Differential Equations

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Contents

0 Introduction

Colours for notes:

- Text will be in black
- Displayed maths will be in blue
- Comments will be in green
- Examples will be in red

This is an *applied* course — we will emphasise methods rather than proofs.

1 Basic Calculus

1.1 Differentiation

Definition (Derivative of a function)**.** Define the *derivative* of a function $f(x)$ with respect to x as a function such that

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0} = \lim_{h\to 0}\frac{f(x_0+h)-f(x)}{h}$$

The derivative can sort of be thought of as the "slope".
For the derivative to exist at x_0, the left-hand and right-hand limits must exist and be equal, i.e.

$$\lim_{h\to O^-}\frac{f(x_0+h)-f(x)}{h} = \lim_{h\to 0^+}\frac{f(x_0+h)-f(x)}{h}$$

Example ($f(x)=|x|$ at $x=0$)**.** Then

$$\lim_{h\to 0^-} = \frac{|h|}{h} = -1 \qquad \text{but} \qquad \lim_{h\to 0^+}\frac{|h|}{h} = +1$$

so $|x|$ is not differentiable at $x=0$ (but it is everywhere else).

Notation (Ways of writing a derivative)**.**

$$\frac{\mathrm{d}f}{\mathrm{d}x} = f'(x) = \dot{f}(x)$$

and we denote the n-th derivative by $f^{(n)}(x)$.

1.2 O and o notation

These are useful concepts to give comparative scalings of a function as they approach some limiting point

Definition (O and o notation)**.** $f(x)$ is $o(g(x))$ as $x \to x_0$ if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

This is often written as $f(x) = o(g(x))$ as $x \to x_0$. Informally you can think of this as "$f(x)$ is much smaller than $g(x)$ as $x \to x_0$."
$f(x)$ is $O(g(x)$ as $x \to x_0$ if $\exists \delta > 0$ and $M > 0$ such that for all x with $0 < |x - x_0| < \delta$, we have

$$|f(x)| \leq M|g(x)|.$$

So $\frac{f(x)}{g(x)}$ is bounded for x sufficiently close to x_0.

Remark (Using = for O and O)**.** This is an abuse of notation, because $O(g(x))$ or $o(g(x))$ does note denote a function: it denotes a class of functions. This can be though of more as $f(x) = o(g(x))$ means $f(x) \in o(g(x))$.

The definition of O can be extended to behaviour at ∞: we say that $f(x)$ is $O(g(x))$ as $x \to \infty$ if $\exists X > 0$ and $M > 0$ such that for all $x > X$ we have

$$|f(x)| \leq M|g(x)|.$$

Remark ($O \subset o$)**.** $f(x) = o(g(x)) \implies f(x) = O(g(x))$, but not the other way around.

Example. In this case we have $f(x) = O(x)$ but $f(x) \neq o(x()$ as $x \to 0$ since the limit is 2 which is not 0.

Example. We have that $x^2 + x = O(x^2)$ as $x \to \infty$, since for $x > 1$, we have that $|x^2 + x| < 2|x^2|$. In general, when we focus on behaviour of a polynomial at infinity, we see that the highest power term dominates.

Example. $\sin 2x = O(x)$ as $x \to 0$ since $\sin 2x \approx 2x$ for small x.

Order parameters are useful to classify remainder terms before taking limits.

Example.

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0} = \frac{f(x_0+h)-f(x_0)}{h} + \frac{o(h)}{h} \qquad \text{as } h \to 0$$

(this equation can be understood by considering the limit as $h \to 0$.) It follows from this equation that

$$f(x_0+h) = f(x_0) + h\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0} + o(h).$$

1.3 Rules for differentiation

Theorem (Chain rule)**.** Given $f(x) = F(g(x))$ we have that

$$\frac{\mathrm{d}f}{\mathrm{d}x} = F'(g(x)) \times \frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\mathrm{d}F}{\mathrm{d}g} \times \frac{\mathrm{d}g}{\mathrm{d}x}$$

Proof. See printed notes. □

Example.

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x^2+x-2) = \underbrace{\cos(x^2+x-2)}_{F'(g(x))}\underbrace{(2x+1)}_{g'(x)}$$

Theorem (Product rule)**.** Given $f(x) = u(x)v(x)$ we have

$$\frac{\mathrm{d}f}{\mathrm{d}x} = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$$

Proof. Left as an exercise. □

Example (Quotient rule)**.** The Quotient Rule is a special case of product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{1}{v}u' + u\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{v}\right) = \frac{u'}{v} - \frac{v'u}{v^2} = \frac{u'v+v'u}{v^2}$$

Remark. For any real number λ, if a function $f(x) = o(g(x))$, then we also have that $\lambda f(x) = o(g(x))$.

Leibniz's rule

Consider $f(x) = u(x)v(x)$; then

$$\begin{aligned}
f' &= u'x + uv' \\
f'' &= u''v + \underbrace{u'v' + u'v'}_{2u'v'} + uv'' \\
f''' &= u'''v + \underbrace{u''v' + 2u''v'}_{3u''v'} + \underbrace{2u'v'' + u'v''}_{3u'v''} + uv'''
\end{aligned}$$

Theorem (Leibniz's rule)**.** Given $f(x) = u(x)v(x)$ then

$$\begin{aligned}
f^{(n)}(x) &= \sum_{r=0}^{n} \binom{n}{r} u^{(n-r)}(x) v^{(r)}(x) \\
&= u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} \\
&\quad + \cdots + \frac{n!}{m!(n-m)!}u^{(n-m)}v^{(m)} + \cdots + uv^{(n)}
\end{aligned}$$

Proof. Induction. □

1.4 Taylor Series

Taylor's Theorem

Previously:

$$f(x_0 + h) = f(x_0) + hf'(x_0) + o(h) \qquad \text{as } h \to 0$$

This extends to the following theorem:

Theorem (Taylor's Theorem)**.** For n-times differentiable $f(x)$, then

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots + \frac{h^{(n)}}{n!}f^{(n)}(x_0) + E_n$$

where E_n is $o(h^n)$ as $h \to 0$.

Stronger version if $f^{(n+1)}(x)$ exists $\forall x \in (x_0, x_0 + h)$ and $f^{(n)}$ is continuous in this range:

$$\begin{aligned}
E_n &= O(h^{n+1}) && \text{as } H \to 0 \\
&= f^{(n+1)}(x_n)\frac{h^{n+1}}{(n+1)!} && \text{for some } x_0 \le x_n \le x_0 + h
\end{aligned}$$

Remark. $E_n = O(h^{n+1})$ is a stronger statement than $o(h^n)$. For example

$$h^{n+1/2} = o(h(n)) \qquad \text{but} \qquad h^{n+1/2} \neq O(h^{n+1}) \quad \text{as } h \to 0$$

Taylor Polynomials

With $x_0 + h = x$, Taylor's theorem gives

$$f(x) = f(x_0) + hf'(x_0) + \cdots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + E_n.$$

This is called the n-th order Taylor polynomial of $f(n)$ about x_0.

- n-th order Taylor polynomial matches first f derivatives of $f(x)$ at x_0.
- Provides local approximation to $f(x)$ in vicinity of x_0, with error $E_n = O(h^{n+1}$.
- If Taylor polynomial converges (as x varies) as $n \to \infty$, gives *Taylor series* of $f(x)$ about x_0.

1.5 L'Hôpital's Rule

L'Hôpital's Rule is used to deal with indeterminate forms in limits.

Theorem (L'Hôpital's Rule)**.** Let $f(n)$ and $g(n)$ be differentiable at x_0, with continuous first derivatives there, and

$$\lim_{x \to x_0} = f(x_0) = 0 \qquad \text{and} \qquad \lim_{x \to x_0} g(x) = g(x_0) = 0$$

Then if $g'(x_0) \neq 0$,

$$\lim x \to x_0 = \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

Proof. Note that

$$f(x) = \underbrace{f(x_0)}_{0} + (x - x_0) f'(x_0) + o(x - x_0) \qquad \text{as } x \to x_0$$

and similarly

$$g(x) = \underbrace{g(x_0)}_{0} + (x - x_0) g'(x_0) + o(x - x_0) \qquad \text{as } x \to x_0.$$

Then

$$\lim x \to x_0 \frac{f(x)}{g(x)} = \frac{f'(x_0) + \frac{o(x-x_0)}{x-x_0}}{\underbrace{g'(x_0)}_{\neq 0} + \frac{o(x-x_0)}{x-x_0}}$$
$$= \frac{f'(x_0)}{f'(x_0)}$$
$$= \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

□

Remark (Generalisation)**.** The proof above also applies more generally, but is easier to understand for the special case mentioned in L'Hôpital's Rule. In particular

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f''(x)}{g''(x)}$$

provided the limit on the right side exists.

