Complex Analysis

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1 Complex Differentiation

Goal: study the theory of complex-valued differentiable functions in one complex variable.

(1) $p(z) = a_d z^d + \cdots + a_1 z + a_0$ polynomial, coefficients in $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{C}$.

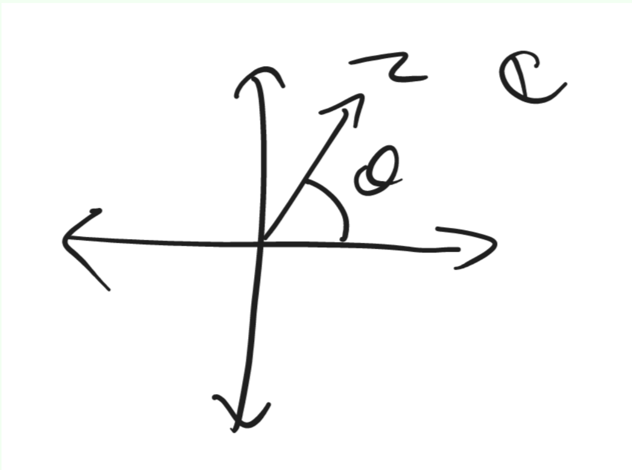
(2) Recall computing the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s > 1$$

We could also consider this as a complex function in complex variable s.

(3) These functions are related to harmonic functions $u(x, y) \colon \mathbb{R}^2 \to \mathbb{R}$, $u_{xx} + u_{yy} = 0$.

Notation. $z \in \mathbb{C}$, $z = x + iy$, real, imaginary parts.
$\overline{z}$ complex conjugate $\overline{z} = x - iy$.
$|z|$, $\arg(z)$ or $\operatorname{Arg}(z)$.



θ with positive real axis, length of vector is $|z|$. $\theta = \arg(z)$, $\theta \in [0, 2\pi)$ then $\operatorname{Arg}(z)$.

Basic Notions

- $\mathcal{U} \subset \mathbb{C}$ is *open* if $\forall u \in \mathcal{U}$, $\exists \varepsilon > 0$ such that

$$D(x, \varepsilon) := \{z \in \mathbb{C} \colon |z - u| < \varepsilon\} \subset \mathcal{U}.$$

(This is sometimes also written as $\mathbb{D}(x, \varepsilon)$ or $B(x, \varepsilon)$).

- a *path* in $\mathcal{U} \subset \mathbb{C}$ is a continuous map $\gamma : [a, b] \to \mathcal{U}$, $\mathcal{C}'$ if γ' exists and is continuous. (one-sided derivatives at endpoints).
  γ is *simple* if it is injective.

- $\mathcal{U} \subset \mathbb{C}$ is *path-connected* if $\forall z, w \in \mathcal{U}$ there exists path in $\mathcal{U}$ with endpoints at z, w.

  Remark. If $\mathcal{U}$ is open, $z, w \in \mathcal{U}$ connected by a path γ in $\mathcal{U}$, then $\exists$ path $\tilde{\gamma}$ in $\mathcal{U}$ connecting z and w consisting of finitely many horizontal and vertical segments.

Definition (Domain)**.** A *domain* is a non-empty, open, path-connected subset of $\mathbb{C}$.

Definition. (1) $f : \mathcal{U} \to \mathbb{C}$ is *differentiable* at $u \in \mathcal{U}$ if

$$f'(u) := \lim_{z \to u} \frac{f(z) - f(u)}{z - u}$$

exists.

(2) $f : \mathcal{U} \to \mathbb{C}$ is *holomorphic* at $u \in \mathcal{U}$ at $u \in \mathcal{U}$ if $\exists \varepsilon > 0$ such that f is differentiable at z for all $z \in D(u, \varepsilon)$ ("analytic").

(3) $f : \mathbb{C} \to \mathbb{C}$ is *entire* if it is holomorphic everywhere.

Remark. All differentiation rules (sum, product, quotient, inverse, chain, ...) hold, by the same proofs.

Identifying $\mathbb{C}$ with $\mathbb{R}^2$ we may write $f : \mathcal{U} \to \mathbb{C}$ as $f(x + iy) = u(x, y) + iv(x, y)$ where u and v are real and imaginary parts of f.

From Analysis and Topology: $u : \mathcal{U} \to \mathbb{R}$ as a function of two real variables is ($\mathbb{R}^2$-)differentiable at $(c, d) \in \mathbb{R}^2$ with $Du|_{(c,d)} = (\lambda, u)$ if

$$\frac{u(x, y) - u(c, d) - [\lambda(x - c) + \mu(y - d)]}{\sqrt{(x - c)^2 + (y - d)^2}} \to 0$$

as $(x, y) \to (c, d)$.

Proposition (Cauchy-Riemann equations)**.** Let $f\colon \mathcal{U} \to \mathbb{C}$ on an open set $\mathcal{U} \subset \mathbb{C}$. Then f is differentiable at $w = c + id \in \mathcal{U}$ if and only if, writing $f = u + iv$, we have u, v ($\mathbb{R}^2$-)differentiable at (c, d) *and*

$$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$$

"Cauchy-Riemann equations". ($u_x = \frac{\partial u}{\partial x}$ and so on).

Proof. f is differentiable at $w \iff f'(w) = p + iq$ exists

$$\iff \lim_{z \to w} \frac{f(z) - f(w) - (z - w)(p + iq)}{|z - w|} = 0.$$

Writing $f = u + iv$ and considering real, imaginary parts in the quotient above, this holds iff

$$\lim_{(x,y)\to(c,d)} \frac{u(x, y) - u(c, d) - [p(x - c) - q(y - d)]}{\sqrt{(x - c)^2 + (y - d)^2}} = 0$$

and

$$\lim_{(x,y)\to(c,d)} \frac{v(x, y) - v(c, d) - [q(x - c) + p(y - d)]}{\sqrt{(x - c)^2 + (y - d)^2}} = 0$$

This holds if and only if u, v are ($\mathbb{R}^2$-)differentiable at (c, d) and $u_x = v_y$ and $u_y = -v_x$ holds. $\square$

Remark (From Analysis and Topology)**.** If the partials u_x, u_y exist and are continuous on $\mathcal{U}$, then u, v are differentiable on $\mathcal{U}$. So it suffices to check partials exist and are continuous and Cauchy-Riemann equations hold to deduce complex differentiability.

Examples

(1) $f(z) = \overline{z}$. f has $u(x, y) = x$, $v(x, y) = -y$ so $u_x = 1$, $v_y = -1$. So $f(z) = \overline{z}$ is *not* holomorphic or differentiable anywhere.

(2) Any polynomial $p(z) = a_d z^d + \cdots + a_1 z + a_0$ with $a_i \in \mathbb{C}$ is entire (holomorphic everywhere).

(3) *rational* functions: a quotient of polynomials $\frac{p(z)}{q(z)}$ is holomorphic on $\mathbb{C} \setminus \{\text{zeroes of } q\}$.

Warning. $f = u + iv$ satisfying Cauchy-Riemann equations at a point does not imply f differentiable; see Example Sheet 1.

Exercise: Let $f : \mathcal{U} \to \mathbb{C}$ on a domain $\mathcal{U}$ with $f'(z) \equiv 0$ on $\mathcal{U}$, then f is constant on $\mathcal{U}$. Sketch: use a nice path and the mean value theorem.

Why are we interested??

structure: Unlike $\mathbb{R}^2$-differentiable functions, holomorphic functions are very constrained: for example, if f is entire and bounded (i.e. $|f(z)| \leq M \ \forall z \in \mathbb{C}$) then f must be constant. (contrasts with sin for example over reals)

analycity: We'll see that f holomorphic on domain $\mathcal{U}$ has holomorphic derivative on $\mathcal{U}$. Hence f is infinitely differentiable, as are u, v. Differentiating Cauchy-Riemann equations:

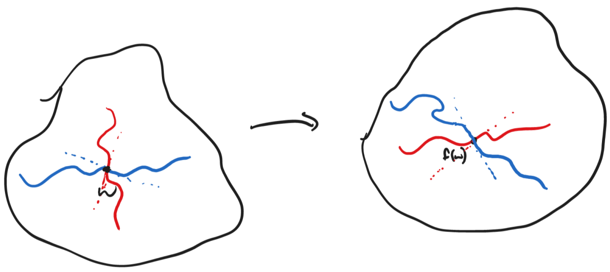
$$u_x = v_y \implies u_{xx} = v_{yx} = v_{xy} = -u_{yy}.$$

So $u_{xx} + u_{yy} = 0$; similarly $v_{xx} + v_{yy} = 0$. The *real* and imaginary parts of a holomorphic function are harmonic.

Start of lecture 2

conformality: Let $f : \mathcal{U} \to \mathbb{C}$ be holomorphic function on an open set $\mathcal{U}$, and $w \in \mathcal{U}$ with $f'(w) \neq= 0$. Geometric behaviour of f at w?

Claim: f is conformal at w:



γ_1, γ_2 $\mathcal{C}'$-paths through w, $\gamma_1, \gamma_2 : [-1, 1] \to \mathcal{U}$. $\gamma_1(0) = \gamma_2(0) = w$, $\gamma_1'(0) \neq 0$. Write $\gamma_j(t) = w + r_j(t)e^{i\theta_j(t)}$, $j = 1, 2$. We have $\operatorname{Arg}(\gamma_j'(0)) = \theta_j(0)$ and

$$\operatorname{Arg}((f \circ \gamma_j)'(0)) = \operatorname{Arg}(\gamma_j'(0)f'(\gamma_j(0))) = \operatorname{Arg}(\gamma_j'(0)) + \operatorname{Arg}(f'(w)) + 2\pi n, n \in \mathbb{Z}$$

so the direction of γ_j at w under application of f is rotated by $\operatorname{Arg}(f'(w))$, independent of γ_j. Since the angle between γ_1, γ_2 is a difference of arguments, the f preserves this angle.

Definition. Let $\mathcal{U}, \mathcal{V}$ be domains in $\mathbb{C}$. A map $f : \mathcal{U} \to \mathcal{V}$ is a *conformal equivalence* of $\mathcal{U}$ and $\mathcal{V}$ if f is a bijective holomorphic map with $f'(z) \neq 0 \ \forall z \in \mathcal{U}$.

Remarks

(1) On Example sheet 1, we will use the real inverse function theorem to show that if $f\colon \mathcal{U} \to \mathcal{V}$ is a holomorphic bijection of open sets with $f'(z) \neq 0\ \forall z \in \mathcal{U}$, then the inverse of f is also holomorphic, so also conformal by the chain rule.

So conformally equivalent domains are the same from the perspective of the holomorphic functions they admit.

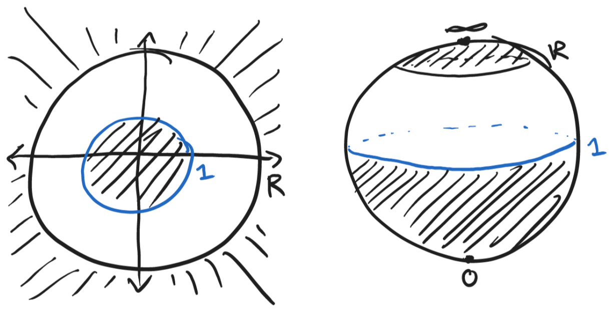
(2) We will see later that injective and holomorphic on a domain implies that $f'(z) \neq 0\ \forall z \in \mathcal{U}$, so this requirement is redundant.

Examples

(1) (Change of coordinates) On $\mathbb{C}$, $f(z) = az + b$, $a \neq 0$ is a conformal equivalence $\mathbb{C} \to \mathbb{C}$. More generally a Möbius map

$$f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0$$

is a conformal equivalence from the Riemann sphere to itself. Riemann sphere: add point ∞ to make a sphere $\mathbb{C}_\infty$ (also sometimes written $\hat{\mathbb{C}}$):



or, imagine giving two copies of the unit disk with coordinates z, $\frac{1}{z}$ (see Part II Riemann Surfaces). If $f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ is continuous then:

(1) If $f(\infty) = \infty$, f holomorphic at $\infty \iff g(z) = \frac{1}{f(\frac{1}{z})}$ is holomorphic at 0.

(2) If $f(\infty) \neq \infty$, f holomorphic at $\infty \iff f\left(\frac{1}{z}\right)$ holomorphic at 0.

(3) If $f(a) = \infty$, $a \in \mathbb{C}$, then f is holomorphic at $a \iff \frac{1}{f(z)}$ is holomorphic at a.

Möbius maps are change of coordinates for the sphere. Choosing $z_1 \mapsto 0$, $z_2 \mapsto \infty$, $z_3 \mapsto 1$ defines a Möbius map:

$$f(z) = \frac{(z - z_1)}{(z - z_2)} \cdot \frac{z_3 - z_2}{z_3 - z_1}$$

for distinct $z_1, z_2, z_3 \in \mathbb{C}$ (recall Part IA Groups).

(2) asdf

Start of lecture 3

Let's recall some facts about functions defined by a power series or other sequences of functions.

(1) A sequence $(f_n)_{n\in\mathbb{N}}$ of functions *converges uniformly* to a function f on some set S if $\forall\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $\forall x \in S$,

$$|f_n(x) - f(x)| < \varepsilon$$

(2) The uniform limit of continuous functions is continuous.

(3) Weierstrass M-test: if $(M_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_{>0}$ and $0 \leq |f_n(x)| \leq M_n$ $\forall x \in S$ and all $n \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} M_n < \infty \implies \sum_{n=1}^{\infty} f_n(x) \text{ converges uniformly on } S \text{ as } N \to \infty$$

(4) Let $(c_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$, and fix $a \in \mathbb{C}$. Then $\exists! R \in [0, 1\infty]$ such that the series

$$z \mapsto \sum_{n=1}^{\infty} c_n (z-a)^n$$

converges absolutely if $|z-a| < R$, diverges if $|z-a| > R$. If $0 < r < R$ then the series converges uniformly on $D(a, r)$. R is the *radius of convergence* of the series. We can compute

$$R = \sup\{r \geq 0 \colon |c_n| r^n \to 0 \text{ as } n \to \infty\}$$

or

$$R = \frac{1}{\lambda}, \qquad \lambda = \lim_{n\to\infty} \sup_{n' \geq n} |c_{n'}|^{1/n'}.$$

Theorem. Let $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ be a complex power series with radius of convergence R. Then

(i) f is holomorphic on $D(a, R)$

(ii) f has derivative

$$f'(z) = \sum_{n=1}^{\infty} n c_n (z-a)^{n-1}$$

with radius of convergence R about a.

(iii) f has derivatives of all orders on $D(a, R)$, and $f^{(n)}(a) = n! c_n$.

Proof. Without loss of generality $a = 0$ by change of variables. Consider the series $\sum_{n=1}^{\infty} nc_n z^{n-1}$. Since $|nc_n| \geq |c_n|$ the radius of convergence of this series is no larger than R. If $0 < R_1 < R$, then for $|z| < R$, we have

$$|nc_n z^{n-1}| \leq n|c_n|R_1^{n-1} \cdot \frac{|z|^{n-1}}{R_1^{n-1}}$$

$n \cdot \left(\frac{|z|}{R_1}\right)^{n-1} \to 0$ as $n \to \infty$. So applying the M-test with $M_n = c_n R_1^{n-1}$ we have convergence of the series. So $\sum nc_n z^{n-1}$ has radius of convergence R.

