)$$

The thin-wire current density can be represented as

$$\mathbf{J}(\mathbf{x}) = I \int_C \delta(\mathbf{x}-\mathbf{x}')\mathrm{d}\mathbf{x}'$$

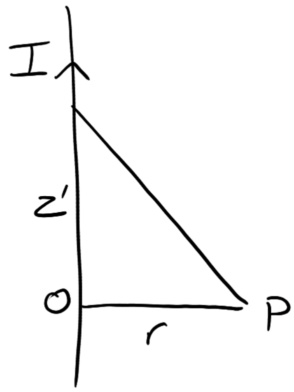
($I = \text{const}$), which gives (3) when substituted in (2) (exercise).

Charge conservation takes the form

$$\begin{aligned}\nabla \cdot \mathbf{J}(\mathbf{x}) &= I \int_C \nabla \delta(\mathbf{x} - \mathbf{x}') \cdot \mathrm{d}\mathbf{x}' \\ &= -I \int_C \nabla' \delta(\mathbf{x} - \mathbf{x}') \cdot \mathrm{d}\mathbf{x}' \\ &= -I[\delta(\mathbf{x} - \mathbf{x}_2) - \delta(\mathbf{x} - \mathbf{x}_1)]\end{aligned}$$

where C runs from $\mathbf{x}_1$ to $\mathbf{x}_2$. If C is closed then $\mathbf{x}_2 = \mathbf{x}_1$, and $\nabla \cdot \mathbf{J} = 0$ as expected. If C is infinite (for example long straight wire) then $\nabla \cdot \mathbf{J} = 0$ for any finite $\mathbf{x}$.

Check that (3) gives the same result as Ampère's law for a long straight thin wire (along the z axis of cylindrical polars).



We have $\mathbf{x} = r\mathbf{e}_r$ (taking $z = 0$ WLOG) and $\mathbf{x}' = z'\mathbf{e}_z$, so

$$\mathbf{x} - \mathbf{x}' = r\mathbf{e}_r - z'\mathbf{e}_z \qquad \text{and} \qquad \mathrm{d}\mathbf{x}' = \mathrm{d}z'\mathbf{e}_z$$

giving

$$\begin{aligned}\mathbf{B}(\mathbf{x}) &= \frac{\mu_0 I}{4\pi}\mathbf{e}_\phi \int_{-\infty}^{\infty} \frac{r\mathrm{d}z'}{(r^2 + z^2)^{3/2}} \\ &= \frac{\mu_0 I}{4\pi}\mathbf{e}_\phi \left[\frac{z'}{r(r^2 + z^2)^{1/2}}\right]_{-\infty}^{\infty} \\ &= \frac{\mu_0 I}{2\pi r}\mathbf{e}_\phi\end{aligned}$$

as expected.

3.5 Magnetic dipoles

For a general current distribution $\mathbf{J}(\mathbf{x})$ confined to a ball $\{V : |\mathbf{x}| < R\}$,

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \mathrm{d}^3\mathbf{x}'$$

The external field for $|\mathbf{x}| = r > R$ can be evaluated by expanding

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r}\left(1 + \frac{\mathbf{x}'\cdot\mathbf{x}}{r^2} + O\left(\frac{R^2}{r^2}\right)\right),$$

leading to a multipole expansion as before. We need to calculate the moments of the current distribution. Since $\mathbf{J} = \mathbf{0}$ on ∂V and $\nabla\cdot\mathbf{J} = 0$, the divergence theorem implies

$$\begin{aligned} 0 &= \int_{\partial V} x_i J_j \mathrm{d}S_j \\ &= \int_V \partial_j (x_i J_j)\mathrm{d}^3\mathbf{x} \\ &= \int_V (\delta_{ij} + x_i \partial_j J_j)\mathrm{d}^3\mathbf{x} \\ &= \int_V J_i \mathrm{d}^3\mathbf{x} \end{aligned}$$

so the moment vanishes. Similarly,

$$\begin{aligned} 0 &= \int_{\partial V} x_i x_j J_k \mathrm{d}S_k \\ &= \int_V \partial_k (x_i x_j J_k)\mathrm{d}^3\mathbf{x} \\ &= \int_V (\delta_{ik} x_j J_k + x_i \delta_{jk} J_k + x_i x_j \partial_k J_k \mathrm{d}^3\mathbf{x} \\ &= \int_V x_j J_i \mathrm{d}^3\mathbf{x} + \int_V x_i J_j \mathrm{d}^3\mathbf{x} \end{aligned}$$

so the first moment is an antisymetric matrix.

The *magnetic dipole moment*

$$\boxed{\mathbf{m} = \frac{1}{2}\int_V \mathbf{x} \times BFJ\mathrm{d}^3\mathbf{x}}$$

$$m_i = \frac{1}{2}\varepsilon_{ijk}\int_V x_j J_k \mathrm{d}^3\mathbf{x}$$

is a vector related to the antisymmetric matrix by

$$\int_V x_i J_j \mathrm{d}^3\mathbf{x} = \varepsilon_{ijk} m_k$$

Returning to the multipole expansion for $\mathbf{A}$, we have

$$\begin{aligned} A_i(\mathbf{x}) &= \frac{\mu_0}{4\pi|\mathbf{x}|}\left(\int_V J_i(\mathbf{x}')\mathrm{d}^3\mathbf{x}' + \frac{x_j}{|\mathbf{x}|^3}\int x'_j J_i(\mathbf{x}')\mathrm{d}^3\mathbf{x}' + \cdots\right) \\ &= \frac{\mu_0}{4\pi|\mathbf{x}|}\left(0 + \frac{x_j \varepsilon_{jik} m_k}{|\mathbf{x}|^2} + \cdots\right) \end{aligned}$$

The leading approximateion is therefore

$$\mathbf{A}(\mathbf{x}) \approx \mathbf{A}_{\text{dipole}}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3}$$

which is the vector potential due to a (point) dipole $\mathbf{m}$ at the origin. The corresponding magnetic field is (exercise)

$$\mathbf{B}_{\text{dipole}} = \nabla\times\mathbf{A}_{\text{dipole}} = \frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m}\cdot\mathbf{x})\mathbf{x} - |\mathbf{x}|^2\mathbf{m}}{|\mathbf{x}|^5}\right)$$

A point dipole $\mathbf{m}$ at the origin corresponds to the current density and vector potential

$$\mathbf{J} = \nabla\times(\mathbf{m}\delta(\mathbf{x})), \qquad \mathbf{A} = \nabla\times\left(\frac{\mu_0\mathbf{m}}{4\pi|\mathbf{x}|}\right)$$

The magnetic dipole moment of a thin wire carrying current I around a closed curve C is

$$\mathbf{m} = \frac{I}{2}\int_C \mathbf{x}\times \mathrm{d}\mathbf{x}$$

To evaluate this, let $\mathbf{a}$ be any constant vector. Then, by Stoke's Theorem,

$$\begin{aligned}\mathbf{a}\cdot\mathbf{m} &= \frac{I}{2}\int_C \mathbf{a}\cdot(\mathbf{x}\times\mathrm{d}\mathbf{x})\\ &= \frac{I}{2}\int_C (\mathbf{a}\times\mathbf{x})\cdot\mathrm{d}\mathbf{x}\\ &= \frac{I}{2}\int_S (\nabla\times(\mathbf{x}\times\mathbf{x}))\cdot\mathrm{d}\mathbf{S}\\ &= I\int_S \mathbf{a}\cdot\mathrm{d}\mathbf{S}\end{aligned}$$

where S is an open surface with boundary C and we use

$$\begin{aligned}\nabla\times(\mathbf{a}\times\mathbf{x}) &= \mathbf{x}\cdot\nabla\mathbf{a} - \mathbf{a}\cdot\nabla\mathbf{x} + (\nabla\cdot\mathbf{x})\mathbf{a} - (\nabla\cdot\mathbf{a})\mathbf{x}\\ &= \mathbf{0} - \mathbf{a} + 3\mathbf{a} - \mathbf{0}\\ &= 2\mathbf{a}\end{aligned}$$

Since $\mathbf{a}$ is arbitrary, we have

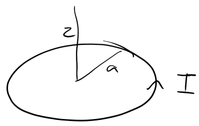
$$\boxed{\mathbf{m} = I\mathbf{S}}$$

where $\mathbf{S} = \int_S \mathrm{d}\mathbf{S}$ is the vector area of S (which depends only on C, not on the choice of S). Simplest example: circular loop, for example $x^2 + y^2 = a^2$, $z = 0$, for which $\mathbf{m} = I\pi a^2\mathbf{e}_z$. On the z-axis, the dipole approximately gives

$$B_z = \frac{\mu_0}{4\pi}\left(\frac{3m_z z^2 - z^2 m_z}{|z|^5}\right) = \frac{\mu_0 I a^2}{2|z|^3}$$

while the exact solution (Example Sheet 2, Question 3) is

$$B_z = \frac{\mu_0 I a^2}{2(z^2 + a^2)^{3/2}}$$



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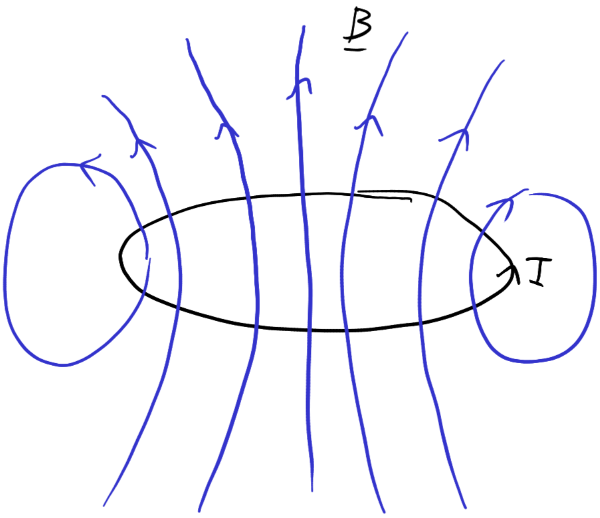
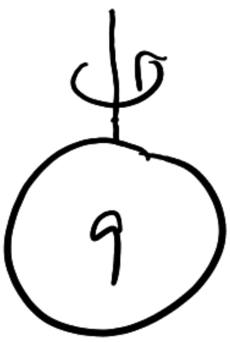


Magnetic field lines are the integral curves of $\mathbf{B}$. Since $\nabla \cdot \mathbf{B} = \mathbf{0}$, they are continuous.

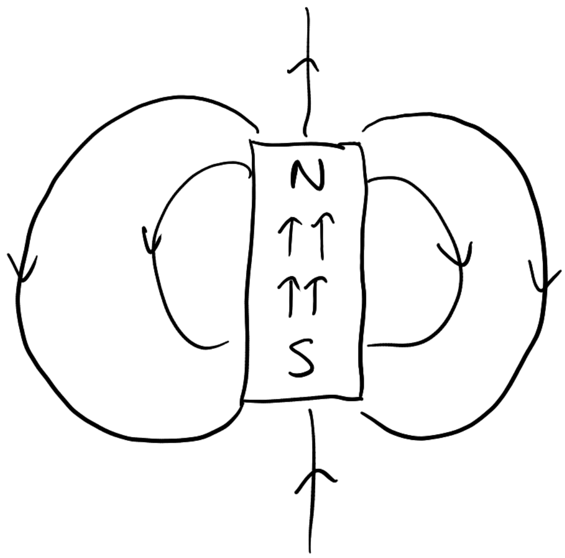
Permanent Magnets

A bar magnet has north and south poles and a dipole moment. THis comes from the superposition of aligned dipoles on the atomic scale. Atoms contain electrons, which are spinning charged particles, with a magnetic dipole moment.