Example. Consider

$$f(x) = 2\sin x - \sin 3x$$
$$g(x) = 2x - \sin 2x$$

Notice that we have $f(0) = g(0) = 0$ and we have

$$f'(x) = 3\cos x - 3\cos 3x \qquad (= 0 \text{ at } x = 0)$$
$$g'(x) = 2 - 2\cos 2x \qquad (= 0 \text{ at } x = 0)$$

Note that

$$f''(0) = g''(0) = 0 \qquad \text{at } x = 0$$

However one can compute that

$$f'''(0) = 24 \qquad \text{and} \qquad g'''(0) = 8$$

From this it follows that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim x \to 0 \frac{f'''(x)}{g'''(x)} = \frac{24}{8} = 3.$$

The following is a non-examinable sketch-proof of Taylor's Theorem: *Proof.* Start from the Fundamental Theorem of Calculus (proved in the next section)

$$
\begin{aligned}
\int_)^x f'(t)\mathrm{d}t &= f(x) - f(0) \\
\implies f(x) &= f(0) = \int_0^x f'(t)\mathrm{d}t \\
&= f(0) + \int_0^x \frac{\mathrm{d}}{\mathrm{d}t}(t-x)f'(t)\mathrm{d}t \\
&= f(0) + \left[(t-x)f'(t)\right]_0^x - \int_0^x (t-x)f''(t)\mathrm{d}t \\
&= f(0) + xf'(0) - \frac{1}{2}\int_)^x \frac{\mathrm{d}}{\mathrm{d}t}(T-x)^2 f''(t)\mathrm{d}t \\
&= f(0) + xf'(0) - \frac{1}{2}\left[(t-x)^2 f''(t)\right]_0^x + \frac{1}{2}\int_0^x (t-x)^2 f'''(t)\mathrm{d}t \\
&\quad\vdots \\
&= f(0) = xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \frac{1}{n!}\int_0^x (x-t)^n f^{(n+1)}(t)\mathrm{d}t
\end{aligned}
$$

And we can note that the integral on the last line is $E_n = o(h^n)$. □

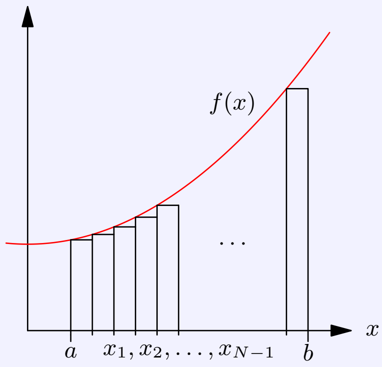
2 Integration

2.1 Integrals as Riemann Sums

The aim of this section is to formalise "area under a curve".

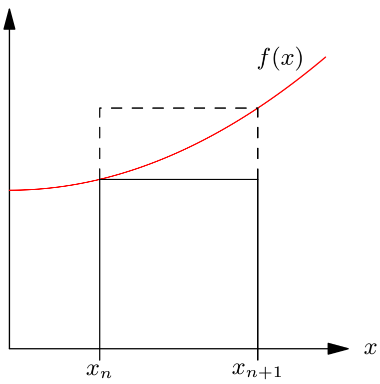
Definition (Integral)**.** The *integral* of a suitably well behaved function $f(x)$ is the limit of a sum:

$$\in_a^b f(x)\mathrm{d}x \equiv \lim_{\Delta x \to 0} \sum_{n=0}^{N-1} f(\underbrace{x_n}_{a+n\Delta x})\underbrace{\Delta x}_{\frac{b-a}{N}}$$



Riemann sum: Limit should not depend on exact choice of rectangles (e.g. Δx does not have to be uniform). Relation to "area under curve"?

Consider one rectangle for finite N:



One can see that the solid rectangle has area less than the area under the curve, and the dashed one has greater area. One might ask whether we could pick a point to construct the rectangle from such that the area of the rectangle and the area of the curve are the same. The mean value theorem states that this is possible.

Theorem (Mean-value theorem)**.** For $f(x)$ continuous, then if A_n is the area under the curve between x_n and x_{n+1}, then we have

$$A_n = (x_{n+1} - x_n) f(c_n)$$

for some c_n satisfying $x_n \leq c_n \leq x_{n+1}$. (See Analysis I for proof). If $f(x)$ is differentiable then

$$\begin{aligned} f(c_n) &= f(x_n) + O(c_n - x_n) \qquad \text{as } c_n - x_n \to 0 \\ &= f(x_n) + O(\Delta x) \qquad \text{since } c_n \leq x_n + \Delta x \end{aligned}$$

It follows that

$$A_n = \Delta x f(x_n) + O(\Delta x^2) \qquad \text{as } \Delta x \to 0$$

Total area between a and b:

$$\begin{aligned} A &= \lim_{N \to \infty} \sum_{n=0}^{N} A_n \\ &= \underbrace{\lim_{N \to \infty} \sum_{n=0}^{N} f(x_n) \Delta x}_{\text{Definition of integral}} + \underbrace{\lim_{N \to \infty} N O\left(\left(\frac{b-a}{N} \right)^2 \right)}_{\lim_{N \to \infty} O\left(\frac{(b-a)^2}{N}\right)} \\ &= \int_a^b f(x) \mathrm{d}x + 0 \end{aligned}$$

2.2 Fundamental Theorem of Calculus

Formalise "inverse of differentiation"

Theorem (FTC)**.** Let $F(x)$ be defined as

$$F(x) = \int_a^x f(Tt) \mathrm{d}t$$

Then

$$\frac{\mathrm{d}F}{\mathrm{d}x} = f(x).$$

Proof.

$$\begin{aligned}
\frac{\mathrm{d}F}{\mathrm{d}x} &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\mathrm{d}t - \int_a^x f(t)\mathrm{d}t\right] \\
&= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t \\
&= \lim_{h\to 0} \frac{1}{h}\left[f(x)h + O(h^2)\right] \qquad \text{(using Mean Value Theorem)} \\
&= f(x) + \lim_{h\to 0} O(h) \\
&= f(x)
\end{aligned}$$

□

Remark. $F(x)$ is the solution of $\frac{\mathrm{d}F}{\mathrm{d}x} = f(x)$ with $F(a) = 0$.

Corollaries:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b f(t)\mathrm{d}t = -f(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{g(x)} f(t)\mathrm{d}t = f(g(x))\frac{\mathrm{d}g}{\mathrm{d}x}$$

Notation (Indefinite Integrals)**.** We may write indefinite integrals as

$$\int f(x)\mathrm{d}x \qquad \text{or} \qquad \int^x f(t)\mathrm{d}t$$

(note that in the second case the lack of lower limit gives rise to integration constant.)

2.3 Methods of integration

Integration by subs

Useful when integrand is "function of a function"

Example. Consider

$$I = \int \frac{1-2x}{\sqrt{x-x^2}}\mathrm{d}x$$

Let $u = x - x^2 \implies \frac{\mathrm{d}u}{\mathrm{d}x} = 1 - 2x$

$$I = \int \frac{1}{\sqrt{u}} = 2\sqrt{u} + c = 2\sqrt{x - x^2} + c$$

Here are some useful trig / hyperbolic subs:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \cosh^2 u \sinh^2 u = 1$$

$$\tan^2\theta + 1 = \sec^2\theta \qquad 1 - \tanh^2 u = \mathrm{sech}^2 u$$

Generally we use the subs $\sin\theta$ or $\tanh u$, $\tan\theta$ or $\sinh u$, $\sec\theta$ or $\cosh u$ for terms $1-x^2$, $1+x^2$, x^2-1 respectively.