For $|z|, |w| < R$, we want to consider $\frac{f(z)-f(w)}{z-w}$. Taking partial sums:

$$\sum_{n=0}^{N} c_N \cdot \frac{z^n - w^n}{z - w} = \sum_{n=0}^{N} c_n \left(\sum_{j=0}^{n-1} z^j w^{n-1-j} \right) \tag{$*$}$$

For $|z|, |w| < P < R$, we have

$$\left| c_n \left(\sum_{j=0}^{n-1} z^j w^{n-1-j} \right) \right| \leq |c_n| \cdot n \cdot P^{n-1}$$

so $(*)$ converges uniformly on $\{(z, w) \colon |z|, |w| < P\}$. So the series converges to a continuous limit on $\{|z|, |w| < R\}$, call it $g(z, w)$. When $z \neq w$, $g(z, w) = \frac{f(z)-f(w)}{z-w}$. When $z = w$, $g(w, w) = \sum_{n=0}^{\infty} nc_n w^{n-1}$, so by continuity of g, (i) and (ii) are proved. (iii) is a simple induction. □

Corollary. Suppose $0 < \varepsilon < R$, where R is the radius of convergence of the complex power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n,$$

and $f(z) = 0 \ \forall z \in F(a, \varepsilon)$. Then $f \equiv 0$ on $D(a, R)$.

Proof. Since $f \equiv 0$ on $D(a, \varepsilon)$, we have $f^{(n)}(a) = 0 \ \forall n$. So by part (iii) of the previous theorem, we have $c_n = 0 \ \forall n$, and $f \equiv 0$ on $D(a, R)$. □

The Exponential and The Logarithm

We define the complex exponential

$$e^z = \exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Properties:

(1) Radius of convergence is ∞, so this function is entire, and we have $\frac{\mathrm{d}}{\mathrm{d}z}e^z = e^z$.

(2) For all $z, w \in \mathbb{C}$, $e^{z+w} = e^z e^w$, and $e^z \neq 0$. Proof: fix $w \in \mathbb{C}$, and consider $F(z) := e^{z+w} \cdot e^{-z}$. We have

$$F'(z) = e^{z+w}e^{-z} - e^{z+w}e^{-z} = 0,$$

so F is constant. Since $e^0 = 1$, $F(z) = e^w$, so $e^{z+w} = e^z e^w$. Since $e^z \cdot e^{-z} = e^0 = 1 \ \forall z \in \mathbb{C}$, $e^z \neq 0$.

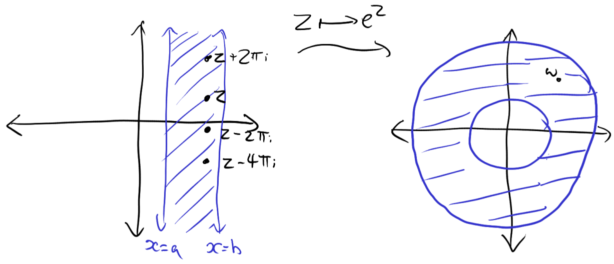
(3) $z = x + iy$. Then $e^z e^{x+iy} = e^x e^{iy}$, $x, y \in \mathbb{R}$.

$$e^{iy} = \cos y + i \sin y;$$

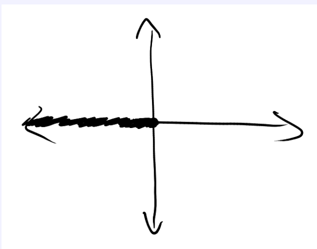
note then $|e^{iy}| = 1$. So

$$e^z = e^x(\cos y + i \sin y),$$

and $|e^z| = e^x = e^{\mathrm{Re}(z)}$. $e^z = 1$ if and only if $x = 0$ and $y = 2\pi k$ for some $k \in \mathbb{Z}$. In fact, $\forall w \in \mathbb{C}^\times$, $\exists$ infinitely many $z \in \mathbb{C}$ such that $e^z = w$, differing by integer multiples of $2\pi i$.



Definition. Let $\mathcal{U} \subseteq \mathbb{C}^\times$ be an open set. We say a continuous function $\lambda \colon \mathcal{U} \to \mathbb{C}$ is a *branch of the logarithm* if $\forall z \in \mathcal{U}$, $\exp(\lambda(z)) = z$. Useful example: $\mathcal{U} = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.



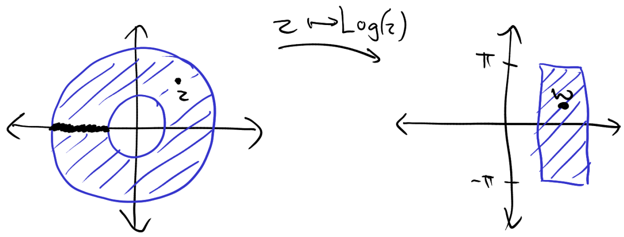
Define $\mathrm{Log} \colon \mathcal{U} \to \mathbb{C}$ by

$$\mathrm{Log}(Z) := \ln|z| + i\theta$$

$\theta = \arg z$, $\theta \in (-\pi, \pi)$. This is the *principal branch of the logarithm.*

Proposition. $\mathrm{Log}(z)$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\le 0}$ with derivative $\frac{1}{z}$. Moreover, if $|z| < 1$, then

$$\mathrm{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}$$



Start of lecture 4

Proof. Since Log is inverse to e^z then using the chain rule, $\mathrm{Log}\, z$ is holomorphic with $\frac{\mathrm{d}}{\mathrm{d}z} \mathrm{Log}\, z = \frac{1}{z}$. We have

$$\frac{\mathrm{d}}{\mathrm{d}z} = \frac{1}{z+1} = 1 - z + z^2 - z^3 + z^4 + \cdots ,$$

which is the derivative of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}$. So $\mathrm{Log}(1+z)$ agrees with this series up to a constant. Since $\mathrm{Log}(1) = 0$ the equality holds. □

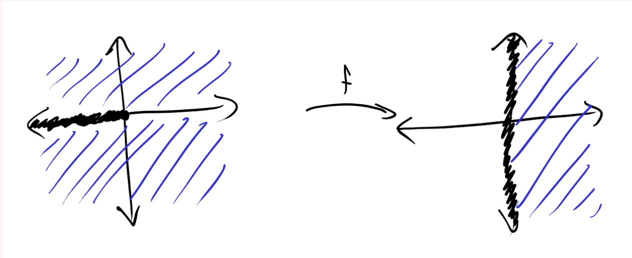
If $\alpha \in \mathbb{C}$, define $z^\alpha := \exp(\alpha \, \mathrm{Log}\, z)$ gives a definition of z^α on $\mathbb{C} \setminus \mathbb{R}_{\le 0}$. Can compute $\frac{\mathrm{d}}{\mathrm{d}z} z^\alpha = \alpha z^{\alpha - 1}$.

Warning. Not necessarily true that $z^\alpha w^\alpha = (zw)^\alpha$.

Note. Note that if $f(z) = z^\alpha$, then the image of f can be "much smaller" than $\mathbb{C}$. For example, $\alpha = \frac{1}{2}$,

$$\begin{aligned} z^{\frac{1}{2}} &= \exp\left(\frac{1}{2}\operatorname{Log} z\right) \\ &= \exp\left(\frac{1}{2}\ln|z| + \frac{1}{2}i\theta\right) \qquad\qquad \theta \in (-\pi, \pi) \end{aligned}$$

So:



1.1 Contour Integration

If $f : [a, b] \to \mathbb{C}$ is continuous (so $\operatorname{Re} f$, $\operatorname{Im} f$ are integrable) we define

$$\int_a^b f(t)\mathrm{d}t := \int_a^b \operatorname{Re}(f(t))\mathrm{d}t + i\int_a^b \operatorname{Im}(f(t))\mathrm{d}t$$

Proposition. Let $f : [a, b] \to \mathbb{C}$ be continuous. Then

$$\left|\int_a^b f(t)\mathrm{d}t\right| \le (b - a) \sup_{a \le t \le b} |f(t)|,$$

with equality if and only if f is constant.

Proof. Write $M = \sup_{a \le t \le b} |f(t)|$, $\theta = \arg\left(\int_a^b f(t)\mathrm{d}t\right)$.

$$\begin{aligned}\left|\int_a^b f(t)\mathrm{d}t\right| &= e^{-i\theta}\int_a^b f(t)\mathrm{d}t \\ &= \int_a^b e^{-i\theta} f(t)\mathrm{d}t \\ &= \int_a^b \mathrm{Re}(e^{-i\theta} f(t))\mathrm{d}t \\ &\le \int_a^b |f(t)\mathrm{d}t \\ &\le M(b-a)\end{aligned}$$

If we have equality, then $|f(t)| \equiv M$, and $\arg f(t) \equiv \theta$, so f is constant. □

Definition. Let $\gamma : [a,b] \to \mathbb{C}$ be a $\mathcal{C}^1$-smooth curve. Then we define the *arc length of* γ to be

$$\mathrm{length}(\gamma) := \int_a^b |\gamma'(t)|\mathrm{d}t.$$

We say γ is simple if $\gamma(t_1) = \gamma(t_2) \implies t_1 = t_2$ or $\{t_1, t_2\} = \{a, b\}$. If γ is simple, then $\mathrm{length}(\gamma) =$ length of the image of γ.

Definition. Let $f : \mathcal{U} \to \mathbb{C}$ be continuous, $\mathcal{U}$ open, and $\gamma : [a,b] \to \mathcal{U}$ be a $\mathcal{C}^1$-smooth curve. Then the integral of f along γ is

$$\int_\gamma f(z)\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t$$

Basic properties

(1) linearity:

$$\int_\gamma c_1 f_1 + c_2 f_2 \mathrm{d}z = c_1 \int_\gamma f_1 \mathrm{d}z + c_2 \int_\gamma f_2$$

(2) additivity: if $a < a' < b$ then

$$\int_{\gamma|_{[a,a']}} f(z)\mathrm{d}z + \int_{\gamma|_{[a',b]}} f(z)\mathrm{d}z = \int_\gamma f(z)\mathrm{d}z$$

(3) inverse path: if $(-\gamma)(t) := \gamma(-t) : [-b, -a] \to \mathcal{U}$, then

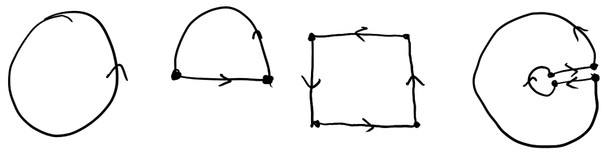
$$\int_{-\gamma} f(z)\mathrm{d}z = -\int_\gamma f(z)\mathrm{d}z$$

(4) independence of paramterisation: if $\phi : [a', b'] \to [a, b]$ is $\mathcal{C}^1$-smooth, $\phi(a') = a$, $\phi(b') = b$, then for $\delta = \gamma \circ \phi$ we have

$$\int_\delta f(z) \mathrm{d}z = \int_\gamma f(z) \mathrm{d}z$$

Note. We can usually assume without loss of generality that $\gamma : [0, 1] \to \mathcal{U}$.

Common types of curves we work with:



We can loosen the $\mathcal{C}^1$-smooth restriction and allow γ to be *piecewise-$\mathcal{C}^1$-smooth*: i.e. $a = a_0 < a_1 < a_2 < \cdots < a_n = b$ such that $\gamma_i := \gamma|_{[a_{i-1}, a_i]}$ is $\mathcal{C}^1$-smooth. Define then

$$\int_\gamma f(z) \mathrm{d}z = \sum_{i=1}^n \int_{\gamma_i} f(z) \mathrm{d}z$$

(which is well-defined by additivity).

Remark. Any piecewise-$\mathcal{C}^1$-smooth curve can be re parametrized to by $\mathcal{C}^1$: for such a γ as above, replace γ_i by $\gamma_i \circ h_i$ where h_i is monotonic $\mathcal{C}^1$-smooth bijection with endpoint derivatives 0. So $\mathcal{C}^1$-smooth paths can have corners. For example,

$$\gamma(t) := \begin{cases} 1 + i \sin(\pi t) & t \in \left[0, \frac{1}{2}\right] \\ \sin(\pi t) + i & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Terminology

- "curve": piecewise-$\mathcal{C}^1$-smooth path.
- "contour": simple *closed* (endpoints are equal) piecewise-$\mathcal{C}^1$-smooth path.

Proposition. For any continuous $f : \mathcal{U} \to \mathbb{C}$, $\mathcal{U}$ open, and any curve $\gamma : [a, b] \to \mathcal{U}$,

$$\left| \int_\gamma f(z) \mathrm{d}z \right| \leq \text{length}(\gamma) \sup_{z \in \gamma} |f(z)|$$

Proof.

$$\begin{aligned}\left|\int_\gamma f(z)\mathrm{d}z\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t\right| \\ &\le \int_a^b |f(\gamma(t))\gamma'(t))|\mathrm{d}t && \text{(by similar trick to previous proof)} \\ &\le \sup_{z\in\gamma} |f(z)| \operatorname{length}(\gamma)\end{aligned}$$

□

Proposition. If $f_n : \mathcal{U} \to \mathbb{C}$ for $n \in \mathbb{N}$ and $f : \mathcal{U} \to \mathbb{C}$ are continuous, and $\gamma : [a, b] \to \mathcal{U}$ is a curve in $\mathcal{U}$ with $f_n \to f$ uniformly on γ, then

$$\int_\gamma f_n(z)\mathrm{d}z \to \int_\gamma f(z)\mathrm{d}z$$

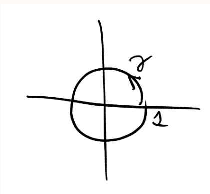
as $n \to \infty$.

Proof. By uniform convergence $\sup_{z\in\gamma} |f(z) - f_n(z)| \to 0$ as $n \to \infty$. By previous proposition,

$$\begin{aligned}\left|\int_\gamma f(z)\mathrm{d}z - \int_\gamma f_n(z)\mathrm{d}z\right| &\le \operatorname{length}(\gamma) \sup_\gamma |f - f_n| \\ &\to 0\end{aligned}$$

as $n \to \infty$. □

Example. $f_n(z) = z^n$, $n \in \mathbb{Z}$, on $\mathbb{C}^* =: \mathcal{U}$, and $\gamma : [0, 2\pi] \to \mathcal{U}$, $\gamma(t) = e^{it}$.



$$\begin{aligned}\int_\gamma f_n(z)\mathrm{d}z &= \int_0^{2\pi} e^{nit} ie^{it}\mathrm{d}t \\ &= i\int_0^{2\pi} e^{(n+1)it}\mathrm{d}t \\ &= \begin{cases} 2\pi i & n = -1 \\ 0 & n \ne -1\end{cases}\end{aligned}$$

Start of lecture 5

Theorem (Fundamental Theorem of Calculus)**.** If $f : \mathcal{U} \to \mathbb{C}$ is a continuous function on open $\mathcal{U} \subseteq \mathbb{C}$ with $F' = f$ an antiderivative of f in $\mathcal{U}$. Then for any curve $\gamma : [a, b] \to \mathcal{U}$,

$$\int_\gamma f(z)\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

In particular, if γ is closed then $\int_\gamma f = 0$.

Proof.

$$\begin{aligned}\int_\gamma f(z)\mathrm{d}z &= \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t \\ &= \int_a^b (F \circ \gamma)'(t)\mathrm{d}t \\ &= F(\gamma(b)) - F(\gamma(a))\end{aligned}$$

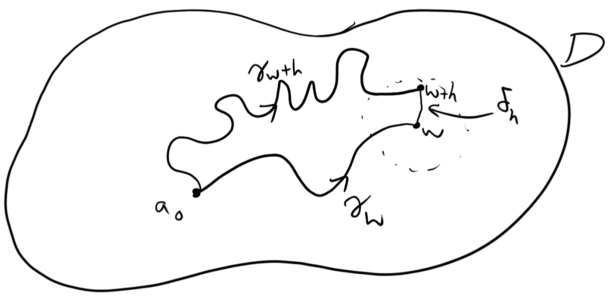
by the real Fundamental Theorem of Calculus. □

Note. In the $z \mapsto z^{-1}$ integral computation and FTC implies that there does not exist a branch of the logarithm on any open neighbourhood of 0.

Theorem. Let $f : D \to \mathbb{C}$ be continuous on a domain D. If $\int_\gamma f = 0$ for all closed curves γ in D, then there exists holomorphic $F : D \to \mathbb{C}$ with $F' = f$.