A classical model of a particle is a spinning charged sphere

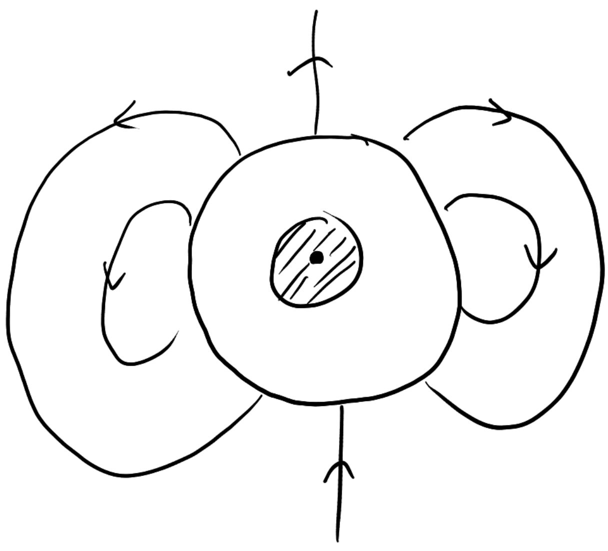


which is a current loop with a magnetic dipole moment proportional to its charge and spin. As far as we know, there are no magnetic charges (monopoles).



The Earth as a Magnet

The liquid iron outer core of the Earth is a conducting fluid in convective motion and supports electric currents that generate a magnetic field. At the Earth's structure, this resembles a dipole field.



3.6 Magnetic Forces

The Lorentz force on a particle of charge q_i at position $\mathbf{x}_i(t)$ is

$$q(\mathbf{E} + \dot{\mathbf{x}}_i \times \mathbf{B}$$

($\mathbf{E}$, $\mathbf{B}$ evaluated at $\mathbf{x}_i(t)$). In the limit of continuous charge and current densities, the Lorentz force per unit volume is then

$$\boxed{\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}}$$

We can recover the discrete version by substituting

$$\rho = \sum_i q_i \delta(\mathbf{x} - \mathbf{x}_i(t))$$
$$\mathbf{J} = \sum_i q_i \dot{\mathbf{x}}_i(t) \delta(\mathbf{x} - \mathbf{x}_i(t))$$

Force between thin wires

Consider two or more thin wires with currents I_i along curves C_i. The total magnetic field is $\mathbf{B} = \sum_i \mathbf{B}$, where

$$\mathbf{B}_i(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int_{C_i} \frac{\mathrm{d}\mathbf{x}_i \times (\mathbf{x} - \mathbf{x}_i)}{|\mathbf{x} - \mathbf{x}_i|^3}$$

is the magnetic field due to wire i. The current density is $\mathbf{J} = \sum_i \mathbf{J}_i$, where

$$\mathbf{J}_i(\mathbf{x}) = I_i \int_{C_i} \delta(\mathbf{x} - \mathbf{x}_i) \mathrm{d}\mathbf{x}_i$$

The total magnetic force acting on a volume V is

$$\mathbf{F} = \int_V \mathbf{J} \times \mathbf{B} \mathrm{d}V$$

The force acting on a volume V is

$$\mathbf{F} = \int_V \mathbf{J} \times \mathbf{B} \mathrm{d}V.$$

The force acting on wire i is

$$\mathbf{F}_i = \int \mathbf{J}_i(\mathbf{x}) \times \mathbf{B}(\mathbf{X}) \mathrm{d}^3\mathbf{x}$$
$$= I_i \int_{C_i} \mathrm{d}\mathbf{x}_i \times \mathbf{B}(\mathbf{x}_i)$$

Since $\mathbf{B} = \sum_i \mathbf{B}_i$, we have

$$\mathbf{F}_i = \sum_j \mathbf{F}_{ij},$$

where

$$\mathbf{F}_{ij} = I_i \int_{C_i} \mathrm{d}\mathbf{x}_i \times \mathbf{B}_j(\mathbf{x}_i)$$

is the force on wire i due to wire j. Using the Biot-Savat Law,

$$\mathbf{F}_{ij} = \frac{\mu_0 I_i I_j}{4\pi} \int_{C_i} \int_{C_j} \mathrm{d}\mathbf{x}_i \times \left(\frac{\mathrm{d}\mathbf{x}_j \times (\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^3} \right)$$

This can be rewritten (see Example 2.4) in a manifestly antisymmetric way that shows that

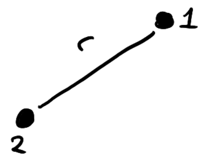
$$\mathbf{F}_{ji} = -\mathbf{F}_{ij}$$

as expected from Newton's third law.

The self-force $\mathbf{F}_{ii}$ vanishes, although the thin-wire integral is singular and it is better to treat the case of thick wires.

Long parallel wires

Consider two infinitely long, parallel, thin wires separated by a distance r. Use cylindrical polars centred on wire 2.



We have $\mathbf{B}_2 = \frac{\mu_0 I_2}{2\pi r}\mathbf{e}_\phi$,

$$\mathbf{F}_{12} = I_1 \int_{-\infty}^{\infty} \mathrm{d}z \mathbf{e}_z \times \mathbf{B}_2$$

Total force is infinite. Force per unit length is

$$I_1 \mathbf{e}_z \times \mathbf{B}_2 = -\frac{\mu_0 I_1 I_2}{2\pi r}\mathbf{e}_r$$

This is directed towards wire 2 if $I_1 I_2 > 0$. So the force is attractive if the currents are aligned and repulsive otherwise.

Force and torque on a magnetic dipole

Consider a localized current distribution (current loop) confined to a ball $\{V : |\mathbf{x}| < R\}$. Place this in an external magnetic field $\mathbf{B}(\mathbf{x})$ that varies slowly over the length scale R.

The magnetic torque (about the origin) on the current loop is

$$\begin{aligned} I &= \int_V \mathbf{x} \times (\mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}))\mathrm{d}^3\mathbf{x} \\ &= \int_V ((\mathbf{x} \cdot \mathbf{B}(\mathbf{x}))\mathbf{J}(\mathbf{x}) - (\mathbf{x} \cdot \mathbf{J}(\mathbf{x}))\mathbf{B}(\mathbf{x}))\mathrm{d}^3\mathbf{x} \end{aligned}$$

(here $\mathbf{B}$ is external field – self-torque of loop vanishes). Within V, $\mathbf{B}(\mathbf{x})$ can be expanded sa a Taylor series

$$B_i(\mathbf{x}) = B_i(\mathbf{0}) + x_j \partial_j B_i(\mathbf{0}) + \cdots$$

Retaining only the zeroth-order term (uniform field), we have

$$T_i \approx B_j(\mathbf{0}) \in_V x_j J_i \mathrm{d}^3\mathbf{x} - B_i(\mathbf{0}) \int_V x_j J_j \mathrm{d}^3\mathbf{x}$$

Recall the first moments of the current distribution:

$$\int_V x_i J_j \mathrm{d}^3\mathbf{x} = \varepsilon_{ijk} m_k$$

Thus $T_i \approx B_j(\mathbf{0})\varepsilon_{jik} m_k$. In general,

$$\boldsymbol{\tau} \approx \mathbf{m} \times \mathbf{B}$$

where $\mathbf{B}$ is evaluated at the dipole's location, and $\boldsymbol{\tau}$ is measured about this point. For the force, we need to go to the first order of the Taylor expansion of $\mathbf{B}$:

$$\begin{aligned} \mathbf{F} &= \int_V \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})\mathrm{d}^3\mathbf{x} \\ F_i &= \int_v \varepsilon_{ijk} J_j(\mathbf{x})(B_j(\mathbf{0}) + x_l \partial_l B_k(\mathbf{0}))\mathrm{d}^3\mathbf{x} \\ &= \varepsilon_{ijk} B_k(\mathbf{0}) \int_V J_j \mathrm{d}^3\mathbf{x} + \varepsilon_{ijk} \partial_l B_k(\mathbf{0}) \int_V x_l J_j \mathrm{d}^3\mathbf{x} \\ &= 0 + \varepsilon_{ijk} \partial_l B_k(\mathbf{0}) \varepsilon_{ljn} m_n \\ &= \partial_i B_k(\mathbf{0} m_k - \partial_k B_k(\mathbf{0}) m_i \\ &= \partial_i (m_k B_k)(\mathbf{0}) \end{aligned}$$

since $\nabla \cdot \mathbf{B} = 0$. In general, $\mathbf{F} \approx \nabla(\mathbf{m} \cdot \mathbf{B})$. Can also be written as $\mathbf{F} = -\nabla U$, where

$$\boxed{U = -\mathbf{m} \cdot \mathbf{B}}$$

is the potential energy of a magnetic dipole in an external field.

As in the electric case, this is minimised when $\mathbf{m}$ is aligned with $\mathbf{B}$.

Start of lecture 8

4 Electrodynamics

4.1 Faraday's law of induction

Maxwell's equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{M3}$$

implies that a time-dependent magnetic field must be accompanied by an electric field. This can induce a current to flow in a conductor – a process known as *electromagnetic induction.*

Faraday's law for a static circuit

Consider a closed curve C that is the boundary of a time-independent open surface S. Integrate (M3) over S and use Stoke's theorem:

$$\int_C \mathbf{E} \cdot \mathrm{d}\mathbf{x} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}$$

This is *Faraday's law of induction* for a static circuit:

$$\boxed{\mathcal{E} = -\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t}}$$

where

$$\boxed{\mathcal{E} = \int_C \mathbf{E} \cdot \mathrm{d}\mathbf{x}}$$

is the *electromotive force* (emf) around C and

$$\boxed{\mathcal{F} = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}}$$

is the *magnetic flux* through S.

Since $\nabla \cdot \mathbf{B} = 0$, the flux $\mathcal{F}$ is the same for any S such that $\partial S = C$, so it can be regarded as the magnetic flux through C.

Using $\mathbf{B} = \nabla \times \mathbf{A}$ and Stoke's theorem, we can write

$$\int_C \mathbf{A} \cdot \mathrm{d}\mathbf{x},$$

which is invariant under a gauge transformation $\mathbf{A}' = \mathbf{A} + \nabla\chi$.

The emf is not actuallly a force. It is the line integral of the Lorentz force on a particle of unit charge confined to C:

$$\mathcal{E} = \frac{1}{q} \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_C (\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{x} = \int_C \mathbf{E} \cdot \mathrm{d}\mathbf{x},$$

since $\dot{\mathbf{x}}$ is tangent to C for a particle confined to a t-independent curve C.

We will see later that if C coincides with a thin wire of resistance R, then the current induced in the wire is $I = \mathcal{E}/R$.

There are several ways in which the magnetic flux through C could change in time:

- A magnet is moved near C.
- A current-carrying circuit is moved near C.
- The current in a nearby circuit is changed.

All these will induce an emg around C and cause a current to flow.