Example.

$$I = \int \sqrt{2x - x^2}\mathrm{d}x$$

Try $x - 1 = \sin\theta \implies \mathrm{d}x = \cos\theta\mathrm{d}\theta$. Then

$$\begin{aligned}
I &= \int \sqrt{1-\sin^2\theta}\cos\theta\mathrm{d}\theta \\
&= \int \cos^2\theta\mathrm{d}\theta \\
&= \frac{1}{2}\int 1 + \cos 2\theta\mathrm{d}\theta \\
&= \frac{1}{2}(\theta + \frac{1}{2}\sin 2\theta) + c \\
&= \frac{1}{2}(\theta + \sin\theta\cos\theta) + c \\
&= \frac{1}{2}\arcsin(x-1) + \frac{1}{2}(x-1)\sqrt{1-(x-1)^2} + 1
\end{aligned}$$

2.4 Integration by parts

Recall the product rule: $uv' = (uv)' - u'v$.

Theorem (Integration by parts)**.** For functions u and v

$$\int uv'\mathrm{d}x = uv - \int u'v\mathrm{d}x$$

Example.

$$
\begin{aligned}
I &= \int_0^\infty xe^{-x}\mathrm{d}x \\
&= [-xe^{-x}]_0^\infty + \int_0^\infty e^{-x}\mathrm{d}x \\
&= 1
\end{aligned}
$$

Example.

$$
\begin{aligned}
I &= \int \ln x \mathrm{d}x \\
&= x\ln x - \int \frac{1}{x} x \mathrm{d}x \\
&= x\ln x - x + c
\end{aligned}
$$

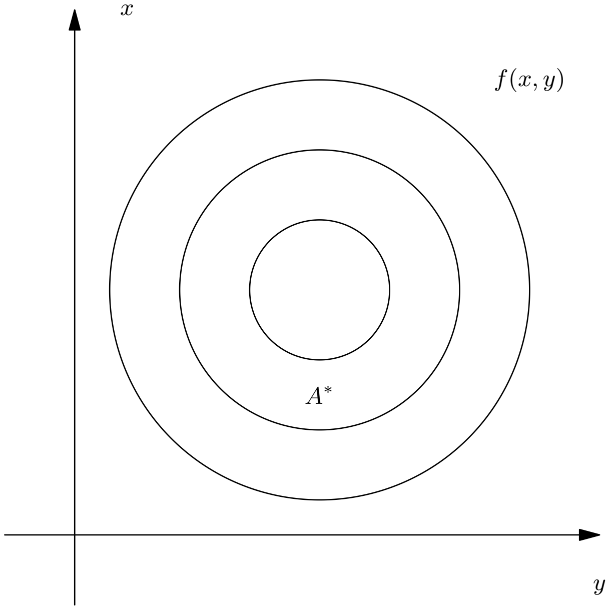
3 Partial Differentiation

3.1 Functions of Several Variables

Definition (Multivariate functions)**.** A function is called *multivariate* if it depends on more than one variable.

Some examples of multivariate functions include

- Height of terrain: $h(\text{latitude}, \text{longitude})$.
- Density of air: $\rho(x, y, z, t)$.



The slope of f at A depends on direction.

3.2 Partial Derivatives

Definition (Partial derivative)**.** Given some function of several variables, for example $f(x, y)$, the *partial derivative* of f with respect to x at fixed y is

$$\left.\frac{\partial f}{\partial x}\right|_y = \lim_{\delta x \to 0}\left[\frac{f(x + \delta x, y) - f(x, y)}{\delta x}\right]$$

"Slope of f when moving in x direction."

Note that $\left.\frac{\partial f}{\partial x}\right|_x$ is defined similarly.

Example. Let f be defined as

$$f(x,y) = x^2 + y^2 + e^{xy^2}$$

Then we have that

$$\left.\frac{\partial f}{\partial x}\right|_y = 2x + 0 + y^2 e^{xy^2}$$

$$\left.\frac{\partial f}{\partial x}\right|_x = 0 + 2y + 2xye^{xy^2}$$

We can also compute second partial derivatives:

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_y = 2 + y^2 \cdot y^2 \cdot e^{xy^2}$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_x = 2 + 2xe^{xy^2} + 2xy \cdot 2xye^{xy^2}$$

and we can compute *mixed* derivatives

$$\left.\frac{\partial}{\partial x}\left(\left.\frac{\partial f}{\partial y}\right|_x\right)\right|_y = 2ye^{xy^2} + 2xy \cdot y^2 \cdot e^{xy^2}$$

$$\left.\frac{\partial}{\partial y}\left(\left.\frac{\partial f}{\partial x}\right|_y\right)\right|_x = 2ye^{xy^2} + y^2 \cdot 2xy \cdot e^{xy^2}$$

Notation. A useful convention that is often used is to omit the $|_y$ symbol if all other variables are being held fixed. For example

$$\frac{\partial^2 f}{\partial x \partial y} = \left.\frac{\partial}{\partial x}\left(\left.\frac{\partial f}{\partial y}\right|_x\right)\right|_y$$

Note that if a function as continuous (mixed) second derivatives then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

This is known as "Schwarz's theorem". (In other words, partial derivatives commute).

Alternative notation:

$$f_x = \frac{\partial f}{\partial x} \qquad ; \qquad f_{x,y} = \frac{\partial^2 f}{\partial y \partial x}$$

3.3 Multivariate Chain Rule

Given path $x(t)$, $y(t)$ and $f(x,y)$, what is $\frac{\mathrm{d}f}{\mathrm{d}t}$?

Consider change in f under $(x,y) \to (x+\delta x, y+\delta y)$:

$$\begin{aligned}\delta f &= f(x+\delta x, y+\delta x) - f(x,y) \\ &= [f(x+\delta x, y+\delta y) - f(x+\delta x, y)] + [f(x+\delta x, y) - f(x,y)]\end{aligned}$$

Then Taylor expanding we get

$$f(x+\delta x, y) - f(x,y) = f_x(f,y)\delta x + o(\delta x)$$

and

$$f(x+\delta x, y+\delta y) - f(x+\delta x, y) = f_y(x+\delta x, y)\delta y + o(\delta y)$$

We also Taylor expand $f_y(x+\delta x, y)$ and find that

$$f_y(x+\delta x, y) = f_y(x,y) + f_{y,x}(x,y)\delta x + o(\delta x)$$

And hence

$$\delta f = [f_y(x,y) + f_{y,x}(x,y)\delta x + o(\delta x)]\delta y + o(\delta y) + f_x(x,y)\delta x + o(\delta x) \qquad (*)$$

Take limit as $\delta x, \delta y \to 0$ to get *differential* of f:

$$\mathrm{d}f = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \delta f$$

Theorem (Chain Rule for Partial Derivatives)**.** Differential $\mathrm{d}f$ of $f(x,y)$ is related to differentials of arguments $\mathrm{d}x$ and $\mathrm{d}y$, as

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial x}\mathrm{d}y$$

This is because the remaining terms in $(*)$ approach 0 faster than $\mathrm{d}x$ or $\mathrm{d}y$.

For the path $x(t), y(t)$ we now have

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

If path parametrised as $y(x)$, we have $f(x, y(x))$ and

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial x}\underbrace{\frac{\mathrm{d}x}{\mathrm{d}f}}_{=1} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Integral Form of the Chain Rule

$$\Delta f = \int \mathrm{d}f = \int \left(\frac{\partial f}{\partial x} \mathrm{d}x = \frac{\partial f}{\partial y} \mathrm{d}y \right)$$

Note that Δf denotes the change in f between endpoints. For a path $x(t), y(t)$ connecting the endpoints, have $f(x(t), y(t))$

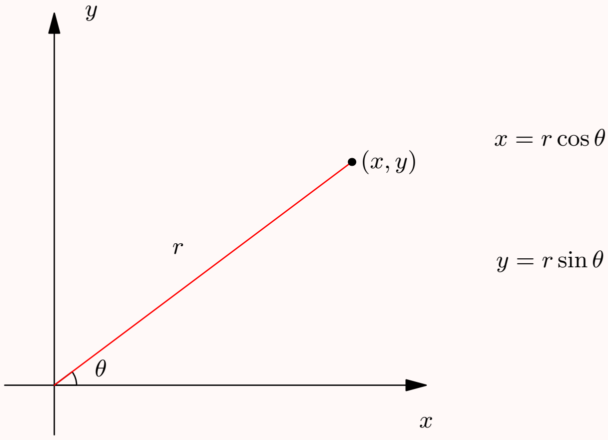
$$\Delta f = \int \left(\frac{\partial f}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}t} \right) \mathrm{d}t$$

Note that the result doesn't depend on path chosen between two given endpoints.