Proof. Fix $a_0 \in D$. If $w \in D$, choose any curve $\gamma_w : [0, 1] \to D$ with $\gamma_w(0) = a_0$, $\gamma_w(1) = w$. Define

$$F(W) := \int_{\gamma_w} f(z)\mathrm{d}z$$



Find $r_w > 0$ such that $\mathbb{D}(w, r_w) \subseteq D$. For $|h| < r$, let $\delta_h : [0,1] \to D$ be the line segment from w to $w+h$. Then

$$F(w+h) = \int_{\gamma_{w+h}} f(z)\mathrm{d}z = \int_{\gamma_w + \delta_h} f(z)\mathrm{d}z$$

So

$$\begin{aligned} F(w+h) &= F(w) + \int_{\delta_h} f(z)\mathrm{d}z \\ &= F(w) + hf(w) + \int_{\delta_h} f(z) - f(w)\mathrm{d}z \end{aligned}$$

So

$$\begin{aligned} \left| \frac{F(w+h) - F(w)}{h} - f(w) \right| &= \left| \frac{1}{h} \int_{\delta_h} f(z) - f(w)\mathrm{d}z \right| \\ &\le \frac{\mathrm{length}(\delta_h)}{|h|} \cdot \sup_{\delta_h} |f(z) - f(w)| \\ &\le \sup_{z \in \mathbb{D}(w, r_w)} |f(z) - f(w)| \\ &\to 0 \end{aligned}$$

as $r_w \to 0$. So $F'(w) = f(w)$. □

Definition. An open subset $\mathcal{U} \subseteq \mathbb{C}$ is *convex* if $\forall a, b \in \mathcal{U}$ the line segment between a and b is in $\mathcal{U}$. $\mathcal{U}$ is *starlike* (sometimes instead called *starshaped*) if $\exists a_0 \in \mathcal{U}$ such that $\forall b \in \mathcal{U}$ the line segment from a_0 to b is in $\mathcal{U}$.

$$\{\text{disks}\} \subseteq \{\text{convex sets}\} \subseteq \{\text{starlike sets}\} \subseteq \{\text{domains}\}$$

A simplification of previous theorem:

Lemma. Suppose $\mathcal{U}$ is starlike domain, and $f : \mathcal{U} \to \mathbb{C}$ continuous with $\int_{\partial T} f(z)\mathrm{d}z = 0$ for all triangles T in $\mathcal{U}$, then f has an antiderivative in $\mathcal{U}$.

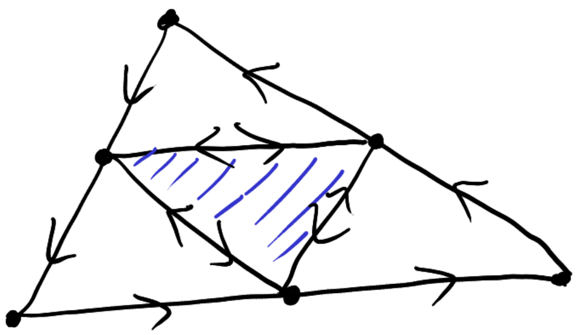
Proof. Exactly the same, choosing γ_w to be the segment from a basepoint a_0 of the starlike. □

Theorem (Cauchy's Theorem in a triangle)**.** If $f : \mathcal{U} \to \mathbb{C}$ is holomorphic on an open $\mathcal{U} \subseteq \mathbb{C}$, and $T \subseteq \mathcal{U}$ is a triangle in $\mathcal{U}$, then

$$\int_{\partial T} f(z)\mathrm{d}z = 0.$$

Remark. Curves are oriented anticlockwise.

Proof. Call $\left|\int_{\partial T} f\right| =: I$, and $L = \text{length}(\partial T)$. We subdivide T by bisecting the sides to obtain T_1, T_2, T_3, T_4:



$\partial T_1 + \partial T_2 + \partial T_3 = \partial T - \partial T_4$, so

$$\int_{\partial T} f(z)\mathrm{d}z = \sum_{i=1}^{4} \int_{\partial T_i} f(z)\mathrm{d}z$$

By triangle inequality, there exists $i \in \{1, 2, 3, 4\}$ such that

$$\left|\int_{\partial T_i} f(z)\mathrm{d}z\right| \ge \frac{1}{4} I$$

call $T^{(1)}$ and note $\text{length}(\partial T^{(1)}) = \frac{1}{2}$.

Proceeding in this way, we obtain triangles

$$T \ge T^{(1)} \ge T^{(2)} \ge T^{(3)} \ge \cdots$$

with $\text{length}(T^{(n)}) = 2^{-N}\,\text{length}(T) = \frac{L}{2^n}$, and

$$\left|\int_{\partial T^{(n)}} f(z)\mathrm{d}z\right| \ge \frac{1}{4^n} I$$

Since $\text{length}(T^{(n)}) \to 0$,

$$\bigcap_{n=1}^{\infty} T^{(n)} = \{\omega\}.$$

Note: z, 1 have holomorphic antiderivatives.

$$\frac{1}{4^n} I \le \left|\int_{\partial T^{(n)}} f(z)\mathrm{d}z\right| = \left|\int_{\partial T^{(n)}} f(z) - f(w) - (z - w)f'(w)\mathrm{d}z\right|$$

Since f is differentiable at w, $\forall \varepsilon > 0$, $\exists \delta.0$ such that $|w - z| < \delta \implies |f(z) - f(w) - (z - w)f'(w)| < \varepsilon |z - w|$. So if $n \gg 1$, we have

$$\left| \int_{\partial T^{(n)}} f(z) - f(w) - (z - w)f'(w) \mathrm{d}z \right| \leq \frac{L}{2^n} \cdot \sup_{z \in \partial T^{(n)}} |z - w| \cdot \varepsilon$$

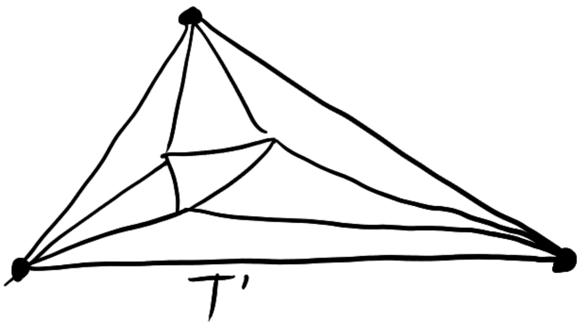
So

$$\frac{I}{4^n} \leq \frac{L}{2^n} \cdot \frac{L}{2^n} \cdot \varepsilon$$

and $I \leq L^2 \varepsilon$. Letting $\varepsilon \to 0$, we have $I = 0$. □

Theorem. Let $S \subseteq \mathcal{U}$ be a finite set and $f : \mathcal{U} \to \mathbb{C}$ be continuous on $\mathcal{U}$ and holomorphic on $\mathcal{U} \setminus S$. Then $\int_{\partial T} f = 0$ for all triangles T in $\mathcal{U}$.

Proof. Using triangle subdivision, assume WLOG that $S = \{a\}$, $a \in T$. If $a \in T' \subseteq T$ for another triangle T', then by triangular subdivision



and previous theorem,

$$\int_{\partial T} f = \int_{\partial T'} = f$$

since f is holomorphic on $T \setminus T'$ we have

$$\left| \int_{\partial T} f \right| = \left| \int_{\partial T'} f \right| \leq \operatorname{length}(T') \cdot \sup_{\partial T'} |f|$$
$$\leq \operatorname{length}(T') \cdot \sup_{T} |f|$$

so letting length$(T') \to 0$, we have $\int_{\partial T} f = 0$. □

Theorem (Cauchy's theorem in a disk)**.** Let D be a disk (or any starlike domain) and $f : D \to \mathbb{C}$ a continuous function, holomorphic away from at most a finite set of points in D, then $\int_\gamma f = 0$ for any closed curve γ in D.

Proof. By previous theorem and converse Fundamental Theorem of Calculus for starlike domains, there exists antiderivative F for f on D. So by FTC, Cauchy's Theorem follows. □

Start of lecture 6

Theorem (Cauchy's Integral Formula)**.** Let $\mathcal{U} \subseteq \mathbb{C}$ be a domain, $f : \mathcal{U} \to \mathbb{C}$ holomorphic, and $\overline{D(a,r)} \subseteq \mathcal{U}$. Then for all $z \in D(a,r)$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f(w)}{w-z} \mathrm{d}w$$

Proof. Define

$$g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} - f'(z) & w \neq z \\ 0 & w = z \end{cases}$$

Then g is continuous at z, holomorphic on $D(a,r)$ except possible at z. Find $r_1 > 0$ such that $D(a,r) \subseteq D(a,r_1) \subseteq \mathcal{U}$. Apply Cauchy's theorem to g on $D(a,r_1$ with curve $\gamma = \partial D(a,r)$, then

$$\int_{\partial D(a,r)} g(w)\mathrm{d}w = 0$$

i.e.

$$\int_{\partial D(a,r)} \frac{f(w)}{w-z} \mathrm{d}w = \int_{\partial D(a,r)} \frac{f(z)}{w-z} \mathrm{d}w$$

Useful expansion: since $|w-a| = r > |z-a|$

$$\frac{1}{w-z} = \frac{1}{(w-a)\left[1 - \frac{z-a}{w-a}\right]} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}},$$

by geometric expansion. So

$$\int_{\partial D(a,r)} \frac{f(z)}{w-z} \mathrm{d}w = \sum_{n=0}^{\infty} \left[f(z)(z-a)^n \int_{\partial D(a,r)} \frac{1}{(w-a)^{n+1}} \mathrm{d}w \right]$$

We have computed that the integral in the brackets on the right is 0 unless $n = 0$, in which case it is $2\pi i$. So

$$\int_{\partial D(a,r)} \frac{f(w)}{w-z} \mathrm{d}w = 2\pi i f(z)$$

as claimed. □

Corollary (Mean Value Property). If $f : \mathcal{U} \to \mathbb{C}$ is holomorphic on domain $\mathcal{U}$, and $\overline{D(a,r)} \subseteq \mathcal{U}$, then

$$f(a) = \int_0^1 f(a + re^{2\pi it})\mathrm{d}t$$

i.e. f takes the average value on a circle about a point.

Proof. Applying Cauchy's Integral Formula, with $t \mapsto a+re^{2\pi it}$ on $[0,1]$ for $\partial D(a,r)$. □

Applications of CIF

Corollary (Local Maximum Principle). Let $f : D(a,r) \to \mathbb{C}$ be holomorphic. If $|f(z)| \leq |f(a)|$ for all $z \in D(a,r)$, then f is constant. "non-constant holomorphic maps cannot achieve maximum on an open set".

Proof. By mean value property, $\forall 0 < \rho < r$ we have

$$\begin{aligned} |f(a)| &= \left| \int_0^1 f(a + \rho e^{2\pi it}\mathrm{d}t \right| \\ &\leq \sup_{|z-a|=\rho} |f(z)| \\ &\leq |f(a)| \end{aligned}$$

Since we have equality at each step, we have $|f(z)| = |f(a)|$ for all $|z-a| = \rho$. So $|f|$ is a constant function on $D(a,r)$. Hence f is constant on $D(a,r)$. □

Theorem (Liouville's Theorem). Every bounded entire function is constant.

Proof. With $|f(z)| \leq M$ for f entire, and $R \gg 1$ we have for any $0 < |z| < \frac{R}{2}$ that .image

$$\begin{aligned} |f(z) - f(0)| &= \frac{1}{2\pi} \left| \int_{\partial D(0,R)} f(w) \left[\frac{1}{w-z} - \frac{1}{w} \right] \mathrm{d}w \right| \\ &= \frac{1}{2\pi} \left| \int_{\partial D(0,R)} f(w) \frac{z}{(w-z)w} \mathrm{d}w \right| \end{aligned}$$

Since $|z - w| > \frac{R}{2}$ and $|w| = R$, we have

$$\begin{aligned} |f(z) - f(0)| &\leq \frac{1}{2\pi} \cdot 2\pi R \cdot \sup_{w \in \partial D(0,R)} |f(w)| \cdot |z| \cdot \frac{1}{R \cdot \frac{R}{2}} \\ &\leq M \cdot |z| \cdot \frac{1}{R/2} \\ &\to 0 \end{aligned}$$

as $R \to \infty$. So $f(z) = f(0)$, so $f \equiv f(0)$ is constant. □

Corollary (Fundamental Theorem of Algebra)**.** Every non-constant polynomial with complex coefficients has a root in $\mathbb{C}$.

Proof. If $p(z)$ has no root in $\mathbb{C}$, then $f(z) := \frac{1}{p(z)}$ is entire. $p(z)$ non-constant implies $a_d \neq 0$, $d \geq 1$. So $\frac{p(z)}{z^d} = a_d + a_{d-1} + \frac{1}{z} + \cdots + a_0 \frac{1}{z^d}$ shows that $|p(z)| \to \infty$ as $|z| \to \infty$. So $|f(z)| \to 0$ as $|z| \to \infty$; so there exists $R > 0$ such that $\forall z \notin D(0,r)$, $|f(z)| \leq 1$. Let $M := \max_{z \in \overline{D(0,r)}} |f(z)|$ Then $|f|$ is bounded by $\max\{1, M\}$, and so by Liouville is constant, contradicting the assumption that p is non-constant. □

Taylor-Expansion

Theorem (Taylor Expansion)**.** Let $f : D(a,r) \to \mathbb{C}$ be holomorphic. Then f is represented by convergent power series on $D(a,r)$:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

with

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(w)}{(w-a)^{n+1}} \mathrm{d}w$$

for $0 \leq \rho < r$.

Proof. For $|z-a| < \rho < r$, CIF gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(w)}{w-z} \mathrm{d}w \\ &= \frac{1}{2\pi i} \int_{\partial D(a,\rho)} f(w) \cdot \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}} \mathrm{d}w \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_{\partial D(a,\rho)} f(w) \cdot \frac{1}{w - a^{n+1}} \mathrm{d}w \right] (z-a)^n \end{aligned}$$

proving the theorem. (We swap the sum and integral since the partial sums give rise to a sequence of functions that converge uniformly on $\partial D(a,\rho)$). □

Remarks

(1) "analytic" = has power series representation on a disk in the domain. So holomorphic $\Longrightarrow$ analytic.

(2) holomorhic functions have derivatives of all orders, which are holomorphic.

Corollary (Morera's Theorem)**.** Let D be a disk and $f : D \to \mathbb{C}$ such that $\int_\gamma f = 0$ for all closed curves γ in D. Then f is holomorphic.

Proof. By converse of Fundamental Theorem of Calculus, there exists holomorphic F on D with $F' = f$. So f is holomorphic. (Because existence of Taylor expansion implies that the derivative of a holomorphic function is holomorphic). □

Start of lecture 7

Corollary (Uniform convergence of holomorphic functions)**.** Let $f_n : \mathcal{U} \to \mathbb{C}$ holomorphic functions on a domain $\mathcal{U}$, and $f_n \to f$ uniformly on $\mathcal{U}$ (sufficient: uniform convergence on compact subsets of $\mathcal{U}$). Then f is holomorphic on $\mathcal{U}$, and $f'(z) = \lim_{n\to\infty} f_n'(z)$.

Proof. $\mathcal{U}$ is a union of open disks, so it suffices to work with $D(z,\varepsilon) \subset \mathcal{U}$. Given γ closed curve in $D(z,\varepsilon)$, $\int_\gamma f_n \to \int_\gamma f$ (A&T), and $\int_\gamma f_n = 0$, so $\int_\gamma f = 0$. Since f is continuous, Morera's theorem applies, so f is holomorphic on $D(z,\varepsilon)$.

