Variational Principles

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Contents

1 Variational Principles

This is not intended to be a complete set of notes. These notes only consist of the things that I wanted to make flashcards of and practise revising a little.

1.1 Euler Lagrange Equations

In this course, almost all of the problems will be about minimising something like (where x is a function of t, so we call F a *functional*):

$$F[x] = \int_a^b f(x, \dot{x}, t)\mathrm{d}t$$

In the case where x must take predefined values at $t = a$ and $t = b$, we can get *Euler Lagrange equations.* The simplest form of Euler Lagrange:

$$\frac{\partial f}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right)$$

The derivation is to consider what we call the *first variation* of F:

$$\begin{aligned}
\delta F &= F[x + \delta x] - F[x] \\
&= \int_a^b \left(f(x + \delta x, \dot{x} + \delta\dot{x}, t) - f(x, \dot{x}, t)\right)\mathrm{d}t \\
&= \int_a^b \left(\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial \dot{x}}\delta\dot{x}\right)\mathrm{d}t \\
&= \int_a^b \frac{\partial f}{\partial x}\delta x\mathrm{d}t + \left[\frac{\partial f}{\partial \dot{x}}\delta x\right]_a^b - \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right)\delta x\mathrm{d}t \\
&= \int_a^b \left(\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right)\right)\delta x\mathrm{d}t
\end{aligned}$$

Since δx is arbitrary, we must have

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0$$

First Integrals

There are also some useful first integrals of the equation which are true under certain conditions on f.

In the case where f has no explicit x dependence (i.e. $\frac{\partial f}{\partial x} = 0$), we get an alternative form of Euler Lagrange:

$$\frac{\partial f}{\partial \dot{x}} = \mathrm{const}$$

In the case where f has no explicit t dependence (i.e. $\frac{\partial f}{\partial t} = 0$), we can also get an alternative form of Euler Lagrange, although the derivation is more involved. Consider $\frac{\mathrm{d}f}{\mathrm{d}t}$ using multivariate chain rule:

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial t}{\partial t}\underbrace{\frac{\partial f}{\partial t}}_{=0} + \frac{\partial x}{\partial t}\frac{\partial f}{\partial x} + \frac{\partial \dot{x}}{\partial t}\frac{\partial f}{\partial \dot{x}} \\
&= \dot{x}\frac{\partial f}{\partial x} + \ddot{x}\frac{\partial f}{\partial \dot{x}}
\end{aligned}$$

Multiplying the Euler Lagrange equation by $\dot{x}$ gives:

$$\begin{aligned}
0 &= \dot{x}\frac{\partial f}{\partial x} - \dot{x}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right) \\
&= \frac{\mathrm{d}f}{\mathrm{d}t} - \ddot{x}\frac{\partial f}{\partial \dot{x}} - \dot{x}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right) \\
&= \frac{\mathrm{d}f}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}\frac{\partial f}{\partial \dot{x}}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(f - \dot{x}\frac{\partial f}{\partial \dot{x}}\right)
\end{aligned}$$

Hence we obtain the first integral:

$$\boxed{f - \dot{x}\frac{\partial f}{\partial \dot{x}} = \text{const}}$$