3.4 Applications of Multivariate Chain Rule

Change of Variables

Example. Cartesian coordinates (x, y) to plane-polar (r, θ).



Original function $f(x, y)$ can be thought of as a function of r, θ:

$$f(x, y) = f(x(r, \theta), y(r, \theta))$$

$$\left.\frac{\partial f}{\partial r}\right|_y = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial r}\right|_\theta + \left.\frac{f}{y}\right|_x \left.\frac{\partial y}{\partial r}\right|_y$$

$$\left.\frac{\partial f}{\partial \theta}\right|_r = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial \theta}\right|_r + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial \theta}\right|_r$$

Implicit Differentiation

Consider $f(x, y, z) = z$ which is a surface in 3D space. Implicitly define

$$z = z(x, y), \qquad x = x(y, z), \qquad y = y(x, z)$$

(Note that we may not be able to find these explicitly but can still determine partial derivatives).

Example. Consider

$$xy + y^2 z + z^5 = 1 \tag{$*$}$$

Take derivative of $(*)$ with respect to x holding y fixed

$$y + y^2 \left.\frac{\partial z}{\partial x}\right|_y + 5z^4 \left.\frac{\partial z}{\partial x}\right|_y = 0$$

$$\implies \left.\frac{\partial z}{\partial x}\right|_y = \frac{-y}{y^2 + 5z^4}.$$

Generally given $f(x, y, z) = c$, MVC gives

$$0 = \mathrm{d}f = \left.\frac{\partial f}{\partial x}\right|_{y,z} \mathrm{d}x + \left.\frac{\partial f}{\partial y}\right|_{x,z} \mathrm{d}y + \left.\frac{\partial f}{\partial z}\right|_{x,y} \mathrm{d}z$$

Note that we can't vary x, y, z independently *and* stay in the surface.

Find $\left.\frac{\partial z}{\partial x}\right|_y$ by

$$\left.\frac{\partial f}{\partial x}\right|_{y,z} \left.\frac{\partial x}{\partial x}\right|_y + \left.\frac{\partial f}{\partial y}\right|_{x,z} \left.\frac{\partial y}{\partial x}\right|_y + \left.\frac{\partial f}{\partial z}\right|_{x,y} \left.\frac{\partial z}{\partial x}\right|_y$$

$$\left.\frac{\partial z}{\partial x}\right|_y = \frac{-\partial f/\partial x|_{y,z}}{\partial f/\partial x|_{x,y}} \tag{$\dagger$}$$

Similarly,

$$\left.\frac{\partial x}{\partial y}\right|_z = \frac{-\partial f/\partial y|_{x,z}}{\partial f/\partial y|_{y,z}}$$

and

$$\left.\frac{\partial y}{\partial z}\right|_x = \frac{-\partial f/\partial z|_{x,z}}{\partial f/\partial y|_{x,z}}$$

Follows that

$$\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_y = -1$$

Theorem (Reciprocal Rule)**.** Applies to partial derivatives provided some variables held fixed:
Similarly to (†)

$$\left.\frac{\partial x}{\partial y}\right|_z = \frac{-\partial f/\partial z|_{x,z}}{\partial f/\partial x|_{y,z}}$$

so that

$$\left.\frac{\partial z}{\partial x}\right|_y = \frac{1}{\partial x/\partial z|_y}.$$

Differentiation of an integral with respect to a parameter

Consider a family of functions $f(x;\alpha)$. Define the integral

$$I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x;\alpha)\mathrm{d}x$$

What is $\mathrm{d}I/\mathrm{d}\alpha$?

Theorem (Differentiation of integral with respect to a parameter)**.**

$$\frac{\mathrm{d}I}{\mathrm{d}\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial \alpha}(x;\alpha)\mathrm{d}x + f(b(\alpha);\alpha)\frac{\mathrm{d}b}{\mathrm{d}\alpha} - f(a(\alpha);\alpha)\frac{\mathrm{d}a}{\mathrm{d}\alpha}$$

Proof.

$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}\alpha} &= \lim_{\delta\alpha\to 0}\frac{1}{\delta\alpha}\left[\int_{a(\alpha+\delta a)}^{b(\alpha+\delta\alpha)} f(\alpha;\alpha+\delta\alpha)\mathrm{d}x - \int_{a(\alpha)}^{b(\alpha)} f(x;\alpha)ddx\right] \\
&= \lim_{\delta x\to 0}\frac{1}{\delta\alpha}\left[\int_{a(\alpha)}^{b(\alpha)} f(x;\alpha+\delta\alpha) - f(x;\alpha)\mathrm{d}x\right] \\
&\quad + \lim_{\delta\alpha\to 0}\frac{1}{\delta\alpha}\int_{b(\alpha)}^{b(\alpha+\delta\alpha)} f(\alpha;\alpha+\delta\alpha)\mathrm{d}x - \lim_{\delta\alpha\to 0}\frac{1}{\delta\alpha}\int_{a(\alpha)}^{b(\alpha+\delta\alpha)} f(x;\alpha+\delta\alpha)\mathrm{d}x \\
&\quad + \lim_{\delta\alpha\to 0}\frac{1}{\delta\alpha}\int_{a(\alpha)}^{a(\alpha+\delta\alpha)} f(\alpha;\alpha+\delta\alpha)\mathrm{d}x \\
&= \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial \alpha}(x;\alpha)\mathrm{d}x + f(b(\alpha);\alpha)\frac{\mathrm{d}b}{\mathrm{d}\alpha} - f(a(\alpha);\alpha)\frac{a}{\alpha}
\end{aligned}$$

□

Example.

$$I(\lambda) = \int_0^\lambda e^{-\lambda x^2} \mathrm{d}x$$

$$\frac{\mathrm{d}I}{\mathrm{d}x} = \int_0^\lambda -x^2 e^{-\lambda x^2} \mathrm{d}x + \frac{\mathrm{d}\lambda}{\mathrm{d}\lambda} e^{-\lambda^2}$$

Example. Suppose we want to evaluate

$$\int_0^\infty x^n e^{-x} \mathrm{d}x \qquad (n \text{ an integer})$$

Let

$$I(\lambda) = \int_0^\infty e^{-\lambda x} \mathrm{d}x = \frac{1}{\lambda}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} I = \int_0^\infty (-x)^n e^{-\lambda n} \mathrm{d}x = (-1)^n \frac{n!}{\lambda^{n+1}}$$

Now set $\lambda = 1$ to get

$$\int_0^\infty x^n e^{-x} \mathrm{d}x = n!$$

TOPIC II

1 First Order Linear Differential Equations

Definition (Exponential function)**.** Exponential function is defined by the infinite series

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{r=0}^{\infty} \frac{x^r}{n!}$$

It can also be written as

$$\exp(x) = \lim_{k\to\infty} \left(1 + \frac{x}{k}\right)^k = \lim_{k\to\infty} \left(1 + k\frac{x}{k} + \frac{k(k-1)}{2!}\left(\frac{x}{k}\right)^2 + \cdots\right)$$

Differentiating the series gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\exp(x) = 1 + 2\frac{x}{2!} + 3\frac{x^2}{3!} + \cdots = \exp(x)$$

As $\exp(0) = 1$, can think of $\exp(n)$ as the solution of ODE

$$\frac{\mathrm{d}f}{\mathrm{d}x} = f \qquad \text{and} \qquad f(0) = 1 \qquad \implies \int_1^{\exp(x)} \frac{\mathrm{d}t}{\mathrm{d}t} = x.$$

Key Property: $\exp(x_1)\exp(x_2) = \exp(x_1 + x_2)$, which can be verified by expanding.

This allows us to write

$$\exp(x) = e^x$$

where

$$e = \exp(1) = \lim_{k\to\infty}\left(1 + \frac{1}{k}\right)^k = 2.718\ldots$$

We can also write the inverse function, ln such that

$$\exp(\ln x) = x$$

Notation. Sometimes ln is written as $\log_e$.

It follows that

$$a^x = (a^{\ln a})^x = e^{x\ln a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = \ln a e^{x\ln a} = (\ln a)a^x$$

Definition (Eigenfunction)**.** An *eigenfunction* of the derivative operator is a function that is unchanged up to a multiplicative scaling by the *eigenvalue* under the action of the operator

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \underbrace{\lambda}_{\text{eigenvalue}} \underbrace{f(x)}_{\text{eigenfunction}}$$

Eigenfunctions of $\frac{\mathrm{d}}{\mathrm{d}x}$ are $\alpha e^{\lambda x}$.