Recall Taylor expansion computation: for $0 < \rho < \varepsilon$,

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D(z,\rho)} \frac{f(\zeta)}{(\zeta - z)^{m+1}} \mathrm{d}\zeta$$

So $f'(z) = \frac{1}{2\pi i} \int_{\partial D(z,\rho)} \frac{f(\zeta)}{(\zeta-z)^2} \mathrm{d}\zeta$.

$$\begin{aligned} |f'(z) - f_n'(z)| &= \frac{1}{2\pi} \left| \int_{\partial D(z,\rho)} \frac{f(\zeta)}{(\zeta - z)^2} - \frac{f_n(\zeta)}{(\zeta - z)^2} \mathrm{d}\zeta \right| \\ &\le \rho \cdot \frac{1}{\rho^2} \cdot \sup_{\zeta \in \partial D(z,\rho)} |f(\zeta) - f_n(\zeta)| \\ &\to 0 \end{aligned}$$

as $n \to \infty$. So $f'(z) = \lim_{n\to\infty} f_n'(z)$. □

Remark. f need not be non-constant; for example, $f_n(z) = z^n$ on $D(0,r)$, $0 < r < 1$. Then $f_n \to 0$ uniformly.

Corollary. If $f : \mathcal{U} \to \mathbb{C}$ is continuous on a domain $\mathcal{U} \setminus S$ for some finite set S, then f is holomorphic on $\mathcal{U}$.

Proof. If $a \in S$, find $D(a,r) \subset \mathcal{U}$ open disk. Cauchy's theorem in a disk implies $\int_\gamma = 0$ for any closed curve γ in $D(a,r)$. Morera's theorem implies f is holomorphic on $D(a,r)$, at a. So f is holomorphic on $\mathcal{U}$. □

zeroes of holomorphic maps

Let $f : D(a, R) \to \mathbb{C}$ be holomorphic, so $f(z) = \sum_{n \geq 0} c_n (z-a)^n$ on $D(a, R)$. If $f \not\equiv 0$ then some c_n is non-zero; let

$$m := \min\{n \in \mathbb{N} \cup \{0\} : c_n \neq 0\}.$$

If $m > 0$ then we say f has a *zero of order m at a*. In this case, we can write

$$f(z) = (z-a)^m g(z)$$

where $g(z)$ is holomorphic on $D(a, R)$, $g(a) \neq 0$.

Theorem (Principle of Isolated Zeroes)**.** If $f : D(a, R) \to \mathbb{C}$ is holomorphic, not identically 0, then there exists $0 < r \leq R$ such that $f(z) \neq 0$ on $0 < |z - a| < r$.

Proof. If $f(a) \neq 0$ then $f(z) \neq 0$ on $D(a, r)$ for some $0 < r \leq R$ by continuity of f. If f has a zero of order m at a, write $f(z) = (z-a)^m g(z)$ where $g(a) \neq 0$, g holomorphic. By continuity of g, there exists $0 < r \leq R$ such that $g(z) \neq 0$ for all $z \in D(a, r)$, so $f(z) \neq 0$ for all $0 < |z - a| < r$. □

Remark. Principle of isolated zeroes says that there is no accumulation point of the zero set of a holomorphic map inside its domain, unless $\equiv 0$.

Remark. It *is* possible for the zeroes of a holomorphic map to accumulate ouside its domain:

$$\sin z := \frac{e^{iz} - e^{-iz}}{2i} = 0 \iff e^{-z} = e^{-iz}$$

i.e. $e^{2iz} = 1$, which holds for all $z = n\pi$, $n \in \mathbb{Z}$. So $\sin\left(\frac{1}{z}\right)$ has zeroes accumulating at 0, on the boundary of its domain $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Remark. Another application of Principle of Isolated Zeroes: since $\cos^2 z + \sin^2 z = 1$ holds for all $z \in \mathbb{R}$, then $\cos^2 z + \sin^2 z - 1$ is entire with $\mathbb{R} \subset \{\text{zero set}\}$. So by PIZ, $\cos^2 z + \sin^2 z = 1$ for all $z \in \mathbb{C}$.

Proposition (Identity theorem for holomorphic functions)**.** Let $f, g : \mathcal{U} \to \mathbb{C}$ be holomorphic on a domain $\mathcal{U}$. Let $S := \{z \in \mathcal{U} : f(z) = g(z)\}$. If S has a non-isolated point (i.e. there exists $w \in S$ such that for all $\varepsilon > 0$, $D(w, \varepsilon) \setminus \{w\} \cap S \neq \emptyset$) then $f(z) = g(z)$ for all $z \in \mathcal{U}$.

Proof. Define $h(z) = f(z) - g(z)$, holomorphic on $\mathcal{U}$, and suppose w is non-isolated in S. Then for $\varepsilon > 0$ with $D(w, \varepsilon) \subseteq \mathcal{U}$, PIZ implies $h \equiv 0$ on $D(w, \varepsilon)$.

Given $z \in \mathcal{U}$, let $\gamma : [0,1] \to \mathcal{U}$ be a path with $\gamma(0) = w$, $\gamma(1) = z$. Consider the set

$$T := \{t \in [0,1] : h^{(n)}(\gamma(t)) = 0 \ \forall n \geq 0\}$$

Note that T is closed by definition. Since $h \equiv 0$ on $D(w, \varepsilon)$, Taylor expansion implies T is non-empty, since $0 \in T$. Define $t_0 := \sup\{t' \in [0,1] : t \in T \ \forall t \leq t'\}$. Then T closed and non-empty so $t_0 \in T$. Since $h^{(n)}(\gamma(t_0)) = 0$ for all $n \geq 0$, $h \equiv 0$ on a neighbourhood of $\gamma(t_0)$, contradicting the maximality of t_0, unless $t_0 = 1$. So $h(\gamma(1)) = 0$, i.e. $h(z) = 0$ as claimed. $\square$

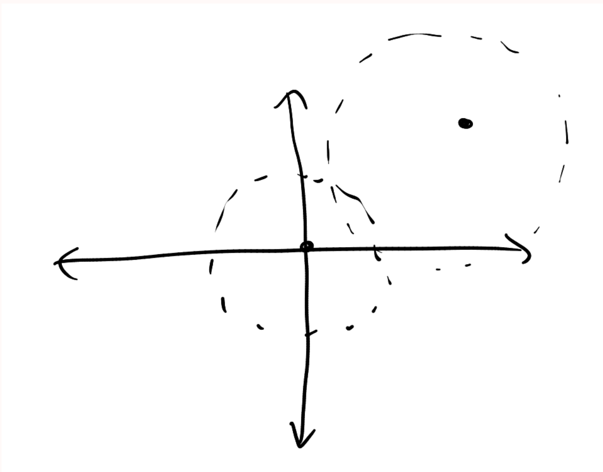
Definition (Analytic Continuation)**.** Let $\mathcal{U} \subseteq V \subseteq \mathbb{C}$ be domains and $f : \mathcal{U} \to \mathbb{C}$ is holomorphic. $g : V \to \mathbb{C}$ is an *analytic continuation* of f if:

(1) g is holomorphic on V

(2) $g|_{\mathcal{U}} = f$.

Example. (1) The series $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} z^n$ converges on $D(0,1)$, and takes the value $\mathrm{Log}(1+z)$ on $D(0,1)$. So $\mathrm{Log}(1+z)$ is an analytic continuation of this series to the domain $\mathbb{C} \setminus (-\infty, -1]$.

(2) $\sum_{n\geq 0} z^n$ has radius of convergence 1 about $a = 0$, and on $D(0,1)$, we have $\frac{1}{1-z} \sum_{n\geq 0} z^n$. So $\frac{1}{1-z}$ is an analytic continuation to $\mathbb{C} \setminus \{1\}$.



(3) Considering $f(z) = \sum_{n\geq 0} z^{2^n}$, f converges on $D(0,1)$ and cannot be analytically continued to any larger domain. We say $\partial D(0,1)$ is *natural boundary* for $f(z)$.

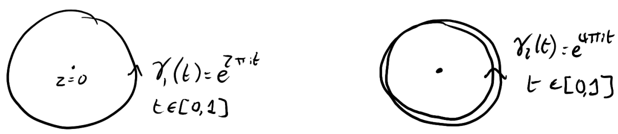
Start of lecture 8

Corollary (Global Maximum principle)**.** Let $\mathcal{U} \subseteq \mathbb{C}$ be a bounded domain, and let $\overline{\mathcal{U}}$ be its closure (the closure of $\mathcal{U}$ is the intersection of all closed supersets of $\mathcal{U}$). If $f : \overline{\mathcal{U}} \to \mathbb{C}$ is continuous and f is holomorphic on $\mathcal{U}$, then $|f|$ attains its maximum on $\overline{\mathcal{U}} \setminus \mathcal{U}$.

Proof. $\mathcal{U}$ bounded implies $\overline{\mathcal{U}}$ is bounded, hence $|f|$ has a maximum on $\mathcal{U}$, call it M. If $|f(z_0)| = M$ for $z_0 \in \mathcal{U}$, then local maximum principle implies $f \equiv f(z_0)$ on any disk $D(z_1, r) \subseteq \mathcal{U}$. By identity theorem, $f \equiv f(z_0)$ on $D(z_0, r)$ hence $f \equiv f(z_0)$ on $\mathcal{U}$, hence $f \equiv f(z_0)$ on $\overline{\mathcal{U}}$. So M is achieved by $|f|$ on $\overline{\mathcal{U}} \setminus \mathcal{U}$. $\square$

Generalise Cauchy's Integral Formula

Goal: generalise CIF by allowing more general closed curves for the integration. We have an issue:

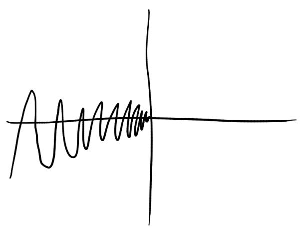


Then

$$\int_{\gamma_2} f = 2 \int_{\gamma_1} f$$

We need to deal with the issue of "winding around" a point more than once; however, once we correctly quantify this notion, we'll see it is the *only* issue to generalising CIF.

Naïve hope: "counting" crossings of a slit in the plane:



can happen infinitely often!

Theorem. Let $\gamma : [a, b] \to \mathbb{C} \setminus \{w\}$ be a continuous curve. Then there exists continuous function $\theta : [a, b] \to \mathbb{R}$ with

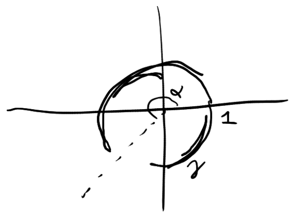
$$\gamma(t) = w + r(t)e^{i\theta t}$$

with $r(t) = |\gamma(t) - w|$.

Proof. WLOG translate to assume $w = 0$. Since $\arg \gamma(t) = \arg \frac{\gamma(t)}{|\gamma(t)|}$, we can replace γ with $\frac{\gamma}{|\gamma|}$ to assume $|\gamma(t)| = 1$ for all $t \in [a, b]$.

Notice that if $\gamma \subseteq \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, then $t \mapsto \operatorname{Arg}(\gamma(t))$ gives a continuous choice of θ. More generally, if γ lies in any slit plane $\mathbb{C} \setminus \left\{z : \frac{z}{e^{-\alpha}} \in \mathbb{R}_{\leq 0}\right\}$, then $\theta(t) := \alpha + \operatorname{Arg}\left(\frac{z}{e^{i\alpha}}\right)$ will do.

Our strategy is to subdivide γ so that its pieces lie in slit-planes, and so θ may be continuously defined on the pieces.



γ is continuous on $[a, b]$, so uniform continuous, and $\exists \varepsilon > 0$ such that $|s - t| < \varepsilon \implies |\gamma(s) - \gamma(t)| < 2$. Subdividing $a = a_0 < a_2 < \cdots < a_{n-1} < a_n = b$ with $a_{j+1} - a_j < 2\varepsilon$, then

$$\left|\gamma(t) - \gamma\left(\frac{a_{j+1} - a_j}{2}\right)\right| < 2 \qquad \forall t \in [a_j, a_{j+1}]$$

So $\gamma([a_{j-1}, a_j])$ lies in a slit plane, and we can define θ_j a continuous choice of argument for $\gamma|_{[a_{j-1}, a_j]}$ for $j \in \{1, \ldots, n\}$. We have

$$\gamma(a_j) = e^{i\theta_j(a_i)} = e^{i\theta_{j+1}(a_j)}$$

for $j \in \{1, \ldots, n-1\}$. So

$$\theta_{j+!}(a_j) = \theta_j(a_j) + 2\pi n_j$$

for some $n_j \in \mathbb{Z}$. Modifying each θ_j, $j \geq 2$, by a suitable integer multiple of 2π ensures the θ_j fit together to a continuous choice of θ on $[a, b]$. □

Remark. θ is not unique, since $\theta(t) + 2\pi n$ is also valid for all $n \in \mathbb{Z}$. If θ_1, θ_2 are two functions as in the theorem, then $\theta_1 - \theta_2$ is continuous, takes values in (discrete) $2\pi\mathbb{Z}$, so constant.

Definition (Winding Number)**.** Let $\gamma : [a, b] \to \mathbb{C}$ be a closed curve, $w \notin \gamma$. The *winding number* or *index* of γ about w is

$$I(\gamma; w) := \frac{\theta(b) - \theta(a)}{2\pi},$$

where $\gamma(t) = w + r(t)e^{i\theta(t)}$ with θ continuous.

Lemma (Winding Number Integral Formula)**.** Let $\gamma : [a, b] \to \mathbb{C} \setminus \{w\}$ be a closed curve. Then

$$I(\gamma; w) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - w}$$

Proof. γ piecewise-C^1 implies $r(t)$ and $\theta(t)$ are piecewise-C^1 as well, where $\gamma(t) = w + R(t)e^{i\theta(t)}$.

$$\begin{aligned}\int_\gamma \frac{\mathrm{d}z}{z - w} &= \int_a^b \frac{\gamma'(t)}{\gamma(t) - w} \mathrm{d}t \\ &= \int_a^b \frac{r'(t)}{r(t)} + i\theta'(t)\mathrm{d}t \\ &= [\ln r(t) + i\theta(t)]_{t=a}^{t=b} \\ &= 2\pi i I(\gamma; w)\end{aligned}$$

since γ is closed and $\theta(b) - \theta(a) = 2\pi I(\gamma; w)$. □

Proposition. If $\gamma : [0, 1] \to D(a, R)$ is a closed curve, then $\forall w \notin D(a, R)$, $I(\gamma; w) = 0$.

Proof. Consider the Möbius map $z \mapsto \frac{z-w}{a-w}$, This takes $a \mapsto 1$, $w \mapsto 0$, so $D(a, R) \mapsto D(1, r)$ for some $r < 1$. So then $D(a, R)$ is contained in the slit plane $\mathbb{C}\setminus\left\{z : \frac{z-w}{a-w} \in \mathbb{R}_{\le 0}\right\}$. So there is a branch of $\arg(z - w)$ defined on $D(a, r)$. And so

$$I(\gamma; w) = \frac{\arg(\gamma(1) - w) - \arg(\gamma(0) - w)}{2\pi} = 0$$ □

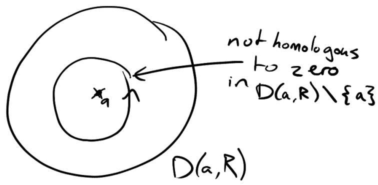
Definition (Homologous to zero)**.** Let $\mathcal{U} \subseteq \mathbb{C}$ be open. Then a closed curve γ in $\mathcal{U}$ is *homologous to zero in* $\mathcal{U}$ if $\forall w \notin \mathcal{U}$, $I(\gamma; w) = 0$.

Definition (Simply Connected)**.** $\mathcal{U}$ is *simply connected* if every closed curve in $\mathcal{U}$ is homologous to zero.

Remark. For $\mathcal{U}$ open this is equivalent to the homotopy definition of simply connected.

(1) Any disk is simply connected by previous proposition.