Faraday's law for a moving circuit

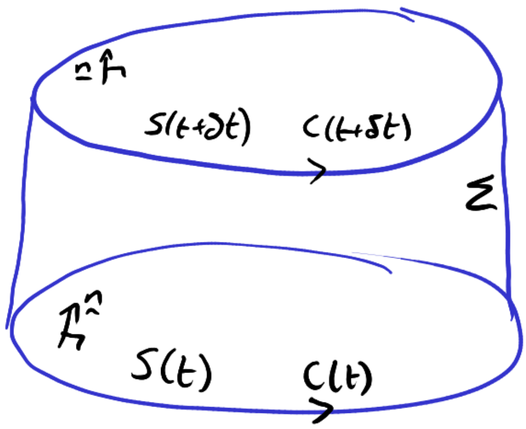
Now let $C(t)$ be a *time-dependent* closed curve that is the boundary of an open surface $S(t)$. How does the magnetic flux through S,

$$\mathcal{F} = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}$$

change in time? We have

$$\begin{aligned}
\mathcal{F}(t+\delta t) - \mathcal{F}(t) &= \int_{S(t+\delta t)} \mathbf{B}(\mathbf{x}, t+\delta t) \cdot \mathrm{d}\mathbf{S} - \int_{S(t)} \mathbf{B}(\mathbf{x}, t) \cdot \mathrm{d}\mathbf{S} \\
&= \int_{S(t+\delta t)} \left(\mathbf{B}(\mathbf{x}, t) + \frac{\partial \mathbf{B}}{\partial t} \delta t + O(\delta t^2) \right) \cdot \mathrm{d}\mathbf{S} - \int_{S(t)} \mathbf{B}(\mathbf{x}, t) \cdot \mathrm{d}\mathbf{S} \\
&= \int_{S(t+\delta t) - S(t)} \mathbf{B}(\mathbf{x}, t) \cdot \mathrm{d}\mathbf{S} + \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S} \delta t + O(\delta t^2)
\end{aligned}$$

Let ∂V be the volume swept out by $S(t)$ in the time interval δt. Its boundary is the closed surface $S(t+\delta t) - S(t) + \Sigma$, where Σ is the surface swept out by $C(t)$ in time δt.



By (M2) and the divergence theorem,

$$
\begin{aligned}
0 &= \int_{\partial V} (\nabla \cdot \mathbf{B}) \mathrm{d}V \\
&= \int_{S(t+\delta t)-S(t)} \mathbf{B} \cdot \mathrm{d}\mathbf{S} + \int_{\Sigma} \mathbf{B} \cdot \mathrm{d}\mathbf{S}
\end{aligned}
$$

To evaluate the last term, parametrise C as $\mathbf{x} = \mathbf{x}(\lambda, t)$, where λ is a parameter around C. An element of C is

$$
\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \lambda} \mathrm{d}\lambda
$$

and has velocity $\mathbf{v} = \frac{\partial \mathbf{x}}{\partial t}$. In time δt it sweeps out the vector area element

$$
\mathrm{d}\mathbf{S} = \mathrm{d}\mathbf{x} \times (\mathbf{v}\delta t)
$$

(points out of δV, as required). Thus

$$
\begin{aligned}
\int_{\Sigma} \mathbf{B} \cdot \mathrm{d}\mathbf{S} &= \int_C \mathbf{B} \cdot (\mathrm{d}\mathbf{x} \times \mathbf{v})\delta t + O(\delta t^2) \\
&= \int_C (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{x} \delta t + O(\delta t^2)
\end{aligned}
$$

We then have

$$
\mathcal{F}(t+\delta t) - \mathcal{F}(t) = -\int_C (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{x}\delta t + \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S}\delta t + O(\delta t^2)
$$

from which

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} &= -\int_C (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{x} + \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S} \\
&= -\int_C (\mathbf{v} \times \mathbf{V}) \cdot \mathrm{d}\mathbf{x} - \int_S (\nabla \times \mathbf{E}) \cdot \mathrm{d}\mathbf{S} \\
&= -\int_C (\mathbf{e} + \mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{x}
\end{aligned}
$$

We recover Faraday's law,

$$
\mathcal{E} = -\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t},
$$

with the redefined emf

$$
\mathcal{E} = \int_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{x}
$$

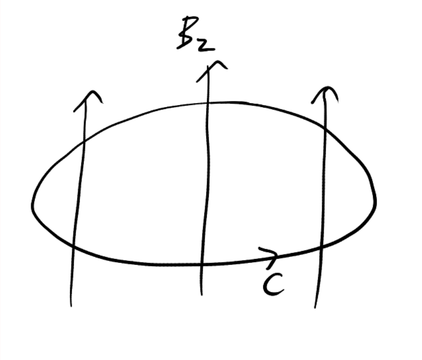
This $\mathcal{E}$ is again the line integral adound C of the Lorentz force on particle of unit charge confined to C (for which the perpendicular components of $\dot{\mathbf{x}}$ must agree with those of the curve velocity $\mathbf{v}$).

Start of lecture 9

Lenz's law

The direction of the induced current is always such as to produce a magnetic field that opposes the change in flux that caused the emf.

Example. A circular wire in the xy-plane. If B_z inside the loop increases in time then $\mathcal{E} = -\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} < 0$. This induces a clockwise current ($I < 0$) that generates a magnetic field with $B_z < 0$ inside the loop.



Hence the minus sign in Faraday's law. This avoids an unstable situation in which the flux grows indefinitely.

Inductance and magnetic energy

If a current I around a circuit C generates a magnetic field with flux $\mathcal{F}$, then the *inductance* of the circuit is defined by

$$L = \frac{\mathcal{F}}{I}$$

and depends only on the geometry.

Example. An ideal solenoid with cross-sectional area A and N turns per unit length. The uniform field $B = \mu_0 N I$ inside the solenoid produces a flux BA per turn, so the inductance per unit length of the solenoid is $\mu_0 N^2 A$.

Exercise: show that the magnetic flux through a thin wire C_i due to a current I_j around another thin wire C_j is $\mathcal{F}_{ij} = L_{ij} I_J$, where the *mutual inductance* is

$$L_{ij} = \frac{\mu_0}{4\pi} \int_{C_i} \int_{C_j} \frac{\mathrm{d}\mathbf{x}_i \cdot \mathrm{d}\mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|} = L_{ji}$$

(Hint: Use $\mathbf{A}$)

When the current I around a circuit C is varied, an emf

$$\mathcal{E} = -\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = -L\frac{\mathrm{d}I}{\mathrm{d}t}$$

is induced. In a small timer interval δt, a charge $\delta Q = I\delta t$ flows around C and the work done on it by the Lorentz force is

$$\delta W = \mathcal{E}\delta Q = -LI\frac{\mathrm{d}I}{\mathrm{d}t}\delta t$$

So the rate at which work is done *by* the current *on* the EM field is

$$-\frac{\mathrm{d}W}{\mathrm{d}t} = LI\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}LI^2)$$

Consider reaching a magnetostatic state by building up the current from 0 to I. The energy stored is

$$\begin{aligned} U &= \frac{1}{2}LI^2 \\ &= \frac{1}{2}I\mathcal{F} \\ &= I\int_C \mathbf{A}\cdot \mathrm{d}\mathbf{x} \\ &= \frac{1}{2}\int \mathbf{J}\cdot\mathbf{A}\mathrm{d}V \end{aligned}$$

analogous to $U = \frac{1}{2}\int \rho\Phi\mathrm{d}V$ in electrostatics.

Now, using $(M3')$ we have

$$U = \frac{1}{2\mu_0}\int(\nabla\times\mathbf{B})\cdot\mathbf{A}\ ddV$$

and $(\nabla\times\mathbf{B})\cdot\mathbf{A} = \nabla\cdot(\mathbf{B}\times\mathbf{A}) + \mathbf{b}\cdot(\nabla\times\mathbf{A})$. If we take the integral over all space then the first term gives zero by the divergence theorem, since

$$|\mathbf{B}| = O\left(\frac{1}{|\mathbf{x}|^3}\right) \qquad \text{and} \qquad |\mathbf{A}| = O\left(\frac{1}{|\mathbf{x}|^2}\right)$$

as $|\mathbf{x}|\to\infty$ for a finite current distribution, leaving

$$\boxed{U = \int\frac{|\mathbf{B}|^2}{2\mu_0}\mathrm{d}V}$$

as the energy stored in the magnetic field.

4.2 Ohm's law

In a stationary conductor

$$\boxed{\mathbf{J} = \sigma \mathbf{E}}$$

where σ is the *electrical conductivity*. This is not a fundamental physical law but a constitutive relation, a macroscopic property of a material.

Inverse relation

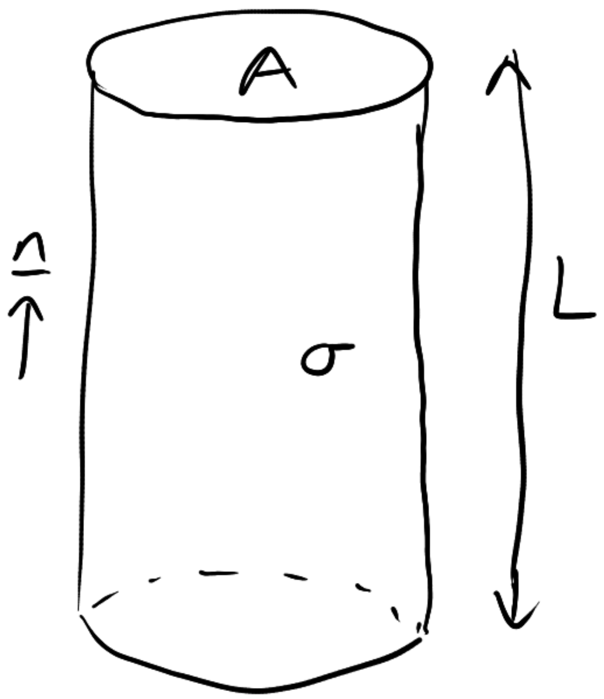
$$\mathbf{E} = \sigma^{-1} \mathbf{J}$$

where σ^{-1} is the *resistivity* (usually denoted ρ – both σ and ρ conflict with notation for charge densities).

A *perfect conductor* corresponds to the limit $\sigma \to \infty$ (so $\mathbf{E} = \mathbf{0}$) and a *perfect insulator* to $\sigma \to 0$ (so $\mathbf{J} = \mathbf{0}$).

Consider a straight wire of length L in the direction of the unit vector $\mathbf{n}$ and with uniform cross-sectional area A and conductivity σ. If the electric field is $\mathbf{E} = E\mathbf{n}$ where $E = \text{const}$ then $\mathbf{J} = \sigma E \mathbf{n}$ and the total current is $I = \sigma E A$. The potential difference (voltage) along the wire is

$$V = \int \mathbf{E} \cdot \mathrm{d}\mathbf{x} = EL = \frac{IL}{\sigma A}$$



$$\boxed{V = IR}$$

where $R = \frac{L}{\sigma A}$ is the *resistance* of the wire.

Accompanying the resistance of a wire is *Joule heating* (or *Ohmic heating*): conversion of EM energy into heat at the rate I^2R. If the voltage V is maintained by a batter, then $VI = I^2R$ is the rate at which the emf of the battery ($\mathcal{E} = V$) does work to maintain the I.