2 First-order Linear ODEs

- *LINEAR*: dependent variable (y, say) and its derivatives only appear linearly.
- *FIRST ORDER*: highest derivative of y that appears in $\frac{\mathrm{d}y}{\mathrm{d}x}$.

2.1 Homogeneous First-order Linear ODEs

- *HOMOGENEOUS*: an ODE in which all terms involve dependent variable or its derivatives, which implies that $y = 0$ is always a solution.
- *CONSTANT COEFFICIENTS*: independent variable (x) does not appear explicitly.

Example.

$$5\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 0$$

Try $y = Ae^{\lambda x}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda Ae^{\lambda x} = \lambda y$$

So ODE $\implies (5\lambda - 3) = 0$, so $\lambda = \frac{3}{5}$. Since linear, homogeneous equation, solution holds for all A.

- Generally, for any linear, homogeneous ODE (doesn't have to be 1st order), any constant multiple of a solution is a solution.
- An n-th order linear ODE (highest derivative is $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$ has exactly n independent solutions, so $y = Ae^{3x/5}$ is *the* general solution.
- To specify unique solution requires giving suitable boundary conditions; e.g. $y(0)$ determines A.

Discrete Equations

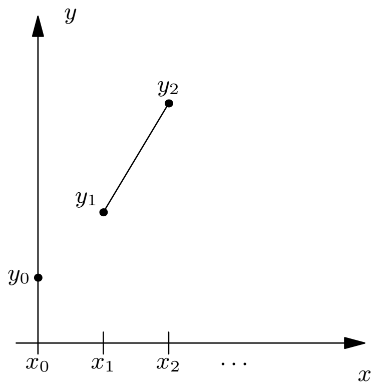
Consider

$$5\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 0$$

with $y(0) = y_0$. Then

$$y(x) = y_0 e^{3x/5}$$

Now approximate the equation by discrete form at $\{x_n\}$ where $x_n = nh$.



To do the approximation, we set

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_n} = \frac{y_{n+1} - y_n}{h}$$

so

$$5\left(\frac{y_{n+1} - y_n}{h}\right) - 3y_n = 0$$

$$\implies y_{n+!} = \left(1 + \frac{3h}{5}\right) y_n$$

which is an example of a recurrence relation.

$$y_n = \left(1 + \frac{3h}{5}\right) y_{n-1} = \left(1 + \frac{3h}{5}\right)^2 y_{n-2} = \left(1 + \frac{3h}{5}\right)^n y_0$$

It follows that

$$y_n = y_0 \left(1 + \frac{3x_n}{5n}\right)^n$$

If we take $x_n \to x$, i.e. take n steps from x_0 to x, as $n \to \infty$ we get

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_0 \left(1 + \frac{3x}{5n}\right)^n = y_0 \exp\left(\frac{3x}{5}\right)$$

(note that this agrees with the continuous case)

Series Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

where we determine a_n by substituting into the ODE.

Example.

$$5\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 0$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \sum_{n=0}^{\infty} a_n n x^{n-1} \\ &= \sum_{n=1}^{\infty} a_n n x^{n-1}\end{aligned}$$

so

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{n=1}^{\infty} a_n n x^n$$

and

$$\begin{aligned}xy &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=1}^{\infty} a_{n-1} x^n\end{aligned}$$

since $5x\frac{\mathrm{d}y}{\mathrm{d}x} - 3xy = 0$, we have

$$\sum_{n=1}^{\infty} (5na_n - 3a_{n-1})x^n = 0$$

$$\implies a_n = \frac{3}{5n} a_{n-1} \qquad (n \geq 1)$$

It follows that

$$\begin{aligned}a_n &= \frac{3}{5n} a_{n-1} \\ &= \left(\frac{3}{5}\right)^2 \frac{1}{n(n-1)} a_{n-2} \\ &= \cdots \\ &= \left(\frac{3}{5}\right)^n \frac{a_0}{n!}\end{aligned}$$

and

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{3x}{5}\right)^n = a_0 \exp\left(\frac{3x}{5}\right).$$

2.2 Forced (inhomogeneous) ODEs

Include terms involving only the independent variable so $y = 0$ no longer a (trivial) solution.

Constant forcing

Example.
$$5\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 10$$

General method of solution:

(1) Find *any* solution of the forced equation $\rightarrow$ *particular integral* $y_p(x)$. In the example, we could choose $y_p(x) = -\frac{10}{3}$, for example.

(2) Write general solution of ODE as

$$y(x) = y_p(x) + y_c(x)$$

where y_c is a complementary function. Then $y(x)$ satisfies the ODE if and only if y_c satisfies the homogeneous version (this only follows because the differential equation is *linear*). So in the example, this means we need

$$5\frac{\mathrm{d}y_c}{\mathrm{d}x} - 3y_c = 0$$
$$\implies y_c(x) = Ae^{3x/5}$$

(3) Combine y_c and y_p to get full general solution.

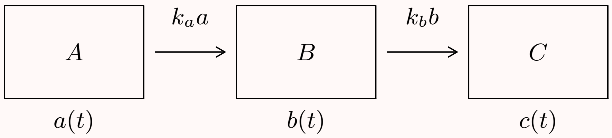
$$y(x_= Ae^{3x/5} - \frac{10}{3}$$

(A could be determined by a boundary condition if given). This method is general for *linear* ODEs.

Eigenfunction Forcing

Forcing term is an eigenfunction of differential operator on LHS.

Example (radioactive decay).



$$\frac{\mathrm{d}a}{\mathrm{d}y} = -k_a a \qquad \text{and} \qquad \frac{\mathrm{d}b}{\mathrm{d}t} = k_a a - k_b b$$

Have $a(t) = a_0 e^{-k_a t}$

$$\frac{\mathrm{d}b}{\mathrm{d}t} + k_b b = k_a a_0 e^{-k_a t} \tag{$*$}$$

Try particular integral

$$b_p(t) = Ce^{-k_a t}$$

$$(*) \qquad (k_s - k_a)Ce^{-k_a t} = k_a a_0 e^{-k_a t}$$

$$\implies C = \frac{k_a a_0}{k_s - k_a} \qquad (k_a \neq k_s)$$

General solution $b(t) = b_c(t) + b_p(t)$ where

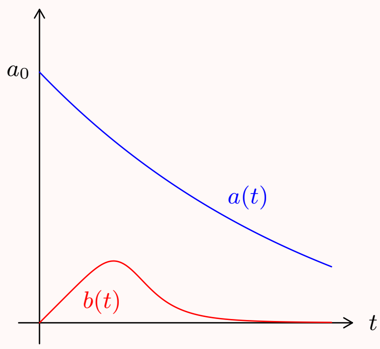
$$\frac{\mathrm{d}b_c}{\mathrm{d}t} + k_s b_c = 0 \implies b_c(t) = De^{-k_s t}$$

Follows that

$$b(t) = De^{-k_s t} + \frac{k_a}{k_s - k_a} a_0 e^{-k_a t}$$

In the case that $b(0) = 0$:

$$b(t) = \frac{b_a a_0}{k_s - k_a}(e^{-k_a t} - e^{k_s t})$$



If $k_a = k_s$, need a different particular integral.

2.3 Non-constant coefficients

General linear first-order ODE

$$a(X)\frac{\mathrm{d}y}{\mathrm{d}x} + b(x)y = c(x)$$

"Standard form"

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = f(x)$$

Solve using *integrating factor* $\mu(x)$; multiply ODE by $\mu(x)$:

$$\underbrace{\mu y' + (\mu p)y)}_{\frac{\mathrm{d}}{\mathrm{d}x}\text{ if } \mu' \mu p} = \mu f(x) \tag{$*$}$$

Requires

$$\underbrace{\int \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}x}\mathrm{d}x}_{\ln\mu} = \int p \mathrm{d}x$$

$$\implies \mu(x) = \exp\left(\int^x p(u)\mathrm{d}u\right)$$

(μ unique up to an irrelevant constant factor)

$$(*) \implies \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu f(x)$$

so

$$\mu y = \int^x \mu f \mathrm{d}u$$

Example.