(2) Any punctured disk $D(a,R) \setminus \{a\}$ is not simply connected, since curves can wind around a:



> **Theorem** (General CIF)**.** Let $f : \mathcal{U} \to \mathbb{C}$ be holomorphic on a domain $\mathcal{U}$, and γ is a closed curve homologous to zero in $\mathcal{U}$. Then $\forall w \in \mathcal{U} \setminus \gamma$,
>
> $$I(\gamma; w) f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w} \mathrm{d}z,$$
>
> and $\int_\gamma f(z) \mathrm{d}z = 0$

Start of lecture 9

Proof. Notice applying the first equality to $g(z) = f(z)(z-w)$ gives $\int_\gamma f = 0$. So suffices to prove the first statement. We have by previous lemma that

$$I(\gamma, w) f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{z-w} \mathrm{d}z$$

so we want to show that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z) - f(w)}{z-w} \mathrm{d}z = 0 \forall w \in U \setminus \gamma$$

Consider the function

$$g(z,w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & z \neq w \\ f'(w) & z = w \end{cases}$$

This is a continuous function on $\mathcal{U} \times \mathcal{U}$. Want to show that

$$\int_\gamma g(z,w) \mathrm{d}z = 0 \forall w \in \mathcal{U} \setminus \gamma$$

Consider the auxiliary function h on $\mathbb{C}$:

$$h(w) := \begin{cases} \int_\gamma g(\zeta, w) \mathrm{d}\zeta & w \in \mathcal{U} \\ \int_\gamma \frac{f(\zeta)}{\zeta - w} \mathrm{d}\zeta & \underbrace{\{w \in \mathbb{C} \setminus \gamma : I(\gamma, w) = 0\}}_{=:V} \end{cases}$$

If $w \in \mathcal{U} \cap V$, then

$$\int_\gamma g(\zeta, w)\mathrm{d}\zeta = \int_\gamma \frac{f(\zeta) - f(w)}{\zeta - w}\mathrm{d}\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - w}\mathrm{d}\zeta$$

so h is well-defined. For any disk $D(0, R)$ with $\gamma \subseteq D(0, R)$, we have $I(\gamma; w) = 0$ for all $w \notin D(0, R)$. In fact, γ homologous to zero in $\mathcal{U}$, so $\mathcal{U} \cup V = \mathbb{C}$. For $w \notin D(0, R)$, we have

$$|h(w)| = \left|\int_\gamma \frac{f(\zeta)}{\zeta - w}\mathrm{d}\zeta\right| \leq \frac{\text{length}(\gamma) \cdot \sup_{\zeta \in \gamma} |f(\zeta)|}{|w| - R} \to 0$$

as $|w| \to \infty$.

Claim: h is holomorphic on $\mathbb{C}$. If so, h is bounded since $|h(w)| \to 0$ as $|w| \to \infty$. Then by Liouville's is constant, taking the value 0 on $\mathbb{C}$, concluding the proof.

Lemma. Let $\mathcal{U} \subseteq \mathbb{C}$ be open and $\phi : \mathcal{U} \times [a, b] \to \mathbb{C}$ continuous with $z \mapsto \phi(z, s)$ holomorphic on $\mathcal{U}$ for every $s \in [a, b]$. Then

$$g(z) := \int_a^b \phi(z, s)\mathrm{d}s$$

is holomorphic on $\mathcal{U}$.

Proof. Idea: Morera. WLOG, $\mathcal{U}$ is a disk. For any closed curve $\gamma : [0, 1] \to \mathcal{U}$ we have

$$\begin{aligned} \int_\gamma g(z)\mathrm{d}z &= \int_0^1 \left[\int_a^b \phi(\gamma(t), s)\mathrm{d}s\right] \gamma'(t)\mathrm{d}t \\ {}^* &= \int_a^b \left[\int_0^1 \phi(\gamma(t), s)\gamma'(t)\mathrm{d}t\right] \mathrm{d}s \end{aligned}$$

$*$ is Fubini's theorem: Suppose $f : [a, b] \times [c, d] \to \mathbb{C}$ is a continuous function. Then we have

$$\int_a^b \left(\int_c^d f(x, y)\mathrm{d}y\right) \mathrm{d}x = \int_c^d \left(\int_a^b f(x, y)\mathrm{d}x\right) \mathrm{d}y$$

and $x \mapsto \int_c^d f(x, y)\mathrm{d}y$ and $y \mapsto \int_a^b f(x, y)\mathrm{d}x$ are continuous. This clearly holds if f is constant, so also when f is a step function. Since $[a, b] \times [c, d]$ is closed and bounded, f is uniformly continuous. So f is a uniform limit of step functions, and so we exchange the order as claimed. End of proof of $*$.

$$\int_\gamma g(z)\mathrm{d}z = \int_a^b \left(\int_\gamma \phi(z, s)\mathrm{d}z\right) \mathrm{d}s$$

Since $z \mapsto \phi(z, s)$ is holomorphic, this is 0 by Cauchy in a disk. So

$$\int_\gamma g = 0$$

and by Morera, g is holomorphic as claimed. ☐

So h is holomorphic as claimed and the generalised CIF follows. $\square$

Corollary (Cauchy's theorem for simply connected domains)**.** Let $f : \mathcal{U} \to \mathbb{C}$ be holomorphic on simply connected domain $\mathcal{U}$. Then $\forall$ closed curves γ in $\mathcal{U}$,

$$\int_\gamma f = 0.$$

Fact: If $\mathcal{U} \subseteq \mathbb{C}$ is open, then $\mathcal{U}$ is simply connected if and only if the complement of $\mathcal{U}$ in $\mathbb{C}_\infty$ Is connected.

Examples

(i) $D(a, R) \subseteq \mathbb{C}$, has disk complement in $\mathbb{C}_\infty$ so simply connected.

(ii) Convex and starlike sets are simply connected.

(iii) Annulus *not* simply connected.

Isolated singularities of holomorphic maps

Definition (Isolated singularity)**.** A point $a \in \mathbb{C}$ is an *isolated singularity* of $f : \mathcal{U} \to \mathbb{C}$ holomorphic if $\exists r > 0$ such that f is holomorphic on $D(a, r) \setminus \{a\}$, denoted $D(a, r)^\times$.

Examples

(i) $a = 0$, $f(z) = \frac{\sin z}{z}$. Use the identity theorem or expansion of e^z to show that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

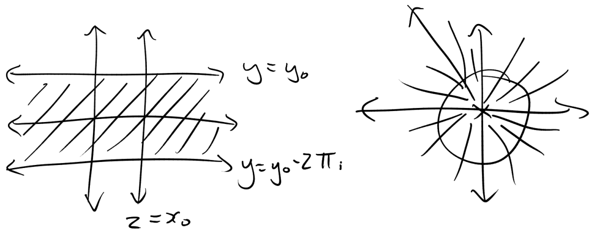
about 0. So

$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

about 0. So f is restriction of a holomorphic function on $\mathbb{C}$, call it f, and $f(0) = 1$.

(ii) $a = 0$, $g(z) = \frac{1}{z^6}$, g holomorphic on $\mathbb{C}^\times$, and $|g(z)| \to \infty$ as $|z| \to 0$, so there doesn't exist continuous extension at 0.

(iii) Recall the action $w \mapsto e^w = e^{\operatorname{Re} w} e^{i \operatorname{Im} w}$



So $h(z) = e^{\frac{1}{z}}$ maps any $D(0, \varepsilon)^\times$ to all of $\mathbb{C}^\times$.

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Theorem (Laurent expansion)**.** Let f be holomorphic on an annulus $A = \{z \in \mathbb{C} : r < |z - a| < R\}$, where $0 \le r < R \le \infty$. Then:

(i) f has a (unique) convergent expansion on A:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n$$

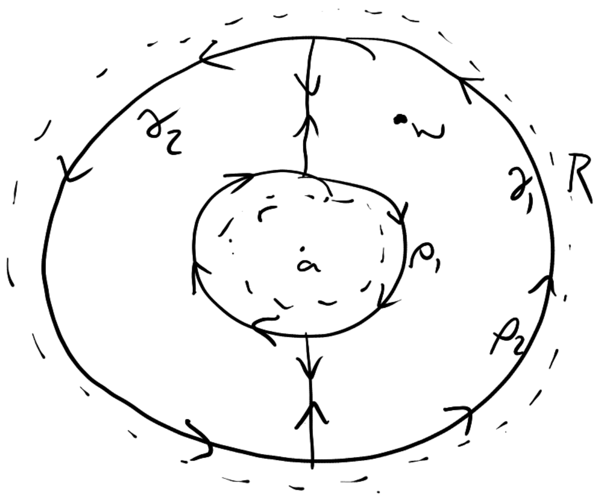
"Laurent series"

(ii) For any $r < \rho < R$, we have

$$c_n = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{n+1}} \mathrm{d}z$$

(iii) If $r < \rho' \le \rho < R$, the Laurent series converges uniformly on $\{z \in \mathbb{C} : \rho' \le |z - a| \le \rho\}$.

Proof. Fix $w \in A$, and choose $r < \rho_1 < |w - a| < \rho_2 < R$. Define two closed curves γ_1, γ_2 by a diameter of the annulus, labelled such that $I(\gamma_1; w) = 1$, $I(\gamma_2; w) = 0$.



γ_1, γ_2 are both homologous to zero in A, so by the generalised CIF we have

$$f(w) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(z)}{z-w}\mathrm{d}z = \frac{1}{2\pi i}\int_{\gamma_1+\gamma_2} \frac{f(z)}{z-w}\mathrm{d}z$$

Travelling $\gamma_1 + \gamma_2$ is the same as travelling $\partial D(a, \rho_2) - \partial D(a, \rho_1)$. So

$$f(w) = \underbrace{\frac{1}{2\pi i}\int_{|z-a|=\rho_1} \frac{f(z)}{z-w}\mathrm{d}z}_{I_2} - \underbrace{\frac{1}{2\pi i}\int_{|z-a|=\rho_1} \frac{f(z)}{z-w}\mathrm{d}z}_{I_1}$$

Using the same geometric series for $\frac{1}{1-\frac{w-a}{z-a}}$ to compute I_2 as in the Taylor series case gives

$$I_2 = \sum_{n=0}^{\infty} c_n (w-a)^n$$

where

$$c_n = \frac{1}{2\pi i}\int_{|z-a|=\rho_2} \frac{f(z)}{(z-a)^{n+1}}\mathrm{d}z$$

for $n \geq 0$. For I_1, using the expansion (since $|z-a| < |w-a|$)

$$-\frac{1}{z-w} = \frac{\frac{1}{w-a}}{1-\frac{z-a}{w-a}} = \sum_{m=1}^{\infty} \frac{(z-a)^{m-1}}{(w-a)^m},$$

gives

$$I_1 = \sum_{m=1}^{\infty} d_m (w-a)^{-m}$$

where

$$d_m = \frac{1}{2\pi i}\int_{|z-a|=\rho_1} \frac{f(z)}{(z-a)^{-m+1}}\mathrm{d}z \qquad \forall m \geq 1$$

Reindex with $n = -m$, we obtain the Laurent expansion for f.

To show (ii), (iii), suppose $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$ on A, and let $r < \rho' \leq \rho < R$. The non-negative power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ has Radius of Convergence $\geq R$, so converges uniformly on $D(a,\rho)$. Similarly, if $u = \frac{1}{z-a}$ the negative part of the Laurent expansion, $\sum_{n=1}^{\infty} c_{-n}u^n$ has Radius of Convergence $\geq \frac{1}{r}$, so converges uniformly on $\rho' \leq |z-a| \leq \rho$, so we can integrate term by term:

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{n+1}}\mathrm{d}z &= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty} c_n \int_{\partial D(a,\rho)} (z-a)^{n-m-1}\mathrm{d}z \\ &= c_m\end{aligned}$$

since this integral $= 0$ unless $n - m - 1 = -1$, in which case it is $2\pi i$. □

Remark. Proof shows $f = f_1 + f_2$, f_1 holomorphic on $D(a,R)$, and f_2 holomorphic on $|z-a| > r$. Applying when $r = 0$, we have three possibilities on a punctured disk domain i.e. an isolated singularity at $z = a$.

(1) $c_n = 0\ \forall n < 0$. Then f is the restriction to $D(a,R)^\times$ of a function holomorphic on $D(a,R)$. We say f has a *removable singularity* at a. For example, $f(z) = \frac{\sin z}{z}$ at $a = 0$.

(2) $\exists k < 0$ such that $c_k \neq 0$ but $c_n = 0$ for all $n < k$¿ We have $(z-a)^{-k}f(z)$ holomorphic and non-zero at a. We say f has a *pole of order* $-k$ *at* a. For example, $g(z) = \frac{1}{z^6}$ at $a = 0$.

(3) $c_n \neq 0$ for infinitely many $n < 0$. f has an essential singularity at a. For example, $e^{\frac{1}{z}}$ at $a = 0$.

Proposition. An isolated singularity at $z = a$ for f is removable if and only if $\lim_{z\to a}(z-a)f(z) = 0$.

Proof. Forwards direction is clear. For backwards direction, consider

$$g(z) = \begin{cases} (z-a)^2 f(z) & z \neq a \\ 0 & z = a \end{cases}$$

$g'(a) = \lim_{z\to a}(z-a)f(z) = 0$, so g is holomorphic at a, with $g(a) = 0$. So $g(z) = \sum_{n=2}^{\infty} c_n(z-a)^n$. So $f(z) = \sum_{n=0}^{\infty} c_{n+2}(z-a)^n$, so is holomorphic at a. □

Proposition. An isolated singularity at $z = a$ for f is a pole $\iff |f(z)| \to \infty$ as $z \to a$.
Furthermore, the following are equivalent:

(1) f has a pole of order k at $z = a$.

(2) $f(z) = (z-a)^{-k}g(z)$, where g is holomorphic and nonzero at a.

(3) $f(z) = \frac{1}{h(z)}$ where h is holomorphic at a with a zero of order k at a.

Proof. (1) $\iff$ (2) is immediate using Laurent expansion. (2) $\iff$ (3) since g is holomorphic, nonzero at a $\iff$ $\frac{1}{g}$ is holomorphic at a.

If f has a pole of order k at $z = a$, then $f(z) = (z-a)^{-k}g(z)$, so $|f(z)| \to \infty$ as $z \to a$. Convsersely if $|f| \to \infty$ as $z \to a$, then there exists $r > 0$ such that $f(z) \neq 0$ for all $0 < |z-a| < r$. So $\frac{1}{f}$ is holomorphic on $D(a,r)^\times$, and $\left|\frac{1}{f}\right| \to 0$ as $z \to a$, so the singularity for $\frac{1}{f}$ is removable, and $\frac{1}{f(z)} = h(z)$, holomorphic h on $D(a,r)$. h has a zero of order k, so $h(z) = (z-a)^k l(z)$ for l holomorphic and nonzero at a, so $f(z) = (z-a)^{-k}g(z)$, i.e. f has a pole of order k. at $z = a$. □

Corollary. An isolated singularity at $z = a$ is essential $\iff |f|$ does not approach a limit in $\mathbb{R} \cup \{\infty\}$ as $z \to a$.

Start of lecture 11

Theorem (Casorati-Weierstrass)**.** $f : D(a,R)^\times \to \mathbb{C}$ with essential singularity at $z = a$. Then f has dense image on any neighbourhood of a; that is, $\forall w \in \mathbb{C}$, $\forall \varepsilon > 0$, $\forall \delta > 0$ then $\exists z \in D(a,\delta)^\times$ such that $|f(z) - w| < \varepsilon$.

Proof. Example Sheet 2. □

More difficult: "great Picard theorem". If $z = a$ is an essential singularity of f, then $\exists b \in \mathbb{C}$ such that $\forall \varepsilon > 0$, $\mathbb{C} \setminus \{b\} \subseteq f((D(a,\varepsilon)^*)$.

Exercise: $f(z) = e^z$ has an essential singularity at ∞, and takes every non-zero value on every neighbourhood of ∞.

Remark. An advantage of the Riemann sphere perspective: if $f : D(a,R)^* \to \mathbb{C}$ has a pole at $z = a$, we can view f as a continuous map $f : D(a,R) \to \mathbb{C}_\infty$, with $f(a) = \infty$. f is "holomorphic at a" in the $\mathbb{C}_\infty$ sense since $\frac{1}{f}$ is holomorphic on a neighbourhood of a, with a zero of the same order as the pole of f.