4.3 Time-dependent electric fields

Scalar and vector potentials

In electrodynamics we can no longer write $\mathbf{E} = -\nabla\Phi$. But (M2) still allows us to write

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

and (M3) then gives

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = \mathbf{0}$$

allowing us to write

$$\boxed{\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}}$$

generalising the electrostatic expression. Under a time-dependent gauge transformation

$$\mathbf{A}' = \mathbf{A} + \nabla\chi, \qquad \Phi' = \Phi - \frac{\partial \chi}{\partial t}$$

where $\chi(\mathbf{x}, t)$ is any scalar field, then both $\mathbf{E}$ and $\mathbf{B}$ are unchanged.

The displacement current

In magnetostatics we used Ampère's law

$$\int_C \mathbf{B} \cdot \mathrm{d}\mathbf{x} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = \mu_0 I$$

or its differential form (M4')

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

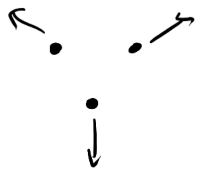
For t-dependent situations (M4)

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

contains an extra term, the *displacement current.*

Why is it needed? Without it, we would have $\nabla \cdot \mathbf{J} = 0$, which describes charge conservation in a situation where ρ is constrained to remain constant.

But suppose we place free particles of positive charge in some locallised region. Repulsice Coulomb forces cause the particles to separate, implying $\nabla \cdot \mathbf{J} > 0$.



We have seen that charge conservation in the correct form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

follows from Maxwell's equations including the displacement current.

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4.4 Electromagnetic waves

The Wave Equation

Consider freely evolving electric and magnetic fields in a vaccuum, in the absence of charges and currents:

$$\nabla \cdot \mathbf{E} = 0 \tag{M1}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{M2}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{M3}$$
$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{M4}$$

Eliminate $\mathbf{B}$ by taking $\frac{\partial}{\partial t}$ (M4) and substituting (M3):

$$\begin{aligned} \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} &= \frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \\ &= \nabla \times \frac{\partial \mathbf{B}}{\partial t} \\ &= -\nabla \times (\nabla \times \mathbf{E}) \\ &= \nabla^2 \mathbf{E} \end{aligned}$$

where we use the identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

and (M1).

Alternatively, eliminate $\mathbf{E}$ by taking $\frac{\partial}{\partial t}$ (M3) and substituting (M4):

$$\begin{aligned}\frac{\partial^2\mathbf{B}}{\partial t^2} &= -\frac{\partial}{\partial t}(\nabla\times\mathbf{E})\\ &= -\nabla\times\frac{\partial\mathbf{E}}{\partial t}\\ &= -\frac{1}{\mu_0\varepsilon_0}\nabla\times(\nabla\times\mathbf{B})\\ &= \frac{1}{\mu_0\varepsilon_0}\nabla^2\mathbf{B}\end{aligned}$$

using the same identity and (M2). So each (Cartesian) component of $\mathbf{E}$ and $\mathbf{B}$ satisfies the *wave equation*

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u}$$

with wave speed

$$\boxed{c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}}$$

which is the *speed of light* (in a vaccuum),

$$c = 2.997923458\times 10^8\,\mathsf{ms}^{-1}.$$

This is of course because light is an *electromagnetic wave* involving oscillations of $\mathbf{E}$ and $\mathbf{B}$. Depending on the wavelength, EM waves can be radio waves, microwaves, infrared, ultraviolet, X-rays, gamma rays, etc.

Plane Electromagnetic Waves

Consider a *plane wave* in which $\mathbf{E}$ and $\mathbf{B}$ depend only on (x,t) and not on (y,z). A simple example is

$$\mathbf{E} = E(x,t)\mathbf{e}_y$$

where $E(x,t)$ satisfies the 1D wave equation

$$\frac{\partial^2 E}{\partial t^2} = c^2\frac{\partial^2 E}{\partial t^2}$$

The general solution

$$E(x,t) = f(x-ct) + g(x+ct)$$

is the sum of a wave travelling without change of form in the $+x$ direction and another travelling in the $-x$ direction. What is the corresponding $\mathbf{B}$? We have

$$\begin{aligned}\frac{\partial\mathbf{B}}{\partial t} &= -\nabla\times\mathbf{E}\\ &= -\frac{\partial E}{\partial x}\mathbf{e}_z\\ &= (-f'(x-ct) - g'(x+ct))\mathbf{e}_z\end{aligned}$$

and so

$$\mathbf{B} = B(x,t)\mathbf{e}_z$$

with

$$B(x,t) = \frac{1}{c}(f(x-ct) - g(x+ct))$$

This also satisfies $\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ (exercise).

Of particular importance is a *monochromatic wave* of a single *angular frequency* ω, for example

$$E = E_0 \cos(kx - \omega t), \qquad B = \frac{E_0}{c}\ cos(kx - \omega t)$$

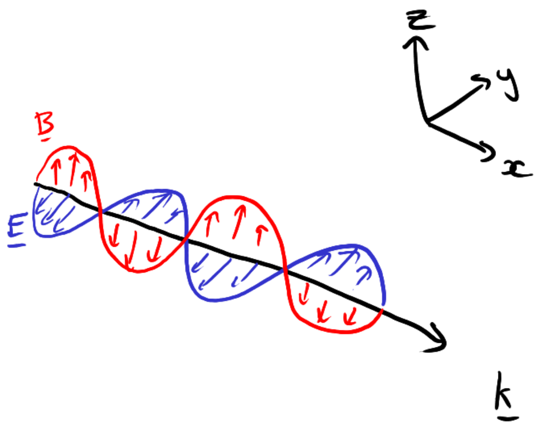
where E_0 is a constant amplitude and $h = \frac{\omega}{c}$ is the *wavenumber*, related to the wavelength λ by $k = \frac{2\pi}{\lambda}$. (The frequency is $\nu = \frac{\omega}{2\pi}$ and the period is $\frac{1}{\nu} = \frac{2\pi}{\omega}$.)

Notes:

- The (angular) frequency and wavenumber are related by the *dispersion relation*

$$\omega^2 = c^2 k^2, \qquad \text{i.e. } \omega = \pm ck$$

- The oscillations and **E** and **B** are in phase but in orthogonal directions.
- The waves are *transverse*: the oscillating fields are orthogonal to the direction in which the wave varies (and propagates).



Because Maxwell's equations are linear, EM waves of different amplitudes, frequencies and directions can be *superposed*. (Light rays can pass through each other!)

Polarization

A more general approach to plane EM waves is to seek solutions of the form

$$\mathbf{E} = \mathrm{Re}(\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)})$$
$$\mathbf{B} = \mathrm{Re}(\mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)})$$

where $\mathbf{E}_0$, $\mathbf{B}_0$ are constant (complex) vector amplitudes, $\mathbf{k}$ is the (real, constant) *wavevector* and ω is the (real, constant) angular frequency. The wavenumer is $k = |\mathbf{k}|$.

The wave equation is satisfied by $\mathbf{E}$ and $\mathbf{B}$ if ω and k satisfy the dispersion relation

$$\boxed{\omega^2 = c^2 k^2}$$

The individual Maxwell equations reduce to algebraic conditions:

$$\begin{aligned}
\nabla\cdot\mathbf{E} &= 0 & \Longrightarrow \quad & \mathbf{k}\cdot\mathbf{E}_0 = 0\\
\nabla\cdot\mathbf{B} &= 0 & \Longrightarrow \quad & \mathbf{k}\cdot\mathbf{B}_0 = 0\\
\nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} & \Longrightarrow \quad & \mathbf{k}\times\mathbf{E}_0 = \omega\mathbf{B}_0\\
\nabla\times\mathbf{B} &= \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} & \Longrightarrow \quad & \mathbf{k}\times\mathbf{B}_0 = -\frac{\omega}{c^2}\mathbf{E}_0
\end{aligned}$$

The fourth equation is redundant because the first and third, together with the dispersion relation, imply

$$\mathbf{k}\times\mathbf{B}_0 = \frac{1}{\omega}\mathbf{k}\times(\mathbf{k}\times\mathbf{E}_0) = -\frac{k^2}{\omega^2}\mathbf{E}_0 = -\frac{\omega}{c^2}\mathbf{E}_0$$

Suppose $\mathbf{E}_0$ is real. Then $\mathbf{B}_0$ is also real and the vectors $\mathbf{k}$, $\mathbf{E}_0$ and $\mathbf{B}_0$ form an orthogonal triad. So $\mathbf{E}$ and $\mathbf{B}$ oscillated in fixed directions, which are perpendicular to each other and to the direction of propagation.

This is similar to the 1D wave considered previously and corresponds to a *linearly polarized wave.*

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Now suppose $\mathbf{E}_0$ is complex and of the form

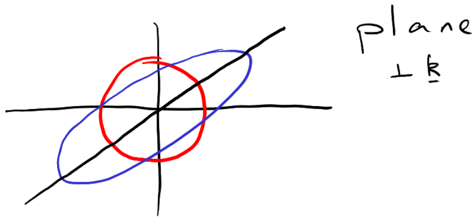
$$\mathbf{E}_0 = \mathbf{a} - i\mathbf{b}$$

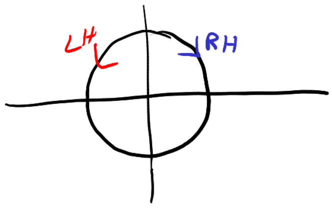
where $\mathbf{a}$ and $\mathbf{b}$ are linearly independent real vectors in the plane perpendicular to $\mathbf{k}$. Then the direction of

$$\mathbf{E} = \mathbf{a}\cos(\mathbf{k}\cdot\mathbf{x}-\omega t) + \mathbf{b}\sin(\mathbf{k}\cdot\mathbf{x}-\omega t)$$

is not fixed but rotates in the plane perpendicular to $\mathbf{k}$. In general it traces out an ellipse and the wave is said to be *elliptically polarized.* The special case of *circular polarization* occurs when $|\mathbf{b}| = |\mathbf{a}|$. Then $\mathbf{E}$ traces out a circle.

The polarization if *right-handed* if $\mathbf{b} = \hat{\mathbf{k}}\times\mathbf{a}$ and *left-handed* if $\mathbf{b} = -\hat{\mathbf{k}}\times\mathbf{a}$.





Exercise: work out how **B** evolves.

Reflection from a conductor

Suppose a perfect conductor occupies the half-space $x > 0$. An *incident wave* approaching the surface $x = 0$ in the $+x$ direction has the form

$$\mathbf{E}_{\text{inc}} = \text{Re}\left(E_0 e^{i(kx-\omega t)}\mathbf{e}_y\right)$$

where we take $k > 0$, $\omega = ck > 0$ and $E_0 \in \mathbb{R}$.

Inside the conductor, we have $\mathbf{E} = \mathbf{0}$. Since the tangential components of **E** are continuous at the surface ($[\mathbf{n} \times \mathbf{E}] = \mathbf{0}$) we require $E_y = 0$ at $x = 0$. This can be satisfied by adding a *reflected wave*

$$\mathbf{E}_{\text{ref}} = \text{Re}\left(-E_0 e^{i(-kx-\omega t)}\mathbf{e}_y\right)$$

propagating in the $-x$ direction.