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + (1-x)y = 1$$

Standard form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{1}{x} - 1\right) y = \frac{1}{x}$$

Integrating factor μ:

$$\begin{aligned}\ln \mu(x) &= \int^x \left(\frac{1}{u} - 1\right) \mathrm{d}u \\ &= \ln x - x \\ &= \ln(xe^{-x})\end{aligned}$$

$$\implies \mu(X) = xe^{-x}$$

Have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(xe^{-x}y\right) = \frac{x}{x}e^{-x} = e^{-x}$$

Integrate to find

$$xe^{-x}y(x) = -e^{-x} + C$$

$$y(x) = -\frac{1}{x} + \frac{c}{x}e^{x}$$

(if for example we wanted y finite at $x \to 0$, then we would set $C = 1$.)

Radioactivity example revisited

Recall

$$\frac{\mathrm{d}b}{\mathrm{d}t} + k_b b = k_a a_0 e^{-k_a t}$$

Integrating factor μ:

$$\ln \mu = \int^t k_b \mathrm{d}t' = k_b t$$

$$\implies \mu = e^{k_b t}$$

Have

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{k_s t}b) = k_a a_0 e^{-k_a t}e^{k_b t}$$

Consider $k_a = k_s (= k)$ Have

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{kt}b) = ka_0$$

$$\implies e^{kt}b(t) = ka_0 t + D$$

$$\implies b(t) = De^{-kt} + ka_0 te^{-kt}$$

Note: te^{-kt} rather than e^{-kt}.

TOPIC III: NON-LINEAR, FIRST-ORDER ODEs

General form

$$Q(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} + P(x,y) = 0 \qquad (*)$$

[Could be non-linear in $\frac{\mathrm{d}y}{\mathrm{d}x}$ but not considered here.]

1 Separable Equations

Definition (Separable Equation)**.** First-order ODE is separable if can be written as

$$q(y)\mathrm{d}y = p(x)\mathrm{d}x$$

Solve by direct integration.

$$\int q(y)\mathrm{d}y = \int p(x)\mathrm{d}x$$

Example.

$$(x^2y - 3y)\frac{\mathrm{d}y}{\mathrm{d}x} - 2xy^2 = 4x$$

Rearrange:

$$y(x^3 - 3)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x(2 + y^2)$$

$$\implies \frac{y}{2 + y^2}\mathrm{d}y = \frac{2x}{x^2 - 3}\mathrm{d}x$$

$$\implies \frac{1}{2}\ln(2 + y^2) = \ln(x^2 - 3) + C$$

$$\implies (y^2 + 2)^{\frac{1}{2}} = A(x^2 - 3)$$

2 Exact Equations

Definition (Exact Equation)**.** ODE $(*)$ is *exact* if $Q(x,y)\mathrm{d}y + P(x,y)\mathrm{d}y$ is an exact differential i.e., $\exists f(x,y)$ such that

$$\mathrm{d}f = P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y$$

If $(*)$ is exact, then if $\mathrm{d}f = 0$, then $f(x,y) = c$ is a solution.
If $P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y$ is exact, use multivariate chain rule

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$$

so $\exists f(x,y)$ such that

$$\frac{\partial f}{\partial x} = P \qquad \text{and} \qquad \frac{\partial f}{\partial y} = Q$$

Partial derivatives commute so

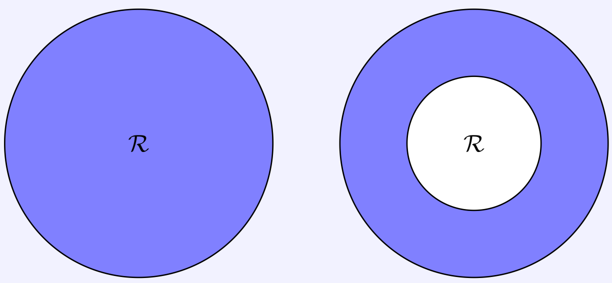
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial P}{\partial y} \qquad \text{and} \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial Q}{\partial x}$$

are equal. Therefore we find the following *necessary* (but not sufficient) condition for exactness:

$$\text{ODE exact} \implies \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Theorem. If $\frac{\partial y}{\partial x} = \frac{\partial Q}{\partial x}$ throughout a *simply-connected* domain $\mathcal{R}$, then $P\mathrm{d}x + Q\mathrm{d}y$ is an exact differential of a single-valued function f in $\mathcal{R}$.

Definition (Simply connected)**.** A domain $\mathcal{R}$ is *simply connected* if it is path connected (every pair of points can be connected by a path in $\mathcal{R}$) and any closed curve can be shrunk continuously to a point in $\mathcal{R}$ without leaving $\mathcal{R}$.



Simply connected　　　Not simply connected

[see notes for example of f in non-simply connected case.]

Example.

$$6y(y-x)\frac{\mathrm{d}y}{\mathrm{d}x} + 2x - 3y^2 = 0$$

Follows that

$$\underbrace{(2x-3y^2)}_{P}\,\mathrm{d}x + \underbrace{6y(y-x)}_{Q}\,\mathrm{d}y = 0$$

Have

$$\frac{\partial P}{\partial y} = -6y, \qquad \frac{\partial Q}{\partial x} = -6y$$

hence *exact* in any simply connected domain. Must have

$$P = \left.\frac{\partial f}{\partial x}\right|_y = 2x - 3y^2$$

$$Q = \left.\frac{\partial f}{\partial y}\right|_x = 6y^2 - 6xy$$

$$\left.\frac{\partial f}{\partial x}\right|_y = 2x - 3y^2 \implies x^2 - 3xy^2 + h(y) = f(x,y)$$

Follows that

$$\left.\frac{\partial f}{\partial y}\right|_x = -6xy + \frac{\mathrm{d}h}{\mathrm{d}y} = Q = 6y^2 - 6xy$$

$$\implies \frac{\mathrm{d}h}{\mathrm{d}y} = 6y^2 \implies h(y) = 2y^2 + c$$

Solution of ODE is

$$f(x,y) = x^2 - 3y^2x + 2y^2 = C.$$

3 Solution Curves and Isoclines

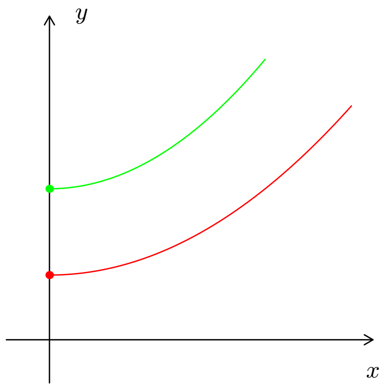
Graphical methods for “solving” of ODEs

3.1 Solution Curves

Consider

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t,y)$$

Each initial condition (e.g. $y(0) = y_0$) generates a distinct solution curve (trajectory).



Can we sketch these solution curves without solving ODE?

Example.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t(1-y^2)$$

[separable, so can solve directly!] Separable:

$$\frac{1}{1-y^2}\mathrm{d} = t\mathrm{d}t$$

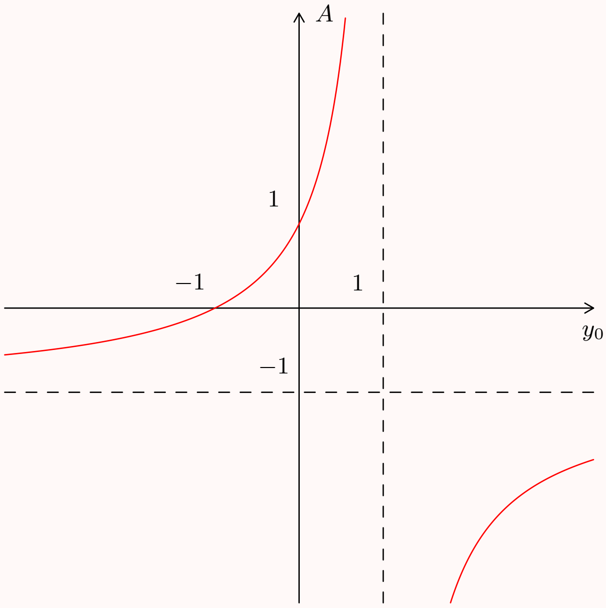
$$\implies \frac{1}{2}\left(\frac{1}{1+y} + \frac{1}{1-y}\right)\mathrm{d}y = t\mathrm{d}t$$

$$\implies \frac{1}{2}\left(\ln(1+y) - \ln(1-y)\right) = \frac{1}{2}t^2 + C$$

$$\implies \frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| = \frac{1}{2}t^2 + C$$

$$y = \frac{A - e^{-t^2}}{A + e^{-t^2}}$$

A parametrises the solution curves.
Initial condition $y(0) = y_0 \implies A = \frac{1+y_0}{1-y_0}$



3.2 Slope fields and Isoclines

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t,y) \implies \text{gradient of solution curve through any point}$$

Slope field represents gradient field by short “sticks”, one centre on each point.