Definition. Suppose D is a domain. A function f is *meromorphic* on D if $f : D \setminus S \to \mathbb{C}$ is holomorphic, where S is a set of isolated singularities for f which are removable or poles.

Definition. Let $f : D(a, R)^* \to \mathbb{C}$ be holomorphic with Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$. The *residue* of f at $z = a$ is

$$\operatorname{Res}_{z=a} f(z) := c_{-1} \in \mathbb{C}$$

Definition. Let $f : D(a, R)^* \to \mathbb{C}$ be holomorphic with Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$. The *principal part* of f at $z = a$ is

$$\sum_{n=-\infty}^{-1} c_n(z-a)^n$$

Proposition. Let γ be a closed curve in $D(a, R)^*$. Then

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i I(\gamma; a) \operatorname{Res}_{z=a} f(z)$$

Proof. Using uniform convergence of Laurent expansion of f, we have that:

$$\int_\gamma f(z)\mathrm{d}z = \sum_{n=-\infty}^{\infty} c_n \left[\int_\gamma (z-a)^n \mathrm{d}z\right]$$

Since

$$\int_\gamma (z-a)^n \mathrm{d}z = \begin{cases} 0 & n \neq -1 \\ 2\pi i I(\gamma; a) & n = -1 \end{cases}$$

the proposition is proved. $\square$

If f is meromorphic on a domain D, and $z = a$ is a pole of f in D, then its principal part at $z = a$ is of the form

$$\frac{c_{-k}}{(z-a)^k} + \frac{c_{-k+1}}{(z-a)^{k-1}} + \cdots + \frac{c_{-1}}{z-a}$$

a polynomial in $\frac{1}{z-a}$, and can be written as $\frac{p(z)}{(z-a)^k}$ for some polynomial p. So the principal part of f at $z = a$ is holomorphic on $\mathbb{C} \setminus \{a\}$.

More generally, if f is meromorphic on D, and $\{a_1, \ldots, a_m\} \subseteq \{\text{poles of } f \text{ in } D\}$, with $p_i(x)$ the principal part of f at $z = a_i$, then the function

$$g(z) = f(z) - \sum_{i=1}^{m} p_i(z)$$

is meromorphic on D, with removable singularities at $a_1, \ldots, a_m$.

Theorem (Residue Theorem)**.** Let f be meromorphic on a domain D, and γ a curve which is homologous to zero in D. Suppose γ does not contain any pole of f, and f has only finitely many poles in D with non-zero winding number for γ; call them $\{a_1, \ldots, a_m\}$. Then

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \sum_{i=1}^{m} I(\gamma; a_i) \operatorname{Res}_{z=a_i} f(z)$$

Proof. Let P_i denote the principal part of f at $z = a_i$, and write $g = f - \sum_{i=1}^{\infty} P_i$. Then by Cauchy's theorem,

$$\int_\gamma g = 0, \qquad \text{i.e.} \qquad \int_\gamma f = \sum_{i=1}^{m} \int_\gamma P_i$$

Each P_i is holomorphic on $\mathbb{C} \setminus \{a_i\}$ as we argued, so by the previous proposition we have

$$\int_\gamma P_i = 2\pi i I(\gamma; a_i) \operatorname{Res}_{z=a_i} P_i(z)$$

By definition, $\operatorname{Res}_{z=a_i} P_i(z) = \operatorname{Res}_{z=a_i} f(z)$, so

$$\int_\gamma f = 2\pi i \sum_{i=1}^{\infty} I(\gamma; a_i) \operatorname{Res}_{z=a_i} f(z)$$

□

Remarks

(∗) If γ is homologous to 0 in a domain D, then $\{z \in \mathbb{C} : I(\gamma; z) \neq 0 \text{ or } z \in \gamma\}$ is a closed set and a bounded set. Notice that the winding number is a continuous function on $\mathbb{C} \setminus \gamma$, taking values in a discrete set, then $\{z \in \mathbb{C} \setminus \gamma : I(\gamma; z) = 0\}$ is open. So the complement is closed. Since the polws of f are isolated, this closed bounded set contains only finitely many of them (Bolzano-Weierstrass).

(1) f holomorphic on D: Residue theorem implies Cauchy's theorem.

(2) $f(z) = \frac{g(z)}{z-a}$. Then $\operatorname{Res}_{z=a} f(z) = g(a)$, so Residue theorem implies CIF.

(3) We say a closed curve γ *bounds a domain* $\mathcal{U}$ if

$$I(\gamma; z) = \begin{cases} 1 & z \in \mathcal{U} \\ 0 & z \notin \mathcal{U} \end{cases}$$

If γ is a closed curve in a domain D which bounds a domain $\mathcal{U}$, and f is holomorphic on D, then $\int_\gamma f = 0$ and

$$\int_\gamma \frac{f(z)}{z-w} \mathrm{d}z = 2\pi i f(w) \qquad \forall w \in \mathcal{U} \setminus \gamma$$

If f is meromorphic on D with no poles on γ, then

$$\int_\gamma f = 2\pi i \sum_{w \text{ poles in } \mathcal{U}} \mathrm{Res}_{z=w} f(z)$$

Start of lecture 12

Remark (Jordan Curve Theorem)**.** Every simple closed continuous curve in the plane separates $\mathbb{C}$ into two connected components, one bounded, one unbounded.

Computing residues

(i) If f has a simple (= order 1) pole at $z = a$, then the Laurent expansion at a is

$$f(z) = \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$$

so

$$\mathrm{Res}_{z=a}(f(z)) = \lim_{z \to a} (z-a) f(z)$$

Example. $f(z) = \frac{1}{1+z^2}$ at $z = i$: $(z-i)f(z) = \frac{1}{z+i}$, so $\mathrm{Res}_{z=i} f(z) = \frac{1}{2i}$.

(ii) If $f = \frac{g(z)}{h(z)}$, where g is holomorphic and non-zero at $z = a$, and h is holomorphic and has a simple zero at $z = a$:

$$g(z) = g(a) + (z-a)\tilde{g}(z)$$

$\tilde{g}$ holomorphic at $z = a$,

$$h(z) = h'(a)(z-a)\tilde{h}(z)$$

$\tilde{h}(a) = 1$ at $z = a$, and is holomorphic at a. So

$$\frac{g(z)}{h(z)} = \frac{g(a)}{h'(a)(z-a)\tilde{h}(z)} + \boxed{\frac{\tilde{g}(z)}{h'(z)\tilde{h}(z)}}$$

(the boxed expression is holomorphic at a). Applying (i) to $\frac{g(a)}{h'(a)(z-a)\tilde{h}(z)}$, we see that

$$\operatorname{Res}_{z=a} f(z) = \frac{g(a)}{h'(a)}$$

Example. $f(z) = \frac{e^z}{z^2+1}$ at $z = i$. (ii) implies

$$\operatorname{Res}_{z=i} f(z) \frac{e^i}{2i}$$

(iii) If $f(z) = \frac{g(z)}{(z-a)^k}$, g holomorphic at a. Then $\operatorname{Res}_{z=a} f(z)$ is the coefficient of $(z-a)^{k-1}$ in the expansion of g, which is

$$\frac{f^{(k-1)}(a)}{(k-1)!}$$

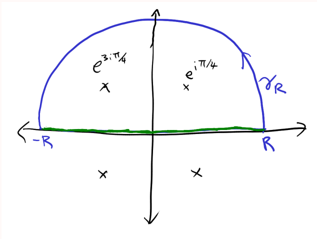
Let's explore applications to real integrals.

Example. Evaluate $\int_0^\infty \frac{1}{1+x^4}\mathrm{d}x$. Note:

(1) $\frac{1}{1+x^4} = \frac{1}{1+(-x)^4}$

(2) $|x| \gg 1 \implies \left|\frac{1}{1+x^4} \ll 1\right|$.

Consider:



$1+x^4$ has 4 simple zeroes: $e^{\pi i/4}$, $e^{3\pi i/4}$, $e^{5\pi i/4}$ and $e^{7\pi i/4}$. γ_R has winding number 1 around $e^{\pi i/4}$, $e^{3\pi i/4}$, and 0 around the others. $\mathrm{Res}_{z=e^{\pi i/4}} \frac{1}{1+z^4} = \frac{1}{4e^{3\pi i/4}}$, and $\mathrm{Res}_{z=e^{3\pi i/4}} \frac{1}{1+z^4} = \frac{1}{4e^{9\pi i/4}} = \frac{1}{4e^{\pi i/4}}$. (Computed using (ii), with $g(z) \equiv 1$, $h(z) = 1+z^4$). We have:

$$\int_{\gamma_R} \frac{1}{z^4+4}\mathrm{d}z = \underbrace{\int_{C_R'} \frac{1}{1+z^4}\mathrm{d}z}_{I_1} + \underbrace{\int_{-R}^{R} \frac{1}{1+z^4}\mathrm{d}z}_{I_2}$$

For I_1, parametrise $z = Re^{i\theta}$, $\theta \in [0,\pi]$. Then

$$I_1 = \int_0^\pi \frac{1}{1+R^4e^{4i\theta}} iRe^{i\theta}\mathrm{d}\theta$$

$|I_1| \le \frac{\pi R}{R^4+1} \to 0$ as $R \to \infty$. So

$$\begin{aligned} I_2 &= \int_{\gamma_R} \frac{1}{1+z^4}\mathrm{d}z - \int_{C_R'} \frac{1}{1+z^4}\mathrm{d}z \\ &= 2\pi i \left[\frac{1}{4e^{3\pi i/4}} + \frac{1}{4e^{\pi i/4}}\right] - \int_{C_R'} \frac{1}{1+z^4}\mathrm{d}z \\ &\to 2\pi i \left[\frac{1}{4e^{3\pi i/4}} + \frac{1}{4e^{\pi i/4}}\right] - 0 \end{aligned}$$

So

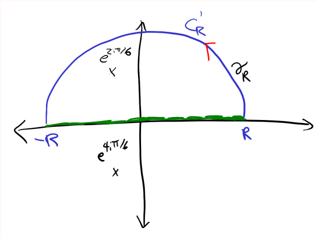
$$I_2 \to \frac{1}{2}\pi i \left(e^{-3\pi i/4} + e^{-\pi i/4}\right) = \frac{1}{2}\pi i \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \frac{\pi}{\sqrt{2}}$$

So $\int_0^\infty \frac{1}{1+z^4}\mathrm{d}z = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{1+z^4}\mathrm{d}z = \frac{\pi}{2\sqrt{2}}$.

Example. Compute $\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x+x^2} \mathrm{d}x$. Note

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{ix-y} + e^{-ix+y}}{2}$$

However, $e^{ix} = \cos x + i \sin x$, so the function $\frac{e^{ix}}{1+x+x^2}$ has real part $\frac{\cos x}{1+x+x^2}$ on $\mathbb{R}$. Notice then $e^{i(x+iy)} = e^{ix-y}$, so this function is bounded above by 1 in modulus for $y \geq 0$.



Roots of $1 + x + x^2$ are $e^{2\pi i/3}$, $e^{4\pi i/3}$, γ_R winds around $e^{2\pi i/3}$ with winding number 1.

$$\int_{\gamma_R} \frac{e^{iz}}{1+z+z^2} \mathrm{d}z = \underbrace{\int_{C_R'} \frac{e^{iz}}{1+z+z^2}}_{I_1} + \underbrace{\int_{-R}^{R} \frac{e^{iz}}{1+z+z^2}}_{I_2}$$

$|I_1| \leq \text{length}(C_R') = \frac{1}{R^2-R-1} = \frac{\pi R}{R^2-R-1} \to 0$ as $R \to \infty$. We have

$$\mathrm{Res}_{z=e^{2\pi i/3}} \frac{e^{iz}}{1+z+z^2} = \frac{e^{ie^{2\pi i/3}}}{1+2e^{2\pi i/3}}$$

so $I_2 \to 2\pi i \left[\frac{e^{ie^{2\pi i/3}}}{1+2e^{2\pi i/3}}\right] - 0$ (as $R \to \infty$). $e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, so $1 + 2e^{2\pi i/3} = \sqrt{3}i$.

$$e^{ie^{2\pi i/3}} = e^{i\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)} = e^{-i/2}e^{-\sqrt{3}/2}$$

so $I_2 \to 2\pi i \left[\frac{e^{-i/2}e^{-\sqrt{3}/2}}{\sqrt{3}i}\right] = \frac{2\pi}{\sqrt{3}}e^{-\sqrt{3}/2}e^{-i/2}$. So

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos x}{1+x+x^2} &= \mathrm{Re}\left(\frac{2\pi}{\sqrt{3}}e^{-\sqrt{3}/2}e^{-i/2}\right) \\ &= \frac{2\pi}{\sqrt{3}}e^{-\sqrt{3}/2}\cos\left(-\frac{1}{2}\right)\end{aligned}$$

Lemma (Jordan's Lemma)**.** Suppose $f(z)$ is holomorphic on $\{|z| > r\}$ for some $r > 0$, and $zf(z)$ is bounded. Then for all $\alpha > 0$, we have

$$\int_{C_R'} f(z)e^{i\alpha z}\mathrm{d}z \to 0$$

as $R \to \infty$, where $C_R' : [0, \pi] \to \mathbb{C}$, $C_R'(t) = Re^{it}$.

Proof. We have for $z = Re^{it}$, that $|e^{i\alpha z}| = e^{-\alpha R \sin t}$, and so using the basic estimate $\frac{\sin t}{t} \ge \frac{2}{\pi}$ on $[0, \pi/2]$ (since $\frac{\sin t}{t}$ decreases on $[0, \pi/2]$), we have

$$|e^{i\alpha z}| \le \begin{cases} e^{-\alpha R \frac{2t}{\pi}} & t \in [0, \pi/2] \\ e^{-\alpha R \frac{2t'}{\pi}} & t' = \pi - t, t \in [\pi/2, \pi] \end{cases}$$

BY hypothesis, there exists $M \in \mathbb{R}$ such that $|zf(z)| \le M$. Putting these together, let $\tilde{C_R'}$ be C_R' for $[0, \pi/2]$. Then

$$\begin{aligned} \left| \int_{\tilde{C_R'}} f(z)e^{i\alpha z}\mathrm{d}z \right| &\le \int_0^{\pi/2} Me^{-\alpha R \frac{2t}{\pi}} \mathrm{d}t \\ &= \left[M \left(\frac{1}{-\alpha R \frac{2}{\pi}} \right) e^{-\alpha R \frac{2t}{\pi}} \right]_{t=0}^{t=\pi/2} \\ &= \frac{(1 - e^{-\alpha R}\pi M}{2R\alpha} \\ &\to 0 \end{aligned}$$

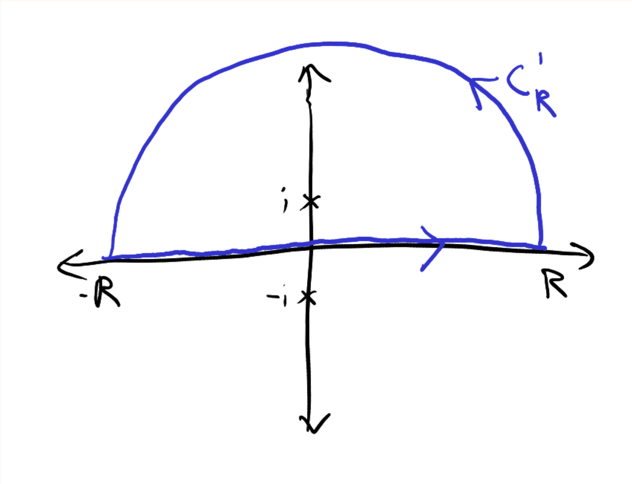
as $R \to \infty$. Similarly for $t \in [\pi/2, \pi]$. □

Start of
lecture 13

Example. Evaluate $\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+1} \mathrm{d}x$, $m \in \mathbb{R}$. $\cos z$ is large for $iR = z$ large R, so instead

$$\cos mx = \mathrm{Re}(\varepsilon(imx)) \implies I = \mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+1} \mathrm{d}x\right)$$

Useful contour:



Call this γ_R. If $m > 0$, Jordan's lemma implies $\int_{C_R'} \frac{e^{imz}}{z^2+1} \mathrm{d}z \to 0$ as $R \to \infty$. Residue theorem gives

$$\begin{aligned}
\int_{\gamma_R} \frac{e^{imz}}{z^2+1} \mathrm{d}z &= 2\pi i \,\mathrm{Res}_{z=i} \frac{e^{imz}}{z^2+1} \\
&= 2\pi i \cdot \frac{e^{im(i)}}{1+i} \\
&= \pi e^{-m}
\end{aligned}$$

So

$$\pi e^{-m} = \int_{C_R'} \frac{e^{imz}}{z^2+1} \mathrm{d}z + \int_{-R}^{R} \frac{e^{imz}}{z^2+1} \mathrm{d}z \implies I = \frac{\pi}{e^m}, \quad m > 0$$

If $m < 0$, $\cos(mx) = \cos(-mx)$, so $I = \frac{\pi}{e^{-m}}$ by previous computation. If $m = 0$, we have

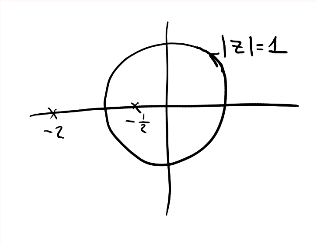
$$\left| \int_{C_R'} \frac{1}{z^2+1} \mathrm{d}z \right| \leq \frac{\pi R}{R^2 - 1} \to 0$$

as $R \to \infty$, so with the residue computation $\mathrm{Res}_{z=i} \frac{1}{z^2+1} = \frac{1}{2i}$ we have $I = \frac{\pi}{e^0} = \pi$. So in all cases, $I = \frac{\pi}{e^{|m|}}$.