The combined solution is

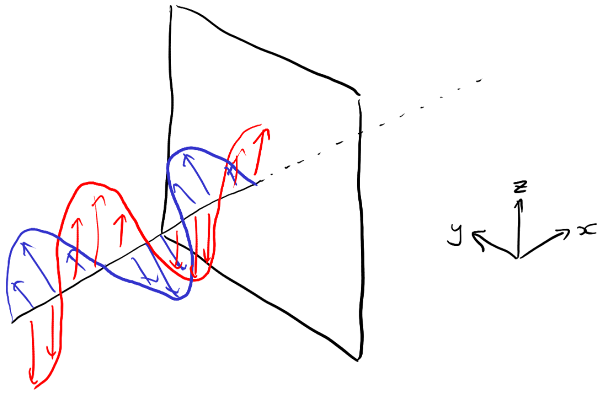
$$\begin{aligned}\mathbf{E} &= \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{ref}} \\ &= \text{Re}\left(2iE_0 \sin(kx) e^{-i\omega t}\mathbf{e}_y\right) \\ &= 2E_0 \sin(kx)\sin(\omega t)\mathbf{e}_y\end{aligned}$$

which is a *standing wave*. The corresponding field is

$$\begin{aligned}\mathbf{B} &= \text{Re}\left(\frac{E_0}{c} e^{i(kx-\omega t)}\mathbf{e}_z\right) + \text{Re}\left(\frac{E_0}{c} e^{i(-kx-\omega t)}\mathbf{e}_z\right) \\ &= \text{Re}\left(\frac{2E_0}{c}\cos(kx) e^{-i\omega t}\mathbf{e}_z\right) \\ &= \frac{2E_0}{c}\cos(kx)\cos(\omega t)\mathbf{e}_z\end{aligned}$$

Inside the conductor, $\mathbf{B} = \mathbf{0}$ (since $\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$ and $\mathbf{B}$ is harmonic in time). So there is a tangential discontinuity in $\mathbf{B}$ and a surface current

$$\mathbf{K} = \frac{1}{\mu_0}[\mathbf{n} \times \mathbf{B}] = \frac{2E_0}{\mu_0 c} \cos(\omega t)\mathbf{e}_y$$



4.5 Electromagnetic Energy

In electrostatics, we saw that the energy per unit volume in $\mathbf{E}$ is $\frac{1}{2}\varepsilon_0|\mathbf{E}|^2$.

In magnetostatics, we saw that the energy per unit volume in $\mathbf{B}$ is $\frac{1}{2\mu_0}|\mathbf{B}|^2$.

ow derive a general conservation law from Maxwell's equations. Consider the rate of change of the electric and magnetic energy per unit volume:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0|\mathbf{E}|^2 + \frac{1}{2\mu_0}|\mathbf{B}|^2\right) &= \mathbf{E} \cdot \left(\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) + \mathbf{B} \cdot \left(\frac{1}{\mu_0}\frac{\partial \mathbf{B}}{\partial t}\right) \\ &= \mathbf{E} \cdot \left(\frac{1}{\mu_0}\nabla \times \mathbf{B} - \mathbf{J}\right) + \mathbf{B} \cdot \left(-\frac{1}{\mu_0}\nabla \times \mathbf{E}\right)\end{aligned}$$

An identity from vector calculus:

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$$

Divide by μ_0 and add to previous equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0|\mathbf{E}|^2 + \frac{1}{2\mu_0}|\mathbf{B}|^2\right) + \nabla \cdots \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0}\right) = -\mathbf{E} \cdot \mathbf{J}$$

This is the energy equation for the EM field. Write it as

$$\boxed{\frac{\partial w}{\partial t} + \nabla \cdot S = -\mathbf{E} \cdot \mathbf{J}} \tag{1}$$

where

$$\boxed{w = \frac{1}{2}\varepsilon_0|\mathbf{E}|^2 + \frac{1}{2\mu_0}|\mathbf{B}|^2}$$

is the *energy density* (energy / unit volume) and

$$\boxed{\mathbf{S} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}}$$

is the *energy flux density* (energy flux / area). **S** is known as the *Poynting vector.* In the absence of currents, equation (1) becomes

$$\frac{\partial w}{\partial t} + \nabla\cdot\mathbf{S} = 0$$

and expresses the conservation of EM energy. Integrate over a time-independent volume V with $\partial V = S$ and use the divergence theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V w\mathrm{d}V + \int_S \mathbf{S}\cdot\mathrm{d}\mathbf{S} = 0$$

(rate of change of energy in V equals $-$ flux of energy through S). On the RHS of equation (1), $\mathbf{E}\cdot\mathbf{J}$ is the rate at which the EM field loses energy (per unit volume) by doin gwork on charged particles via the Lorentz force. Recall

$$\begin{aligned}
\rho &= \sum_{i=1}^{N} q_i\delta(\mathbf{x}-\mathbf{x}_i)\\
\mathbf{J} &= \sum_{i=1}^{N} q_i\dot{\mathbf{x}}_i\delta(\mathbf{x}-\mathbf{x}_i)\\
\mathbf{F}_i &= q_i(\mathbf{E}(\mathbf{x}_i,t) + \dot{\mathbf{x}}_i\times\mathbf{B}(\mathbf{x}_i,t))
\end{aligned}$$

so the rate of working is

$$\begin{aligned}
\sum_{i=1}^{N}\mathbf{F}_i\cdot\dot{\mathbf{x}}_i &= \sum_{i=1}^{N}\mathbf{E}(\mathbf{x}_i,t)\cdot(q_i\dot{\mathbf{x}}_i)\\
&= \int\mathbf{E}\cdot\mathbf{J}\mathrm{d}V
\end{aligned}$$

In a conductor, Ohm's law $\mathbf{J} = \sigma\mathbf{E}$ implies $\mathbf{E}\cdot\mathbf{J} = \frac{|\mathbf{J}|^2}{\sigma}$ which is positive definite, and is the rate of Joule heating / volume. (EM energy $\to$ heat)

Energy density and flux for an EM wave

For

$$\begin{aligned}
\mathbf{E} &= \mathrm{Re}(\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)})\\
\mathbf{B} &= \mathrm{Re}(\mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)})
\end{aligned}$$

then using the identity

$$\operatorname{Re}(A)\operatorname{Re}(B) = \operatorname{Re}\left(\frac{AB + A\overline{B}}{2}\right),$$

we have

$$\begin{aligned} w &= \frac{1}{2}\varepsilon_0|\mathbf{E}|^2 + \frac{1}{2\mu_0}|\mathbf{B}|^2 \\ &= \frac{1}{4}\varepsilon_0 \operatorname{Re}((\mathbf{E}_0\cdot\mathbf{E}_0)e^{2i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + |\mathbf{E}_0|^2) + \frac{1}{4\mu_0}\operatorname{Re}((\mathbf{B}_0\cdot\mathbf{B}_0)e^{2i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + |\mathbf{B}_0|^2) \end{aligned}$$

The $e^{2i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ terms average to zero, leaving the average energy density

$$\langle w\rangle = \frac{1}{4}\varepsilon_0|\mathbf{E}_0|^2 + \frac{1}{4\mu_0}|\mathbf{B}_0|^2$$

Since $\mathbf{B}_0 = \frac{1}{c}\hat{\mathbf{k}} \times BfE_0$ and $\hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$, the two contributions are equal and we have

$$\langle w\rangle = \frac{1}{2}\varepsilon_0|\mathbf{E}_0|^2.$$

Similarly, the Poynting vector is

$$\begin{aligned} \mathbf{S} &= \frac{\mathbf{E}\times\mathbf{B}}{\mu_0} \\ &= \frac{1}{2\mu_0}\operatorname{Re}\Big((\mathbf{E}_0\times\mathbf{B}_0)e^{2i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + (\mathbf{E}_0\times\overline{\mathbf{B}}_0)\Big) \end{aligned}$$

Average flux density

$$\langle\mathbf{S}\rangle = \frac{|\mathbf{E}_0|^2}{2\mu_0 c}\hat{\mathbf{k}} = \frac{1}{2}\varepsilon_0|\mathbf{E}_0|^2 c\hat{\mathbf{k}} = \langle w\rangle c\hat{\mathbf{k}}$$

Implies that the EM wave transports energy at speed c in the direction $\hat{\mathbf{k}}$.

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5 Electromagnetism and Relativity

Review of special relativity

Lorentz transformation

In the special theory of raltivity we use spacetime coordinates

$$X^\mu = (ct, x, y, z) = (ct, \mathbf{x})$$

(where c is the speed of light) to describe time and position in an inertial frame S.

Greek indices such as μ and ν run from 0 to 3 to cover the dimensions of spacetime. Roman indices such as i and j run from 1 to 3 to cover the dimensions of space.

X is the *position 4-vector*. Under a *Lorentz transformation* (LT) from S to another inertial frame S', it transforms according to

$$X' = \Lambda X$$

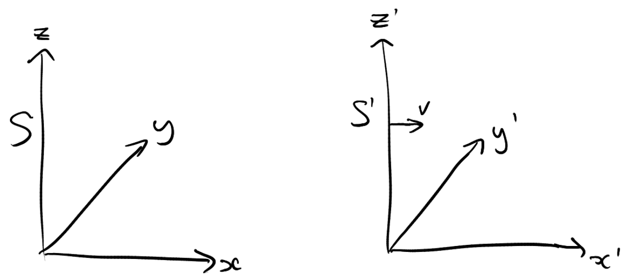
where Λ is the 4×4 *LT matrix*. In index notation,

$$X'^\mu = \Lambda^\mu{}_\nu X^\nu$$

where the summation convention implies a sum from 0 to 3 over the repeated index ν. We have

$$\frac{\partial X'^\mu}{\partial X^\nu} = \lambda^\mu{}_\nu.$$

An important example of an LT is a *Lorentz boost* in standard configuration: S' moves relative to S with velocity v ($|v| < c$) in the x-direction.



Then

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the *Lorentz factor*. The corresponding matrix is

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\beta = \frac{v}{c} \in (-1, 1)$, $\gamma = \frac{1}{\sqrt{1-\beta^2}} \geq 1$. A more general LT may combine a boost with spatial isometries (rotations and reflections).

Four-vectors

Any *4-vector* V transforms as (1) under a LT:

$$V' = \Lambda V$$

The *inner product* of two 4-vectors V and W is

$$V \cdot W = \eta_{\mu\nu} V^\nu W^\mu$$

where $\eta_{\mu\nu}$ is the *Minkowski metric tensor*. This is isotropic, having the same components

$$\eta_{\mu\nu} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathrm{diag}(-1, 1, 1, 1)$$

in any inertial frame. Thus

$$V \cdot W = -V^0 W^0 + V^1 W^1 + V^2 W^2 + V^3 W^3.$$

The inner product is *invariant* under a LT:

$$\begin{aligned} V' \cdot W' &= \eta'_{\mu\nu} V'^\mu W'^\nu \\ &= \eta_{\mu\nu} \Lambda^\mu{}_\rho V^\rho \Lambda^\nu{}_\sigma W^\sigma \\ &= \eta_{\rho\sigma} V^\rho W^\sigma \\ &= V \cdot W \end{aligned}$$

because the LT matrix satisfies the equation

$$\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$$

In matrix notation:

$$\Lambda^\top \eta \Lambda = \eta$$

which implies $\det(\Lambda) = \pm 1$.