Example.

$$\frac{dy}{dt} = t(1-y^2)$$

For $t > 0$, $\dot{y} > 0$ for $|y| < 1$ and $\dot{y} < 0$ for $|y| > 1$.

Isoclines sometimes useful: curves along which $\dot{y} = \text{const}$.

Example.

$$\frac{dy}{dt} = t(1-y^2)$$

Isosclines:

$$t(1-y^2 = D$$

$$y^2 = 1 - \frac{D}{t}$$

Slope field is tangent to solution curves. [If $f(t,y)$ is single-valued, solution curves cannot cross.]

4 Fixed (Equilibrium) Points and Stability

Definition (Fixed / equilibrium points)**.** Fixed point of ODE $\frac{dy}{dt} = f(t,y)$ is a constant solution $y = c$, i.e.

$$\frac{dy}{dt} = 0 \; \forall t \text{ at } y = C.$$

Example.

$$\frac{dy}{dt} = t(1-y^2)$$

Fixed points at $y = \pm 1$.

Definition (Stability of fixed points)**.** Fixed point $y = C$ is *stable* (*unstable*) if whenever y deviates slightly from C, y converges (diverges) to (from) $y = C$ as $t \to \infty$.

4.1 Perturbation analysis and stability

Suppose $y = C$ is a fixed point of

$$\frac{dy}{dt} = f(t,y)$$

[so $f(t,c) = 0\ \forall t$]
Consider some small perturbation from $y = C$:

$$y(t) = C + \epsilon(t)$$

so

$$\begin{aligned} ODE \implies \frac{\mathrm{d}\epsilon}{\mathrm{d}t} &= f(t, C+\epsilon) \\ &= f(t,C) + \epsilon \left.\frac{\partial f}{\partial y}\right|_t (t,C) + O(\epsilon^2) \end{aligned}$$

Linear for small ϵ:

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} \approx \epsilon \frac{\partial f}{\partial y}(t,C)$$

(note that this is a linear ODE so should be easier to solve!) [If $\frac{\partial f}{\partial y} = 0$ at the fixed point, then we need higher-order terms in Taylor expansion to determine stability.]

Example.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t(1-y^2)$$

Fixed points at $y = \pm 1$

$$\frac{\partial f}{\partial y} = -2ty = \begin{cases} -2t & \text{at } y = +1 \\ 2t & \text{at } y = -1 \end{cases}$$

Near $y = +1$,

$$\begin{aligned} \frac{\mathrm{d}\epsilon}{\mathrm{d}t} &= -2t\epsilon \\ \implies \epsilon(t) &= \epsilon_0 e^{-t^2} \end{aligned}$$

As $t \to \infty$, $\epsilon \to 0 \implies$ *stable* fixed point.
Near $y = -1$,

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = 2t\epsilon \implies \epsilon = \epsilon_0 e^{t^2}$$

Now $|\epsilon(t)| \to \infty$ as $t \to \infty$ so trajectories diverge from $y = -1 \implies$ *unstable* fixed point.

4.2 Autonomous systems and phase portraits

Special case:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$$

(so no explicit t dependence) then near a fixed point $y = C$ ($f(C) = 0$) we have

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \epsilon \frac{\mathrm{d}f}{\mathrm{d}y}(C) = k\epsilon$$

$$\implies \epsilon(t) = \epsilon_0 e^{kt}$$

For autonomous ODE:

$$f'(c) < 0 \implies \text{stable fixed point}$$
$$f'(c) > 0 \implies \text{unstable fixed point}$$

Example (Chemical kinetics)**.**

$$A + B \to C + D$$

$$a(t) = a_0 - c(t) \qquad b(t) = b_0 - c(t)$$

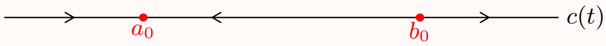
Assume rate of reaction α a b (e.g. dilute gas)

$$\begin{aligned} \frac{\mathrm{d}c}{\mathrm{d}t} &= \lambda a(t) b(t) \\ &= \underbrace{\lambda[a_0 - c(t)][b_0 - c(t)]}_{f(c)} \end{aligned}$$

Fixed points are $C = a_0$ and $C = b_0$. Assume $a_0 < b_0$: $c = b_0$ is unphysical.
Perturbation analysis:

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}c} &= \lambda(2c - a_0 - b_0) \\ &= \begin{cases} \lambda(a_0 - b_0) & \text{at } c = a_0 \\ \lambda(b_0 - a_0) & \text{at } c = b_0 \end{cases} \end{aligned}$$

For $a_0 < b_0$, $C = a_0$ is a *stable* fixed point (and $C = b_0$ is unstable)
Phase portrait:



Since there is no time dependence, we can shift any solution in time and it will remain a solution. This is because:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y) \implies \int^y \frac{\mathrm{d}u}{f(u)} = t + t_0$$

so if $y(T)$ is a solution, so is $y(t - t_0)$ $\forall t_0$.

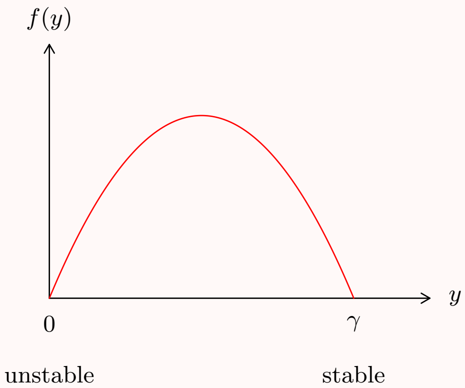
Example (Population dynamics)**.** "Logistic eq"
Population size $y(t)$
Birth rate $\alpha y(t)$ with $\alpha > 0$.
Death rate $\beta y + \gamma y^2$

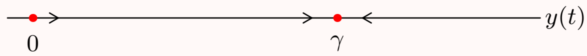
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= (\alpha - \beta)y - \gamma y^2 \\ &= \lambda y(1 - y/\gamma) = f(y)\end{aligned}$$

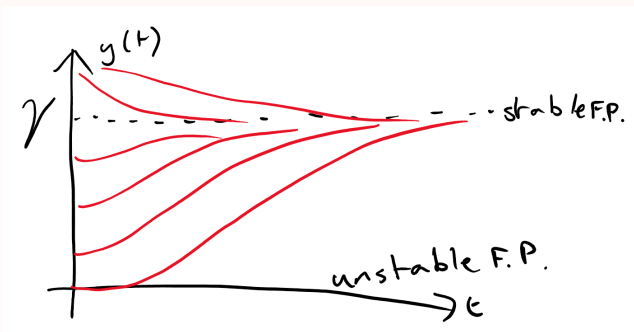
Autonomous ODE fixed points: $y = 0$ or $y = \gamma$.

$$\frac{\mathrm{d}f}{\mathrm{d}y} = \lambda(1 - 2y/\gamma) = \begin{cases}\lambda & \text{at } y = 0 \\ -\lambda & \text{at } y = \gamma\end{cases}$$

For $\lambda > 0$, $y = 0$ is unstable fixed point; $y = \gamma$ is stable.







4.3 Fixed points in discrete equations

Consider *first order* (links x_{n+1} only to x_n) discrete equation

$$x_{n+1} = f(x_n)$$

Definition (Fixed point of discrete equation)**.** A *fixed point* of a first-order discrete equation is a value of x_n such that $x_{n+1} = x_n$, i.e.,

$$f(x_n) = x_n$$

Stability now perturbation analysis.