Example. Evaluate $\int_0^{2\pi} \frac{1}{5+4\cos\theta}\mathrm{d}\theta$. Let's use $\cos\theta = \frac{1}{2}[e^{i\theta} + e^{-i\theta}]$, so $\cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right]$ for $z = e^{i\theta}$. So $\mathrm{d}z = ie^{i\theta} = iz\mathrm{d}\theta$.

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{5+4\cos\theta}\mathrm{d}\theta &= \int_{|z|=1} \frac{1}{5+4\left(\frac{z+\frac{1}{z}}{2}\right)} \cdot \frac{\mathrm{d}z}{iz} \\
&= \frac{1}{i}\int_{|z|=1} \frac{1}{2z^2+5z+2}\mathrm{d}z \\
&= \frac{1}{i}\int_{|z|=1} \frac{1}{(2z+1)(z+2)}\mathrm{d}z
\end{aligned}$$

So we have



with winding number 1 around $z = -\frac{1}{2}$. CIF applied to $\frac{1}{2(z+2)}$ says

$$\frac{1}{2\left(-\frac{1}{2}+2\right)} = \frac{1}{2\pi i}\int_{|z|=1} \frac{1}{2(z+2)\left(z+\frac{1}{2}\right)}\mathrm{d}z$$

so

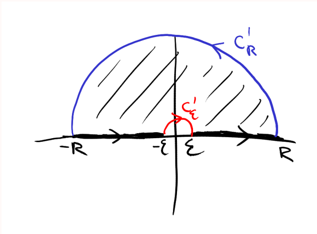
$$\frac{2\pi i}{3} = \int_{|z|=1} \frac{1}{2z^2+5z+2}\mathrm{d}z = i\int_0^{2\pi} \frac{1}{5+4\cos\theta}\mathrm{d}\theta$$

so $\int_0^{2\pi} \frac{1}{5+4\cos\theta}\mathrm{d}\theta = \frac{2\pi}{3}$.

Example. Evaluate $\int_0^\infty \frac{\sin x}{x} \mathrm{d}x$. Consider

$$\begin{aligned}\frac{1}{2i}\int_0^\infty \frac{e^{ix}-e^{-ix}}{x}\mathrm{d}x &= \frac{1}{2i}\int_0^\infty \frac{e^{ix}}{x}\mathrm{d}x - \frac{1}{2i}\int_0^{-\infty}\frac{e^{it}}{-t}(-\mathrm{d}t) \\ &= \frac{1}{2i}\int_{-\infty}^\infty \frac{e^{ix}}{x}\mathrm{d}x\end{aligned}$$

Modify by considering $\gamma_{R,\varepsilon}$ contour.



Cauchy's theorem gives $\int_{\gamma_{R,\varepsilon}} \frac{e^{iz}}{z}\mathrm{d}z = 0$. Jordan's lemma gives $\int_{C'_R} \frac{e^{iz}}{z}\mathrm{d}z \to 0$ as $R \to \infty$. On C'_ε, $z = \varepsilon e^{i\theta}$, $\mathrm{d}z = i\varepsilon e^{i\theta}\mathrm{d}\theta = iz\mathrm{d}\theta$, so

$$\int_{C_\varepsilon} \frac{e^{iz}}{z}\mathrm{d}z = \int_0^\pi e^{i\varepsilon e^{i\theta}} i\mathrm{d}\theta \to i\int_0^\pi 1\mathrm{d}\theta = \pi i$$

as $\varepsilon \to 0$. So

$$\int_{\gamma_{R,\varepsilon}} \frac{e^{iz}}{z}\mathrm{d}z = \int_{C'_R}\frac{e^{iz}}{z}\mathrm{d}z + \int_{-R}^{-\varepsilon}\frac{e^{iz}}{z}\mathrm{d}z - \int_{C'_\varepsilon}\frac{e^{iz}}{z}\mathrm{d}z + \int_\varepsilon^R \frac{e^{iz}}{z}\mathrm{d}z$$

As $\varepsilon \to 0$, $R \to \infty$, we obtain

$$0 = \int_{-\infty}^\infty \frac{e^{iz}}{z}\mathrm{d}z - \pi i$$

So

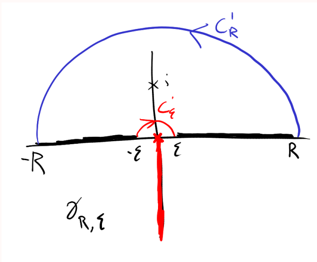
$$\int_0^\infty \frac{\sin x}{x}\mathrm{d}x = \frac{1}{2i}\pi i = \frac{\pi}{2}$$

Example. Evaluate $\int_0^\infty \frac{x^\alpha}{x^2+1}\mathrm{d}x$, $\alpha \in (0,1)$. $z^\alpha = \exp(\alpha \log z)$, branch of log.

Claim: Let $\log z = \ln|z| + i \arg z$, $\arg z \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Then for $x > 0$, we have $(-x)^\alpha = (-1)^\alpha x^\alpha$. Proof of the claim: $\log(-x) = \ln|x| + \pi i = \ln x + \pi i$ since $x > 0$. In particular, $\log(-1) = \pi i$. So $\log x + \log(-1) = \ln x + \pi i = \log(-x)$. So

$$\exp(\alpha \log x)\exp(\alpha \log(-1)) = \exp(\alpha \log(-x))$$

as claimed.



So consider $\gamma_{R,\varepsilon}$ as in previous example. Can show integrals along $C_R', C_\varepsilon' \to 0$ as $R \to \infty$, $\varepsilon \to 0$. Residue theorem:

$$\mathrm{Res}_{z=i} \frac{\exp(\alpha \log z)}{(z+i)(z-i)} = \frac{i^\alpha}{2i}$$

So

$$\begin{aligned} 2\pi i\, \mathrm{Res}_{z=i} \frac{\exp(\alpha \log z)}{(z+i)(z-i)} &= \int_{\gamma_{R,\varepsilon}} \frac{z^\alpha}{1+z^2}\mathrm{d}z \\ &= \int_{C_R'} \frac{z^\alpha}{1+z^2}\mathrm{d}z - \int_{C_\varepsilon'} \frac{z^\alpha}{z^2+1}\mathrm{d}z + \int_{-R}^{-\varepsilon} \frac{z^\alpha}{z^2+1} + \int_\varepsilon^R \frac{z^\alpha}{z^2+1}\mathrm{d}z \end{aligned}$$

By substitution $t = -z$, we have

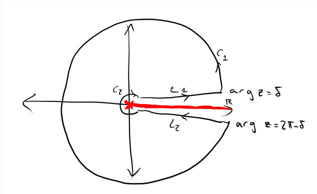
$$\int_{-R}^{-\varepsilon} \frac{z^\alpha}{1+z^2}\mathrm{d}z = (-1)^\alpha \int_\varepsilon^R \frac{z^\alpha}{z^2+1}\mathrm{d}z$$

So taking $\varepsilon \to 0$, $R \to \infty$, we have

$$2\pi i \frac{i^\alpha}{2i} = 0 - 0 + [(-1)^\alpha + 1] \int_0^\infty \frac{x^\alpha}{1+x^2}\mathrm{d}x$$

so $\int_0^\infty \frac{x^\alpha}{1+x^2}\mathrm{d}x = \frac{\pi i^4}{1+(-1)^\alpha}$.

Example. Evaluate $\int_0^\infty \frac{x^{1/3}}{(x+2)^2}\mathrm{d}x$. Let's define $\log z = \ln|z| + i\arg z$, $\arg z \in (0, 2\pi)$. We'll consider a "keyhole contour", γ. Integral on γ of $\frac{z^{1/3}}{(z+2)^2}$.



On C_1:

$$\left|\int_{C_1} \frac{z^{1/3}}{(z+2)^2}\mathrm{d}z\right| \le (2\pi - 2\delta)R \cdot \frac{R^{1/3}}{(R-2)^2} \to 0$$

as $R \to \infty$. On C_2:

$$\left|\int_{C_2} \frac{z^{1/3}}{(z-2)^2}\mathrm{d}z\right| \le (2\pi - 2\delta)\varepsilon\frac{\varepsilon^{1/3}}{(2-\varepsilon)^2} \to 0$$

as $\varepsilon \to 0$. On L_1, $z = te^{i\delta}$, $t \in [\varepsilon, R]$, $\mathrm{d}z = e^{i\delta}\mathrm{d}t$.

$$\int_\varepsilon^R \frac{t^{1/3}e^{i\delta/3}}{(te^{i\delta}+2)^2}e^{i\delta}\mathrm{d}t \to \int_\varepsilon^R \frac{t^{1/3}}{(t+2)^2}\mathrm{d}t$$

as $\delta \to 0$. On L_2, $z = te^{i(2pi-\delta)}$,

$$\int_\varepsilon^R \frac{t^{1/3}e^{i\frac{2\pi-\delta}{3}}}{(te^{i(2\pi-\delta)}+2)^2}e^{i(2\pi-\delta)}\mathrm{d}t \to e^{2\pi i/3}\int_\varepsilon^R \frac{t^{1/3}}{(t+2)^2}\mathrm{d}t$$

So we have by residue theorem,

$$O\left(\frac{1}{R^{2/3}}\right) - e^{2pii/3}\int_\varepsilon^R \frac{t^{1/3}}{(t+2)^2}\mathrm{d}t - O\left(\varepsilon^{4/3}\right) + \int_\varepsilon^R \frac{t^{1/3}}{(t+2)^2}\mathrm{d}t = \mathrm{Res}_{z=-2}\frac{z^{1/3}}{(z+2)^2}\cdot 2\pi i$$

Then taking $\varepsilon \to 0$ and $R \to \infty$ we get

$$(1 - e^{2\pi i/3})\int_0^\infty \frac{t^{1/3}}{(t+2)^2}\mathrm{d}t = 2\pi i\,\mathrm{Res}_{z=-2}\frac{z^{1/3}}{(z+2)^2}$$

Using residue computation trick (iii), this residue is

$$\left.\frac{\mathrm{d}}{\mathrm{d}z}\right|_{z=-2} = \left.\frac{\mathrm{d}}{\mathrm{d}z}\right|_{z=-2}\exp\left(\frac{1}{3}\log z\right) = \frac{1}{3z}\exp\left(\frac{1}{3}\log z\right)\bigg|_{z=-2}$$

so $\mathrm{Res}_{z=-2}\frac{z^{1/3}}{(z+2)^2} = -\frac{1}{6}\sqrt[3]{2}e^{\pi i/3}$. Can compute $\frac{e^{\pi i/3}}{(1-e^{2\pi i/3}} = \frac{i}{\sqrt{3}}$, so $\int_0^\infty \frac{t^{1/3}}{(t+2)^2}\mathrm{d}t = \frac{\pi}{3\sqrt{3}}\sqrt[3]{2}$.

Start of lecture 14

Proposition. Let f have a zero (respectively pole) of order $k > 0$ at $z = a$. Then $\frac{f'(z)}{f(z)}$ has a simple pole at $z = a$, of residue k (respectively $-k$).

Remark. By Example Sheet 2, if $f : \mathcal{U} \to \mathbb{C}$ with $f(\mathcal{U})$ contained in a simply connected set which omits 0, then there exists holomorphic function $g(z) = \log f(z)$ on $\mathcal{U}$, so $\frac{f'(z)}{f(z)}$ has holomorphic antiderivative $\log f$ on $\mathcal{U}$. We call $\frac{f'(z)}{f(z)}$ the "logarithmic derivative" of f.

Proof. Suppose $f(z) = (z-a)^k g(z)$ near a, with $g(a) \neq 0$, then $f'(z) = k(z-a)^{k-1}g(z) + (z-a)^k g'(z)$, so

$$\frac{f'(z)}{f(z)} = \frac{k}{z-a} + \frac{g'(z)}{g(z)}$$

Since $g(a) \neq 0$, $\frac{g'}{g}$ is holomorphic at a. So $\mathrm{Res}_{z=a} \frac{f'}{f} = k$. (Similarly for the pole case). □

Theorem (Argument Principle)**.** Let γ be a closed curve bounding a domain D, and f a function meromorphic on an open neighbourhood of $D \cup \gamma$. If f has no zeroes or poles on γ, then

$$I(f \circ \gamma; 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = \#\text{ of zeroes of } f \text{ in } D \; - \; \#\text{ of poles of } f \text{ in } D$$

where zeroes and poles are counted with multiplicity.

Proof. We have

$$\begin{aligned} I(f \circ \gamma; 0) &= \frac{1}{2\pi i} \int_{f \circ \gamma} \frac{\mathrm{d}w}{w} \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z \end{aligned}$$

By residue theorem, this is

$$\sum_{\substack{\text{poles } \alpha \text{ in } D \\ \text{of } f'/f}} \mathrm{Res}_{z=\alpha} \frac{f'}{f}$$

but by previous proposition this equals

$$\text{number of zeroes of } f \text{ in } D - \text{number of poles of } f \text{ in } D$$

counting multiplicity. □

Remarks

(1) Recall γ is compact, $f \circ \gamma$ is also a closed curve (and compact).

(2) Morally: this says

$$2\pi(\# \text{ of zeroes of } f \text{ in } D \;-\; \# \text{ of poles of } f \text{ in } D)$$

is the change in $\arg f(z)$ as z travels γ.

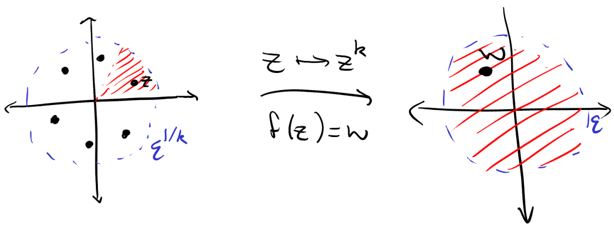
The argument principle has important consequences for local behaviour of f.

Definition. If f is holomorphic and non-constant near $z = a$, then the *local degree* of $f(z)$ at $z = a$ is $\deg_{z=a} f(z)$, the order of the zero of $f(z) - f(a)$ at $z = a$.