The *square* of a 4-vector V,

$$\begin{aligned} V \cdot V &= \eta_{\mu\nu} V^\mu V^\nu \\ &= -V^0 V^0 + V^i V^i \\ &= -(V^0)^2 + (V^1)^2 + (V^2)^2 + (V^3)^2 \end{aligned}$$

is also Lorentz-invariant. V is:

- *timelike* if $V \cdot V < 0$.
- *spacelike* if $V \cdot V > 0$.
- *null* (or *lightlike*) if $V \cdot V = 0$.

An important example is the invariant *spacetime interval* between two events

$$\Delta s^2 = \Delta X \cdot \Delta X = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

This is also useful in differential form

$$\mathrm{d}s^2 = \mathrm{d}X \cdot \mathrm{d}X = -c^2 \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

Proper time and four-velocity

A particle traces out a curve in spacetime, called its *world line.*

Let the world line in S be $X^\mu(\lambda)$, parametrised by the real variable λ. Then $V^\mu = \frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda}$ is a 4-vector.

The curve is timelike if V^μ is timelike, etc, and this property is independent of how the curve is parametrised.

MAssive particles travel slower than light and have timelike world lines. We can use the *proper time* τ (the time experienced by the particle) to parametrise the world line. (Analogous to arclength in Euclidean space).

$\tau(\lambda)$ can be defined by

$$\begin{aligned} c\frac{\mathrm{d}\tau}{\mathrm{d}\lambda} &= \sqrt{-V \cdot V} \\ \implies \tau &= \frac{1}{c} \int \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda} \frac{X^\nu}{\lambda}} \mathrm{d}\lambda \end{aligned}$$

(with an arbitrary additive constant). By the chain rule, the RHS, and therefore τ, are invariant under a reparametrisation $\lambda \mapsto \tilde{\lambda}(\lambda)$.

The differential form is

$$c^2 \mathrm{d}\tau^2 = -\mathrm{d}X \cdot \mathrm{d}X = -\mathrm{d}s^2 > 0$$

The *4-velocity* of a massive particle is

$$U = \frac{\mathrm{d}X}{\mathrm{d}\tau}$$

and satisfies $U \cdot U = -c^2$ (Lorentz-invariant, as expected). Since $X^\mu = (ct, \mathbf{x})$,

$$U^\mu = \frac{\mathrm{d}t}{\mathrm{d}\tau}(c, \mathbf{u})$$

where $\mathbf{u} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$ is the 3-velocity. Thus

$$U \cdot U = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 (-c^2 + |\mathbf{u}|^2) = -c^2,$$

from which

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - \frac{|\mathbf{u}|^2}{c^2}}} = \gamma_\mathbf{u}.$$

Thus $U^\mu = \gamma_\mathbf{u}(c, \mathbf{u})$.

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The *4-momentum* of a massive particle with rest mass $m > 0$ is

$$\begin{aligned} P^\mu &= mU^\mu \\ &= m\gamma_\mathbf{u}(c, \mathbf{u}) \\ &= \left(\frac{E}{c}, \mathbf{p}\right) \end{aligned}$$

where E is the energy and $\mathbf{p}$ is the 3-momentum. Its square is

$$P \cdot P = -\left(\frac{E}{c}\right)^2 + |\mathbf{p}|^2 = -m^2c^2,$$

consistent with the relation

$$E = \sqrt{m^2c^4 + |\mathbf{p}|^2c^2}$$

between energy and momentum. In the Newtonian limit $|\mathbf{u}|^2 \ll c^2$, we have $\mathbf{p} = m\gamma_\mathbf{u}\mathbf{u} \approx m\mathbf{u}$ and

$$E = m\gamma_\mathbf{u}c^2 \approx m\left(1 + \frac{1}{2}\frac{|\mathbf{u}|^2}{c^2}\right)c^2$$

$$E \approx mc^2 + \frac{1}{2}m|\mathbf{u}|^2$$

In the rest frame of the particle, $\mathbf{u} = \mathbf{0}$ and $E = mc^2$ which is the *rest energy.*

Massless particles such as photons travel at the speed of light (in any inertial frame) and have null world lines. They have $E = |\mathbf{p}|c$.

Covectors and tensors

Let

$$Y_\mu = \frac{\partial f}{\partial X^\mu}$$

be the *derivative* of a scalar field $f(X)$ on spacetime. How does this transform under a LT? By the chain rule,

$$\begin{aligned}\frac{\partial f}{\partial X'^\mu} &= \frac{\partial X^\nu}{\partial X'^\mu}\frac{\partial f}{\partial X^\nu} \\ Y'_\mu &= (\Lambda^{-1})^\nu{}_\mu Y_\nu \end{aligned} \tag{2}$$

where

$$(\Lambda^{-1})^\nu{}_\mu = \frac{\partial X^\nu}{\partial X'^\mu}$$

is the inverse LT matrix, satisfying

$$\Lambda^\mu{}_\nu(\Lambda^{-1})^\nu{}_\mu = \frac{\partial X'^\mu}{\partial X^\nu}\frac{\partial X^\nu}{\partial X'^\rho} = \frac{\partial X'^\mu}{\partial X'^\rho} = \delta^\mu{}_\rho$$

The derivative is an example of a *covector*. Generally a covector transforms as (2):

$$Y' = \Lambda^{-1}Y.$$

A (4-)tensor of type (r, s) is an object whose 4^{r+s} components transform as

$$T'^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} = \Lambda^{\mu_1}{}_{\rho_1}\cdots\Lambda^{\mu_r}{}_{\rho_r}(\Lambda^{-1})^{\sigma_1}{}_{\nu_1}\cdots(\Lambda^{-1})^{\sigma_3}_{\nu_3}T^{\rho_1\cdots\rho_r}{}_{\sigma_1\cdots\sigma_s}$$

under a LT. (Any ordering of the up and down indices is possible, for example $A_\mu{}^\nu$ or $B^{\mu\sigma}_{\nu\rho}$).

Each "up" index transforms like a vector. Each "down" index transforms like a co-vector.

- A $(1, 0)$ tensor is a (4-)vector.
- A $(0, 1)$ tensor is a covector.
- A $(0, 0)$ tensor is a *scalar* (a Lorentz invariant).

It is straightdorward to show the following:

- A linear combination of (r, s) tensors is also an (r, s) tensor, for example

  $$A^{\mu\nu}{}_{\rho\sigma} = aB^{\mu\nu}{}_{\rho\sigma} + bC^{\mu\nu}{}_{\rho\sigma}$$

  where a and b are scalars. The free indices must match.
- The *outer product* of tensors of type (r_1, s_1) and (r_2, s_2) is a tensor of type $(r_1 + r_2, s_1 + s_2)$, for example

  $$A^{\mu\nu}{}_{\rho\sigma} = B^{\mu\nu}C_{\rho\sigma}.$$

- The *derivative* of a tensor field of type (r, s) is a tensor of type $(r, s+1)$, for example
$$A_\mu{}^{\nu\rho}{}_\sigma = \partial_\mu B^{\nu\rho}{}_\sigma$$
where
$$\partial_\mu = \frac{\partial}{\partial X^\mu}$$

- The *contraction* of an (r, s) tensor (with $r, s > 0$) on a pair of its indices produces an $(r-1, s-1)$ tensor, for example
$$A^\mu{}_\rho = B^{\mu\nu}{}_{\rho\nu}$$
This works only if the pair consists of one "up" and one "down" index.

The summation convention on spacetime requires the repeated Greek index to appear once up and once down.

Note that, for example
$$\sum_{\mu=0}^{3} X^\mu X^\mu = c^2t^2 + |\mathbf{x}|^2$$
is *not* a scalar (Lorentz invariant), unlike
$$X \cdot X = -c^2t^2 + |\mathbf{x}|^2.$$

A tensor is *isotropic* if its components are the same in every inertial frame. The metric tensor
$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$$
is an isotropic $(0, 2)$ tensor, and its inverse
$$\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$$
is an isotropic $(2, 0)$ tensor (*exercise*). The Kronecker delta symbol
$$\delta^\mu{}_\nu = \mathrm{diag}(1, 1, 1, 1)$$
is an isotropic $(1, 1)$ tensor (*exercise*). The Levi-Civita epsilon symbol
$$\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } (\mu, \nu, \rho, \sigma) \text{ is an even permutation} \\ -1 & \text{if } (\mu, \nu, \rho, \sigma) \text{ is an odd permutation} \\ 0 & \text{otherwise} \end{cases}$$
is an isotropic *pseudo-tensor* of type $(0, 4)$ (see Example Sheet 3 Question 5). A pseudo-tensor of type (r, s) satisfies the same transformation law as a tensor but with an extra factor of $\det(\Lambda) = \pm 1$.

The LT matrix Λ is *not* a tensor. A *tensor equation* is one in which every term is a tensor of the same type and transforms identically under a LT. Suh an equation is said to be *covariant* and is valid in any inertial frame.

Vector calculus in spactime

The spacetime derivative of a scalar field $f(X)$ is a covector field

$$\partial_\mu f = (\partial_0 f, \partial_i f) = \left(\frac{1}{c}\frac{\partial f}{\partial t}, \nabla f\right)$$

The spacetime divergence of a (4-)vector field $V(X) = (V^0, \mathbf{v})$ is a scalar field:

$$\begin{aligned}\partial_\mu V^\mu &= \partial\partial_0 V^0 + \partial_i V^i \\ &= \frac{1}{c}\frac{\partial V^0}{\partial t} + \nabla \cdot \mathbf{v}\end{aligned}$$

In $\mathbb{R}^3$ the curl involves the antisymmetric part of the derivative of a vector field. The spacetime equivalent is the antisymmetric derivative of a covector field $Y(X)$, a tensor of type $(0, 2)$:

$$\partial_\mu Y_\nu \ - \partial_\nu Y_\nu \ .$$

The generalisation of the Laplacian ∇^2 is the *d'Alembertian*, a scalar differential operator

$$\begin{aligned}\Box &= \eta^{\mu\nu}\partial_\mu\partial_\nu \\ &= -\partial_0\partial_0 + \partial_i\partial_i \\ &= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\end{aligned}$$

The wave equation for a scalar field $f(X)$ is $\Box f = 0$. This is a scalar equation only for waves that travel at c.

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Raising and Lowering

The metric tensor provides an isomorphism between vectors (V^μ) and covectors (V_μ):

$$V^\mu = \eta^{\mu\nu} V_\nu, \qquad V_\mu = \eta_{\mu\nu} V^\nu$$

These operations are known as *raising and lowering indices.*

Raising or lowering simply changes the sign of the 0 (time) component, while leaving the spatial components unchanged. So raising or lowering a Roman index changes nothing. For example: for the 4-velocity:

$$U^\mu = \gamma_\mathbf{u} = \gamma_\mathbf{u}(c, \mathbf{u}), \qquad U_\mu = \gamma_\mathbf{u}(-c, \mathbf{u})$$

or the derivative:

$$\partial_\mu f = \left(\frac{1}{c}\frac{\partial f}{\partial t}, \nabla f\right), \qquad \partial^\mu f = \left(-\frac{1}{c}\frac{\partial f}{\partial t}, \nabla f\right)$$

The inner product of two 4-vectors can be written variously as

$$\begin{aligned} V \cdot W &= \eta_{\mu\nu} V^\mu W^\nu \\ &= V^\mu W_\mu \\ &= V_\mu W^\mu \\ &= \eta^{\mu\nu} V_\mu W_\nu \end{aligned}$$

Raising and lowering can be applied to any tensor index, for example

$$\begin{aligned} T_\mu{}^\nu &= \eta_{\mu\rho} T^{\rho\nu} \\ T^\mu{}_\nu &= \eta_{\nu\rho} T^{\mu\rho} \end{aligned}$$

Exercise: show that $\eta^\mu{}_\nu = \delta^\mu{}_\nu$.

A tensor may be *symmetric* or *antisymmetric* on a pair of its indices if they are both "down" or both "up". Exercise: Show that this property is preserved if *both* indices are raised or lowered.

5.1 Charge conservation

The equation of charge conservation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

can be written in the form of a spacetime divergence,

$$\boxed{\partial_\mu J^\mu = 0} \qquad (1)$$

by defining the *charge-curent density* 4-vector (or 4-current density)

$$\boxed{J^\mu = (\rho c, \mathbf{J})}$$

Equation (1) is a scalar (Lorentz invariant) equation, i.e. valid in any inertial frame.

Consider a Lorentz boost in standard configuration. Suppose, in frame S, we have uniform distribution of charge particles at rest, with charge q and (proper) number density (number per unit volume) n. Then $\rho = nq$ and $\mathbf{J} = \mathbf{0}$, giving

$$J^\mu = (nqc, \mathbf{0}) \qquad \text{in } S$$

In frame S', the particles have 3-velocity $(-v, 0, 0)$. Their number density increases to γn because of length contraction in the x-direction. So $\rho' = \gamma nq$ and $\mathbf{J}' = \gamma nq(-v, 0, 0)$, giving

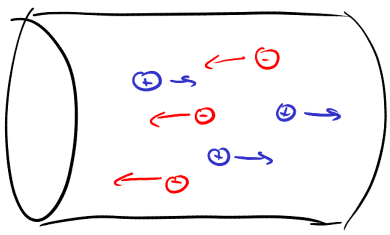
$$J'^\mu = (\gamma nqc, -\gamma nqv, 0, 0) \qquad \text{in } S'$$

These are consistent with the Lorentz Transformation of a 4-vector.

Now suppose that, in S, we have a wire in which particles of charge $+q$ and number density n_+ move along the wire with 3-velocity $(u, 0, 0)$ and particles of charge $-q$ and number density $n_- = n_+$ move with 3-velocity $(-u, 0, 0)$. The net charge density is $\rho = n_+ q - n_- q = 0$ and the net current density is

$$\begin{aligned} \mathbf{J} &= n_+ q(u, 0, 0) - n_- q(-u, 0, 0) \\ &= (2n_+ qu, 0, 0) \end{aligned}$$

The wire is uncharged but carries a current. So $\mathbf{E} = \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$.



In S', the $+$ and $-$ particles have different x-velocities

$$\frac{\partial \pm -v}{\partial 1 \mp \frac{uv}{c^2}}$$

and therefore different charge densities because of length contraction. The wire has a net charge and therefore $\mathbf{E}' = \mathbf{0}$.

So a magnetic field in one frame can produce an electric field in another (and vice versa).

5.2 Electromagnetic tensor

The relativistic equation of motion for a massive particle

$$\frac{\mathrm{d}P^\mu}{\mathrm{d}\tau} = f^\mu \tag{2}$$

where τ is the proper time and f^μ is the 4-force. Since $P \cdot P = -m^2c^2$ is constant, f^μ must satisfy $f \cdot P = 0$. In detail,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}(\eta_{\mu\nu} P^\mu P^\nu) &= \eta_{\mu\nu} \left(\frac{\mathrm{d}P^\mu}{\mathrm{d}\tau} P^\nu + P^\mu \frac{\mathrm{d}P^\nu}{\mathrm{d}\tau} \right) \\ &= \eta_{\mu\nu}(f^\mu P^\nu + P^\mu f^\nu) \\ &= 2\eta_{\mu\nu} f^\mu P^\nu \\ &= 2 f_\mu P^\mu \\ &= 0 \end{aligned}$$

How can we express the Lorentz force as a 4-vector (or covector)? (All known charged particles are massive). It should be $\propto q$ and linear in U^μ. Assume

$$f_\mu = qF_{\mu\nu}U^\nu \tag{3}$$

for some $(0, 2)$ tensor $F_{\mu\nu}$. Then

$$f \cdot P = 0 \iff qmF_{\mu\nu}U^\mu U^\nu = 0$$

and is satisfied for all U if $F_{\mu\nu}$ is *antisymmetric*. The spatial components of (3) are

$$\begin{aligned} f_i &= q(F_{i0}U^0 + F_{ij}U^j) \\ &= \gamma_{\mathbf{u}} q(F_{i0}c + F_{ij}u_j) \end{aligned}$$

For this to agree with $\gamma_{\mathbf{u}} q(E_i + \varepsilon_{ijk}u_j B_k)$ (the factor of $\gamma_{\mathbf{u}}$ comes from converting $\frac{\mathrm{d}}{\mathrm{d}t}$ into $\frac{\mathrm{d}}{\mathrm{d}\tau}$) we require

$$F_{i0} = \frac{E_i}{c}, \qquad F_{ij} = \varepsilon_{ijk}B_k.$$

By antisymmetry,

$$F_{00} = 0, \qquad F_{0i} = -\frac{E_i}{c}$$

Thus,

$$\boxed{F_{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & B_z & -B_y \\ \frac{E_y}{c} & -B_z & 0 & B_x \\ \frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}}$$

is the *electromagnetic (field) tensor.*

Neither $\mathbf{E}$ nor $\mathbf{B}$ is the spatial part of a 4-vector. However, for a particle with 4-velocity U, $F_{\mu\nu}U^\nu$ is a covector that equals $(0, \mathbf{E})$ in the particle's rest frame.

By construction, the spatial components of (2) give the relativistic equation of motion

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = \gamma_{\mathbf{u}} q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

where $\mathbf{p} = \gamma_{\mathbf{u}} m\mathbf{u}$ is the relativistic 3-momentum. Equivalently,

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) = \mathbf{F}$$

which is familiar. The time component of (2) is

$$\begin{aligned} \frac{\mathrm{d}P^0}{\mathrm{d}\tau} &= f^0 \\ &= -f_0 \\ &= -qF_{0i}U^i \\ &= q\frac{E_i}{c}\gamma_{\mathbf{u}} u_i \end{aligned}$$

equivalent to

$$\frac{\mathrm{d}E}{\mathrm{d}\tau} = q\gamma_{\mathbf{u}}\mathbf{E}\cdot\mathbf{u}$$

or

$$\frac{\mathrm{d}E}{\mathrm{d}t} = q\mathbf{E}\cdot\mathbf{u} = \mathbf{F}\cdot\mathbf{u}$$

which is also familiar.

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5.3 The 4-potential

We previously wrote

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t},$$

in terms of scalar and vector potentials. What does this mean for $F_{\mu\nu}$?

$$\begin{aligned} F_{i0} &= \frac{E_i}{c} \\ &= v - \frac{1}{c}\frac{\partial\phi}{\partial x_i} - \frac{1}{c}\frac{\partial A_i}{\partial t} \\ &= \partial_i\left(-\frac{\phi}{c}\right) - \partial_0 A_i \\ F_{ij} &= \varepsilon_{ijk}B_k \\ &= \varepsilon_{ijk}\varepsilon_{klm}\partial_l A_m \\ &= \partial_i A_j - \partial_j A_i \end{aligned}$$

We can interpret $F_{\mu\nu}$ as the antisymmetric derivative

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$$

of the covector

$$A_\mu = \left(-\frac{\phi}{c}, \mathbf{A}\right)$$

The related 4-vector is the *4-potential*

$$A^\mu = \left(\frac{\phi}{c}, \mathbf{A}\right).$$

A gauge transformation

$$\tilde{\mathbf{A}} = \mathbf{A} + \nabla\chi, \qquad \tilde{\phi} = \phi - \frac{\partial\chi}{\partial t}$$

corresponds to replacing A_μ with

$$\tilde{A}_\mu = A_\mu + \partial_\mu\chi$$

i.e. adding $\partial_\mu\chi = \left(\frac{1}{c}\frac{\partial\chi}{\partial t}, \nabla\chi\right)$ to $A_\mu = \left(-\frac{\phi}{c}, \mathbf{A}\right)$. This leaves $F_{\mu\nu}$ unaffected, because

$$\partial_\mu(\partial_\nu\chi) - \partial_\nu(\partial_\mu\chi) = 0.$$

The spacetime equivalent of the Coulomb gauge is the *Lorentz gauge* in which

$$\partial_\mu A^\mu = 0.$$

If $\partial_\mu A^\mu \neq 0$ then we can make a gauge transformation such that $\partial_\mu \tilde{A}^\mu = 0$ by choosing χ to be the solution of the forced wave equation

$$-\Box\chi = \partial_\mu A^\mu$$

(Note that $\partial_\mu \partial^\mu \chi = \eta^{\mu\nu}\partial_\mu\partial_\nu\chi = \Box\chi$).

5.4 Lorentz transformation of E and B

Since $F_{\mu\nu}$ is a $(0,2)$ tensor, it transforms as

$$F'_{\mu\nu} = (\lambda^{-1})^\rho{}_\mu (\lambda^{-1})^\sigma{}_\nu F_{\rho\sigma}$$

In matrix notation,

$$F' = (\lambda^{-1})^\top F(\Lambda^{-1})$$

For a Lorentz boots in standard configuration,

$$\Lambda^{-1} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence (exercise):

$$\begin{aligned} F'_{01} &= \gamma^2(F_{01} + \beta^2 F_{10} + \beta(F_{00} + F_{11}) \\ &= \gamma^2(1-\beta^2)F_{01} \\ &= F_{01} \\ F'_{02} &= \gamma(F_{02} + \beta F_{12}) \\ F'_{03} &= \gamma(F_{03} + \beta F_{13}) \\ F'_{12} &= \gamma(F_{12} + \beta F_{02}) \\ F'_{13} &= \gamma(F_{13} + \beta F_{03}) \\ F'_{23} &= F_{23} \end{aligned}$$

From which we deduce

$$\boxed{\begin{array}{ll} E'_x = E_x & B'_x = B_x \\ E'_y = \gamma(E_y - vB_z) & B'_y = \gamma\left(B_y + \dfrac{v}{c^2}E_z\right) \\ E'_z = \gamma(E_z + vB_y) & B'_z = \gamma\left(B_z - \dfrac{v}{c^2}E_y\right) \end{array}}$$