If f is non-constant, we can write $f(z) - f(a) = (z-a)^k g(z)$, g holomorphic at a, and the zero at $z = a$ of $f(z) - f(a)$ is isolated. So $0 < |z - a|$ sufficiently small implies $f(z) - f(a) \neq 0$. SO for small $\varepsilon > 0$, the circle $\gamma(t) = a + \varepsilon e^{it}$, $t \in [0, 2\pi]$, about a gives

$$\begin{aligned} I(f \circ \gamma; f(a)) &= I(f(\gamma(t)) - f(a); 0) \\ &= \# \text{ of zeroes in } D(a,\varepsilon) \text{ of } f(z) - f(a) \;-\; \# \text{ of poles of } f(z) - f(a) \text{ in } D(a,\varepsilon) \\ &= \deg_{z=a} f(z) \end{aligned}$$

Consider the local behaviour of $f(z) = z^k$ at $z = 0$ for $k > 0$. We have $\deg_{z=0} f(z) = k$.



Note that $\forall w \in D(0, \varepsilon)$, w has k preimages under f in $D(0, \varepsilon^{1/k})$.

Theorem (Local mapping degree theorem)**.** Let $f : D(a, R) \to \mathbb{C}$ be holomorphic and non-constant, with local degree $k > 0$. Then for $r > 0$ sufficiently small, there exists $\varepsilon > 0$ such that if $0 < |w - f(a)| < \varepsilon$, then $f(z) = w$ has exactly k (simple) roots in $D(a, r)$.

Proof. Choose $r > 0$ such that $f(z) - f(a)$ has no zeroes for $0 < |z - a| \leq r$ and $f'(z) \neq 0$ for $0 < |z - a| \leq r$; r exists by identity principle. Let γ be the circle of radius r about a. Then $f \circ \gamma$ doesn't contain $f(a)$, so there exists $\varepsilon > 0$ such that $D(f(a), \varepsilon) \cap f \circ \gamma = \emptyset$. For $w \in D(f(a), \varepsilon)$, the number of zeroes of $f(z) = w$ in $D(a, r)$ is $I(f \circ \gamma; w)$. But $I(f \circ \gamma; w) = I(f \circ \gamma, f(a)) = k$. Since $f(z) - w$ has nonzero derivative in $D(a, r)^\times$, so the preimages of w are simple. $\square$

Note $I(f \circ \gamma; w) = I(f \circ \gamma; f(a))$ because the winding number is constant on connected components of $\mathbb{C} \setminus f \circ \gamma$.

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Corollary (Open mapping theorem)**.** A nonconstant holomorphic function maps open sets to open sets.

Proof. Want to show that if $f : D \to \mathbb{C}$ then $\forall a \in D$, $\forall r > 0$ sufficiently small, $f(D(a,r)) \supset D(f(a), \varepsilon)$ for some ε. By previous theorem, if r and ε are sufficiently small, then $\forall w \in D(f(a), \varepsilon)$ we have that the number of zeroes of $f(z) - w$ in $D(a,r)$ is $\deg_{z=a} f(z) > 0$. □

Theorem (Rouché's theorem)**.** Let γ bound a domain D, and f, g are holomorphic on a neighbourhood of $D \cup \gamma$. If $|f(z)| > |g(z)|$ for all $z \in \gamma$, then f and $f + g$ have the same number of zeroes in D.

Proof. Define $h(z) = \frac{f(z)+g(z)}{f(z)} = 1 + \frac{g(z)}{f(z)}$. Note h is meromorphic on a neighbourhood of $D \cup \gamma$. Since $|f(z)| > |g(z)|$ $\forall z \in \gamma$¡ $f + g$ and f are nonzero on γ, so h has no zeroes or poles on γ. By argument principle, we have

$$\# \text{ zeroes of } f + g \text{ on } D - \# \text{ zeroes of } f \text{ on } D = I(h \circ \gamma; 0)$$

By hypothesis, $h \circ \gamma \subset D(1,1)$. So $I(h \circ \gamma; 0)$. □

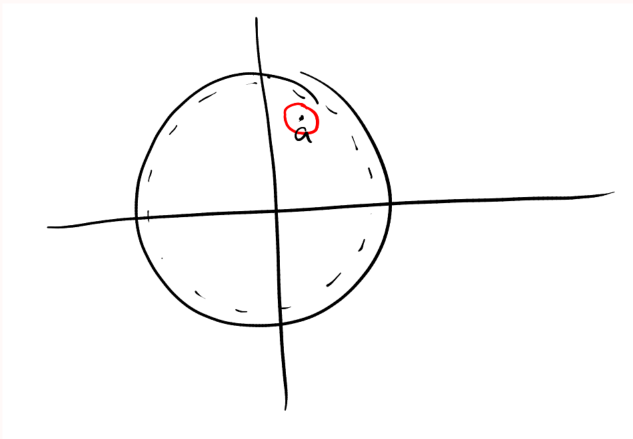
Example. Consider $p(z) = z^4 + 6z + 3$. If $|z| \ge 2$, then $\left|z^3 + 6 + \frac{3}{z}\right| \ge |z|^3 - 6 - \frac{3}{|z|} > 0$, so $p(z) = z\left(z^2 + 6 + \frac{3}{z}\right) \neq 0$. We could instead apply Rouché's with $\gamma : |z| = 2$, $f(z) = z^4$, $g(z) = 6z + 3$, so $|z|^4 = 16 > 15 = 6|z| + 3 \ge |6z + 3|$. By Rouché's, $p(z)$ has 4 zeroes inside $D(0,2)$. For $|z| = 1$, $|6z| = 6$ and $|z^4 + 3| \le 4$. So using $\gamma : |z| = 1$, $f(z) = 6z$, $g(z) = z^4 + 3$, we see $p(z)$ has 1 zero inside $D(0,1)$. (Note that this implies that $p(z)$ has a real root, since roots come in conjugate pairs for polynomials over $\mathbb{R}$).

Example (Rouché's $\implies$ open mapping)**.** If $f : D \to \mathbb{C}$ is holomorphic and nonconstant and $a \in D$, we can choose $r > 0$ such that $D(a, 2r)^\times$ has no zeroes of $f(z) - f(a)$. Let γ be $|z - a| = r$, and let $0 < \varepsilon, \min_{z \in \gamma} |f(z) - f(a)|$. Then for $w \in D(f(a), \varepsilon)$, $f(z) - w = f(a) - w + f(z) - f(a)$, and we have by $|f(a) - w| < \varepsilon < |f(z) - f(a)|$ for all $z \in \gamma$. By Rouché's, zeroes in $D(a,r)$ of $f(z) - w$ is equal to number of zeroes in $D(a,r)$ of $f(z) - f(a) > 0$. So $f(D(a,r)) \supset D(f(a), \varepsilon)$.

Uniform limits of holomorphic functions

Definition (Converging locally uniformly)**.** Let $\mathcal{U} \subset \mathbb{C}$ be open, and $f_n : \mathcal{U} \to \mathbb{C}$ a sequence of functions. Then $f_n \to f$ *converges locally uniformly on* $\mathcal{U}$ if $\forall u \in \mathcal{U}$, $\exists D(a,r) \subset \mathcal{U}$ on which $f_n \to f$ uniformly.

Example. $f_n(z) = z^n$ on $\mathcal{U} = D(0,1)$. As $n \to \infty$, f_n tends to constant zero function pointwise. For $a \in D(0,1)$, $\overline{D\left(a, \frac{1-|a|}{2}\right)} \subset D(0,1)$, and $f_n \to 0$ uniformly on $D\left(a, \frac{1-|a|}{2}\right)$. So $f_n \to 0$ locally uniformly on $D(0,1)$.



However, for any $\varepsilon > 0$, $|f_n(z)| < \varepsilon \iff |z|^n < \varepsilon \iff |z| < \varepsilon^{1/n}$, so no uniform bound can hold for all $|z| < 1$.

Proposition. $\{f_n\} : \mathcal{U} \to \mathbb{C}$ is locally uniformly convergent on $\mathcal{U} \iff \{f_n\}$ converges uniformly on any compact subset of $\mathcal{U}$.

Recall: $K \subset \mathbb{C}$ is compact $\iff$ K is closed and bounded $\iff$ every open cover has a finite subcover.

Proof. If $f_n \to f$ locally uniformly on $\mathcal{U}$, and $K \subset \mathcal{U}$ is compact, then $\forall a \in K$, there exists $r_a > 0$ such that $\{f_n\}$ converges uniformly on $D(a, r_a)$. $\bigcup_{a \in K} D(a, r_a)$ is an open cover of K, so there exists $a_1, \ldots, a_l$ such that

$$K \subset D(a_1, r_{a_1}) \cup \cdots \cup D(a_l, r_{a_l}).$$

Taking the max of constants of uniform convergence on these discs, $f_n \to f$ uniformly on K.

If $f_n \to f$ on every compact subset of $\mathcal{U}$ then if $a \in \mathcal{U}$, find a closed disc $\overline{D(a,r)} \subset \mathcal{U}$. Then $f_n \to f$ converges uniformly on $D(a,r)$. □

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Theorem. Let $\{f_n\}$ be a sequence of analytic functions on $\mathcal{U}$, converging locally uniformly to f. Then f is holomorphic, with $f_n' \to f'$ locally uniformly.

Proof. Fix $a \in \mathcal{U}$ and $\overline{D(a,r)} \subset \mathcal{U}$. For $r \ll 1$, $f_n \to f$ uniformly of $\overline{D(a,r)}$. So

$$|f(z) - f(w)| = |f(z) - f_n(z) + f_n(z) - f_n(w) + f_n(w) - f(w)|$$

so uniform convergence implies f continuous no $\overline{D(a,r)}$. Given γ a closed curve in $D(a,r)$, we have

$$\int_\gamma f = \lim_{n\to\infty} \int_\gamma f_n = 0$$

by Cauchy's theorem. So Morera's theorem implies f is holomorphic on $D(a,r)$. By Cauchy's integral formula we have:

$$|f'(w) - f_n'(w)| = \frac{1}{2\pi} \left| \int_{|z-a|=r} \frac{f(z) - f_n(z)}{(z-w)^2} \mathrm{d}z \right|$$

for $|w - a| \le \frac{r}{2}$ we have

$$\begin{aligned} |f'(w) - f_n'(w)| &\le r \cdot \frac{1}{\left(\frac{r}{2}\right)^2} \cdot \sup_{|z-a|=r} |f(z) - f_n(z)| \\ &\to 0 \end{aligned}$$

as $n \to \infty$ by uniform convergence. $f_n \to f$ uniformly on $\overline{D(a,r)}$ implies $|f_n' \to f'|$ uniformly on $D\left(a, \frac{r}{2}\right)$. □

Remark. The assumption of locally uniform convergence is necessary; a construction with non-holomorphic limit can be done via Runge's theorem (see Topics in Analysis).

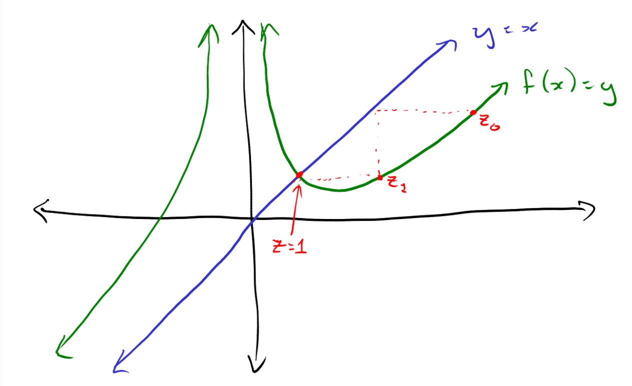
Application 1: Newton's method and complex dynamics

Recall Newton's method, an iterative root-finding algorithm, takes a polynomial $p(z)$ and an initial z_0 for a root of $p(z)$, and compute a sequence $z_1, z_2, \dots, z_n := f^n(z_0), \dots$ where

$$f(z) = z - \frac{p(z)}{p'(z)};$$

sometimes(??) this sequence limits to a root of p.

Example. $p(z) = z^3 - 1$, $f(z) = \frac{2z^3+1}{3z^2}$. In $\mathbb{R}$:



$f^n(z)$ is a sequence of meromorphic functions, so if $f^n(z_0)$ approaches a limit, for some region $\mathcal{U}$ of initial guesses, then $f^n|_{\mathcal{U}}$ has holomorphic limit.

Definition. A family $\mathcal{F} = \{f_i\}_{i \in I}$ of holomorphic functions on a domain D is *normal* if every sequence in $\mathcal{F}$ has a locally uniformly convergent subsequence. (Note: we allow convergence to ∞).

Deep theorem ("Montel's theorem"): If $\exists a, b, c \in \mathbb{C}_\infty$ such that $\forall f \in \mathcal{F}$, $f(D) \cap \{a, b, c\} = \emptyset$, then $\mathcal{F}$ is a normal family.

Definition. The *Fatou set* of a rational map f is

$$F(f) := \{z \in \mathbb{C}_\infty : \exists \text{ neighbourhood } \mathcal{U} \text{ of } z \text{ s.t. } \{f^n|_{\mathcal{U}}\} \text{ forms a normal family}\}$$

Riemann mapping theorem

Theorem (RMT)**.** Lt $\mathcal{U} \subsetneq \mathbb{C}$ be a nonempty, proper, open, simply connected subset of $\mathbb{C}$. Then there exists conformal isomorphism $f : \mathcal{U} \to \mathbb{D} = D(0, 1)$.

Sketch of proof. Fix $z_0 \in \mathcal{U}$, and consider

$$\mathcal{F} := \{f : \mathcal{U} \to \mathbb{D}, f \text{ holomorphic, injective and } f(z_0) = 0\}$$

Steps:

(1) $\mathcal{F}$ is non-empty.

(2) Show there exists $g \in \mathcal{F}$ such that $|g'(z_0)|$ is finite and maximal among elements of $\mathcal{F}$.

(3) Prove g is a conformal isomorphism.

Now we actually prove these claims:

(1) $\mathcal{U} \neq \mathbb{C}$ implies $\exists a \in \mathbb{C} \setminus \{\mathcal{U}\}$, so by Example Sheet 2 there exists holomorphic branch of the logarithm $\log(z-a)$ on $\mathcal{U}$. So there exists holomorphic branch $h(z) = \sqrt{z-a}$ on $\mathcal{U}$. Show: h is injective on $\mathcal{U}$, and $h(\mathcal{U}) \cap -h(\mathcal{U}) = \emptyset$. By open mapping theorem, $h(D)$ contains some $D(h(z_0), \varepsilon)$, so $|h(z) + h(z_0)| \geq \varepsilon$ for all $z \in D$. Can then check that:

$$f_0(z) = \frac{\varepsilon}{4} \cdot \frac{|h'(z_0)|}{|h(z_0)|^2} \cdot \frac{h(z_0)}{h'(z_0)} \cdot \frac{h(z) - h(z_0)}{h(z) + h(z_0)} \in \mathcal{F}$$

(2) Let $A = \sup_{f \in \mathcal{F}} |f'(z_0)|$, and choose $\{f_n\}$ in $\mathcal{F}$ such that $f_n'(z_0) \to A$. By Montel's, $\mathcal{F}$ is a normal family, so there exists f_{n_k} converging locally uniformly to some g, holomorphic. Show g is in the family (injectivity requires argument).

(3) If g is not surjective then can construct an element of $\mathcal{F}$ violating maximality of g: if $c \in D(0,1) \setminus g(\mathcal{U})$, then choose (Example Sheet 2) a holomorphic branch

$$k(z) := \sqrt{\frac{g(z) - c}{1 - cg(z)}}.$$

Then

$$F(z) = \frac{e^{i\theta}(k(z) - k(z_0))}{1 - k(z_0)k(z)}, \qquad \frac{k'(z_0)}{|k'(z_0)|} = e^{-i\theta}$$

is in $\mathcal{F}$, with $|F'(z_0)| > |g'(z_0)|$, contradiction. □