More generally if $\mathbf{v}$ is the velocity of S' with respect to S,

$$\boxed{\begin{array}{ll} \mathbf{E}'_\parallel = \mathbf{E}_\parallel & \mathbf{B}'_\parallel = \mathbf{B}_\parallel \\ \mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v}\times\mathbf{B}_\perp) & \mathbf{B}'_\perp = \gamma\left(\mathbf{B}_\perp - \dfrac{1}{c^2}\mathbf{v}\times\mathbf{E}_\perp\right) \end{array}}$$

where we decompose $\mathbf{E}$ and $\mathbf{B}$ into components parallel and perpendicular to $\mathbf{v}$:

$$\mathbf{E} = \mathbf{E}_\parallel + \mathbf{E}_\perp \qquad \mathbf{B} = \mathbf{B}_\parallel + \mathbf{B}_\perp$$

$F_{\mu\nu}$ can be double raised to make

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0B_z & -B_y & \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} = -F^{\mu\nu}$$

Thus

$$\begin{aligned} F^{00} &= 0 \\ F^{0i} &= \frac{E_i}{c} \\ F^{i0} &= -\frac{E_i}{c} \\ F^{ij} &= \varepsilon_{ijk}B_k \end{aligned}$$

We can find two Lorentz-invariant combinations of $\mathbf{E}$ and $\mathbf{B}$ by forming scalars from $F_{\mu\nu}$:

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= F_{00}F^{00} + F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij} + F_{ij} \\ &= 0 + \left(-\frac{E_i}{c}\right)\frac{E_i}{c} + \frac{E_i}{c}\left(-\frac{E_i}{c}\right) + \varepsilon_{ijk}B_k\varepsilon_{ijl}B_l \\ &= -2\frac{|\mathbf{E}|^2}{c^2} + 2\delta_{kl}B_kB_l \\ &= -\frac{2}{c^2}\left(|\mathbf{E}|^2 - c^2|\mathbf{B}|^2\right) \\ \varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} &= \varepsilon_{0ijk}F^{0i}F^{jk} + \varepsilon_{i0jk}F^{i0}F^{jk} + \varepsilon_{ij0k}F^{ij}F^{0k} + \varepsilon_{ijk0}F^{ij}F^{k0} \\ &= \varepsilon_{ijk}\frac{E_i}{c}\varepsilon_{jkl}B_l - \varepsilon_{ijk}\left(-\frac{E_i}{c}\right)\varepsilon_{jkl}B_l + \varepsilon_{ijk}\varepsilon_{ijl}B_l\frac{E_k}{c} - \varepsilon_{ijk}\varepsilon_{ijl}B_l\left(-\frac{E_k}{c}\right) \\ &= 4\varepsilon_{ijk}\frac{E_i}{c}\varepsilon_{jkl}B_l \\ &= \frac{8}{c}\delta_{il}E_iB_l \\ &= \frac{8}{c}\mathbf{E}\cdot\mathbf{B} \end{aligned}$$

where we used $\varepsilon_{0ijk} = \varepsilon_{ijk}$ and permutations. Thus $|\mathbf{E}|^2 - c^2|\mathbf{B}|^2$ is a scalar and $\mathbf{E}\cdot\mathbf{B}$ is a pseudoscalar. For example, if $\mathbf{E} \perp \mathbf{B}$ in one inertial frame then they are perpendicular in any frame.

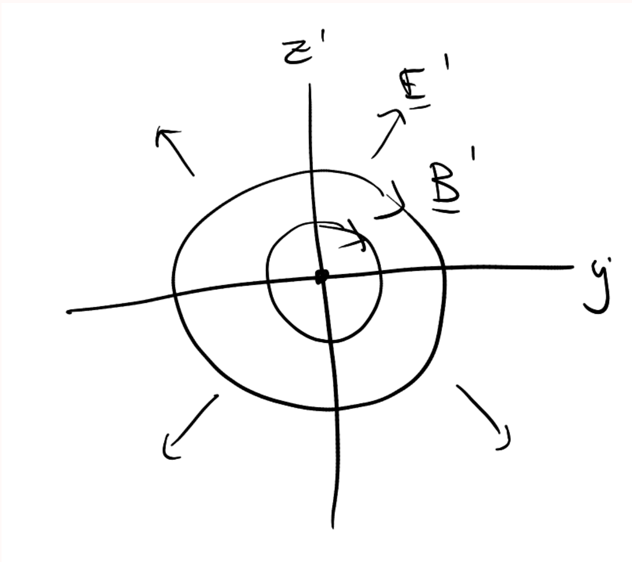
Example (Boosted line charge)**.** From Section 2.1.2, the electric field of a line charge λ per unit length along the x-axis is

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0(y^2+z^2)}(0, y, z)$$

while $\mathbf{B} = \mathbf{0}$ for static charge. Under a standard Lorentz boost,

$$\begin{aligned}\mathbf{E}' &= \frac{\gamma\lambda}{2\pi\varepsilon_0(y^2+z^2)}(0, y, z)\\ &= \frac{\gamma\lambda}{2\pi\varepsilon_0(y'^2+z'^2}(0, y', z')\\ \mathbf{B}' &= \frac{\gamma v}{c^2}\frac{\lambda}{2\pi\varepsilon_0(y^2+z^2)}(0, z, -y)\\ &= \frac{\mu_0\gamma\lambda v}{2\pi(y'^2+z'^2)}(0, z', -y')\end{aligned}$$

$\mathbf{E}'$ can be understood as the electric field of a line charge $\lambda' = \gamma\lambda$ per unit length along the x'-axis. (Length contraction enhances the charge density). $\mathbf{B}'$ can be understood as the magnetic field due to a current $I' = -\lambda' v$ along the x'-axis. (The line charge moves with velocity $-v$ in S').



Start of lecture 16

5.5 Maxwell's equations

How do we write Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{M1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{M2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{M3}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \tag{M4}$$

in a covariant form? Using the relations:

$$F^{0i} = -F^{i0} = \frac{E_i}{c}, \quad F^{ij} = \varepsilon_{ijk} B_k, \quad J^0 = \rho c, \quad J^i = J_i, \quad c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

(M1) becomes

$$c\partial_i^{0i} = \frac{J^0}{\varepsilon_0 c} \implies \partial_i F^{0i} = \mu_0 J^0$$

and (M4) becomes

$$\varepsilon_{ijk} \partial_j B_k = \mu_0 J_0 + \frac{1}{c} \partial_0 E_i \implies \partial_0 F^{i0} + \partial_j F^{ij} = \mu_0 J^i$$

These are the components of the 4-vector equation

$$\boxed{\partial_\nu F^{\mu\nu} = \mu_0 J^\mu}$$

Now

$$F_{ij} = \varepsilon_{ijk} B_k \implies B_i = \frac{1}{2} \varepsilon_{ijk} F_{jk}$$

So (M2) becomes

$$\begin{aligned} \partial_i B_i &= \frac{1}{2} \varepsilon_{ijk} \partial_i F_{jk} \\ &= \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} \\ &= 0 \end{aligned} \tag{1}$$

equivalent to

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0 \tag{3}$$

(trivially satisfied if any of i, j, k coincide because F is antisymmetric).

(M3) becomes

$$\varepsilon_{ijk} \partial_j E_k = -c\partial_0 B_i \implies c\varepsilon_{ijk} \partial_j F_{k0} + \frac{c}{2} \varepsilon_{ijk} \partial_0 F_{jk} = 0 \tag{3}$$

Multiply by ε_{ilm} and divide by c:

$$\partial_l F_{m0} - \partial_m F_{l0} + \partial_0 F_{lm} = 0.$$

Using antisymmetry and relabelling:

$$\partial_i F_{j0} + \partial_j F_{0i} + \partial_0 F_{ij} = 0 \tag{4}$$

Combine (2) and (4) into the covariant form

$$\boxed{\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0} \tag{5}$$

Alternatively, we can write this as a 4-vector equation

$$\boxed{\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0} \tag{6}$$

The 0-component of (6) is

$$\varepsilon^{0ijk}\partial_i F_{jk} = -\varepsilon_{ijk}\partial_i F_{jk} = 0$$

equivalent to (1). The i-component of (6) is

$$\begin{aligned}\varepsilon^{i0jk}\partial_0 F_{jk} + \varepsilon^{ij0k}\partial_j F_{0k} + \varepsilon^{ijk0}\partial_j F_{k0} &= \varepsilon_{ijk}\partial_0 F_{jk} - \varepsilon_{ijk}\partial_j F_{0k} + \varepsilon_{ijk}\partial_j F_{k0} \\ &= \varepsilon_{ijk}\partial_0 F_{jk} + 2\varepsilon_{ijk}\partial_j F_{k0} \\ &= 0\end{aligned}$$

equivalent to (3) $\times\frac{2}{c}$. In summary, Maxwell's equations can be written in covariant form as

$$\boxed{\partial_\nu F^{\mu\nu} = \mu_0 J^\mu} \tag{7}$$

and either

$$\boxed{\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0} \tag{8}$$

or

$$\boxed{\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}} \tag{9}$$

The equation of motion of a charged particle is

$$\frac{\mathrm{d}P^\mu}{\mathrm{d}\tau} = f^\mu \quad \text{with} \quad P^\mu = mU^\mu, f_\mu = qF_{\mu\nu}U^\nu$$

Charge conservation follows easily from (7):

$$\frac{1}{\mu_0}\partial_\mu\partial_\nu F^{\mu\nu} = \partial_\mu J^\mu = 0,$$

since $\partial_\mu\partial_\nu$ is symmetric while $F^{\mu\nu}$ is antisymmetric. If we write $F_{\mu\nu}$ in terms of the 4-potentials as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

then (8) and (9) are automatically satisfied:

$$\partial_\mu\partial_\nu A_\rho - \partial_\mu\partial_\rho A_\nu + \partial_\nu\partial_\rho A_\mu - \partial_\nu\partial_\mu A_\rho + \partial_\rho\partial_\mu A_\nu - \partial_\rho\partial_\nu A_\mu = 0,$$

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu\partial_\rho A_\sigma - \varepsilon^{\mu\nu\rho\sigma}\partial_\nu\partial_\sigma A_\rho = 0,$$

while (7) becomes

$$\partial_\nu(\partial^\mu A^\nu) - \partial_\nu(\partial^\nu A^\mu) = \mu_0 J^\mu$$

In the Lorentz gauge, the first term vanishes:

$$\partial_\nu(\partial^\mu A^\nu) = \partial^\mu(\partial_\nu A^\nu) = 0,$$

leaving

$$\boxed{-\Box A^\mu = \mu_0 J^\mu} \qquad (10)$$

This is an inhomogeneous wave equation for the 4-potential, with a source α charge-current density:

- In the absence of charges and currents, (10) describes free EM waves.
- In a time-independent situation, (10) reduces to

  $$-\nabla^2\Phi = \frac{\rho}{\varepsilon_0}, \qquad -\nabla^2\mathbf{A} = \mu_0\mathbf{J}$$

  as we found in electrostatics and magnetostatics. (Lorentz gauge becomes Coulomb gauge when there is no time-dependence).
- More generally, (10) describes how EM waves are generated by time-dependent charge and current distributions. (See Part II Electrodynamics).

Non-examinable application

Motion in a uniform magnetic field

$$\begin{aligned}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} &= q\mathbf{u}\times\mathbf{B}\\ &= \frac{q}{\gamma m}\mathbf{p}\times\mathbf{B}\end{aligned}$$

($\mathbf{p} = m\gamma\mathbf{u}$). Circular gyromotion in plane perpendicular to $\mathbf{B}$ with frequency $\frac{qB}{\gamma m}$.