Let x_f be a fixed point, write

$$x_n = x_f + \epsilon_n$$

$$\begin{aligned} x_f + \epsilon_{n+1} &= f(x_f + \epsilon_n) \\ &= f(x_f) + \epsilon_n \frac{\mathrm{d}f}{\mathrm{d}x}(x_f) + O(\epsilon_n^2) \\ \epsilon_{n=1} &\approx \epsilon_n \frac{\mathrm{d}f}{\mathrm{d}x}(x_f) \end{aligned}$$

Follows that

$$\begin{aligned} |f'(x_f)| < 1 &\implies \text{stable F.P.} \\ |f'(x_f)| > 1 &\implies \text{unstable F.P.} \end{aligned}$$

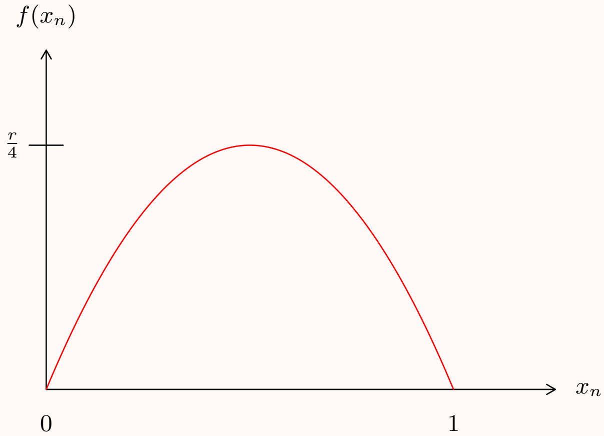
Example (Logistic map).

$$x_{n+1} = \underbrace{rx_n(1-x_n)}_{f(x_n)}$$

"Discrete logistic equation" or "logistic map"
Complimentary function:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda x\left(1-\frac{x}{X}\right)$$

Interested in $x_n \geq 0$. If $r < 5$, $\{x_n\}$ stay in range $0 \leq x_n \leq 1$:



Fixed points: $x_n = rx_n(1-x_n)$

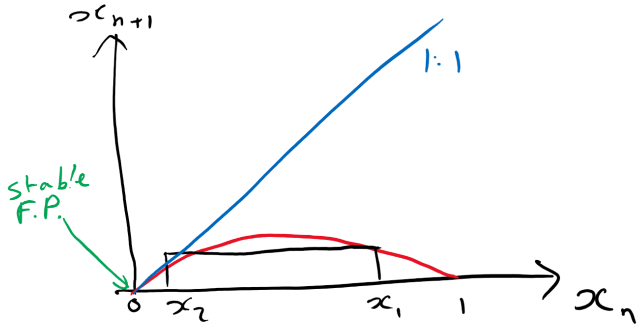
$$\implies x_n = 0 \wedge x_n = 1 - \frac{1}{r}$$

(note that the second solution only works for $x_n > 0$ if $r > 1$)
Stability?

$$f'(x) = r(1-2\alpha) = \begin{cases} r & \text{at } x = 0 \\ 2-r & \text{at } x = 1 - \frac{1}{r} \end{cases}$$

Fixed point at $x = 0$: stable for $0 < r < 1$; unstable for $r > 1$.
Fixed point at $x = 1 - \frac{1}{r}$: stable for $1 < r < 3$; unstable for $r > 3$.

Illustrate with "cobweb diagrams" (Details are non-examinable)
$0 < r < 1$:



Converge to $x_n = 0$ for any starting points $0 \leq x_1 \leq 1$.

TOPIC IV: HIGHER-ORDER LINEAR ODEs

1 Second order ODEs

General form

$$\underbrace{a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy}_{\mathcal{D}y}=f(x) \tag{$*$}$$

where $\mathcal{D}$ is a *differential operator* defined by

$$\mathcal{D}\equiv a\frac{\mathrm{d}^2}{\mathrm{d}x^2}+b\frac{\mathrm{d}}{\mathrm{d}x}+c$$

Note that this is *linear*.

Definition (Linear operator)**.** $\mathcal{D}$ is *linear* if for any $y_1(x)$ and $y_2(x)$, and constants α and β,

$$\mathcal{D}(\alpha y_1+\beta y_2)=\alpha\mathcal{D}y_1+\beta\mathcal{D}y_2$$

Exploit linearity to solve ODE in 2 steps:

(1) Find complimentary functions that satisfy homogeneous equation

$$a\frac{\mathrm{d}^2y_c}{\mathrm{d}x^2}+b\frac{\mathrm{d}y_c}{\mathrm{d}x}+cy_c=0$$

(2) Find *any* particular integral that satisfies the full equation.

Then the solution to full equation from y_c+y_p:

$$\mathcal{D}(y_c+y_p)=\mathcal{D}(y_c)+\mathcal{D}(y_p)=0+f(x)$$

If y_{c_1} and y_{c_2} are linearly independent complimentary functions, then $y_{c_1}+y_p$ and $y_{c_2}+y_p$ are linearly independent solutions of the full equation.

Definition (Linear independence of functions)**.** Set of N functions $\{f_i(x)\}$ is *linearly dependent* if

$$\sum_{i=1}^{N}c_if_i(x)=0$$

for N constants c_i, not all of which are zero. Otherwise the functions are *linearly independent*. [Equivalently, if any function can be written as a linear combination of the others they are linearly dependent.]

1.1 Complementary functions

Recall

$$\frac{\mathrm{d}}{\mathrm{d}x}e^{\lambda x} = \lambda e^{\lambda x}$$

(λ is an eigenvalue and $e^{\lambda x}$ is a eigenfunction) Also $e^{\lambda x}$is an eigenfunction of $\mathcal{D}$:

$$\mathcal{D}(e^{\lambda x}) = a\frac{\mathrm{d}^2}{\mathrm{d}x^2}e^{\lambda x} + b\frac{\mathrm{d}}{\mathrm{d}x}e^{\lambda x} + ce^{\lambda x} = \underbrace{(a\lambda^2 + b\lambda c)}_{\text{eigenvalue}}e^{\lambda x}$$

Complimentary functions of $(*)$ satisfy $\mathcal{D}y_c = 0$: eigenfunctions of $\mathcal{D}$ with eigenvalue zero:

$$y_c = Ae^{\lambda x}$$

provided $a\lambda^2 + b\lambda + c = 0$. (*Characteristic equation* of $ay'' + by' + cy = 0$)
Two roots λ_1 and λ_2.

- If $\lambda_1 \neq \lambda_2$ then we have two linearly independent complimentary functions

  $$y_{c_1} \propto e^{\lambda_1 x} \qquad \text{and} \qquad y_{c_2} \propto e^{\lambda_2 x}$$

  Most general complimentary function is a linear combination of y_{c_1} and y_{c_2}:

  $$y_c = c_1 y_{c_1}(x) + c_2 y_{c_2}(x)$$

  (so y_{c_1} and y_{c_2} form a basis for the solution subspace of the homogeneous ODE)

  Note. Roots may be complex → oscillations.

- $\lambda_1 = \lambda_2$: (degenerate case). Only generates one linearly independent complimentary function of form $e^{\lambda_1 x}$. How to find second complimentary function? See example below.

Example (non-degenerate, real roots)**.**

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0$$

Characteristic equation:

$$\lambda^2 - 5\lambda + 6 = 0$$
$$\implies (\lambda - 2)(\lambda - 3) = 0$$

Complimentary function:

$$y_c(x) = Ae^{2x} + Be^{3x}$$

Example (complex roots).

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4y = 0$$

Characteristic equation:

$$\lambda^2 + 4 = 0 \implies \lambda = \pm 2i$$

General complimentary function:

$$\begin{aligned} y_c &= A \underbrace{e^{2ix}}_{\cos 2x + i \sin 2x} + Be^{-2ix} \\ &= \underbrace{(A+B)}_{\alpha} \cos 2x + \underbrace{i(A-B)}_{\beta} \sin 2x \end{aligned}$$

Example (degeneracy and detuning)**.**

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0$$

Characteristic equation:

$$\begin{aligned} \lambda^2 - 4\lambda + 4 = 0 \\ \implies (\lambda - 2)^2 = 0 \end{aligned}$$

Degenerate: $\lambda = 2, 2$ so only *one* linearly independent solution of the form $e^{\lambda x}$. Now *detuning* is removing degeneracy by considering "detuned" (modified) equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + (4 - \epsilon^2)y = 0 \qquad (\epsilon \ll 1)$$

in $\lim_{\epsilon \to 0}$, detuned equation $\to$ one we really want to solve. Try $y = e^{\lambda x}$:

$$\lambda^2 - 4\lambda + (4 - \epsilon^2) = 0$$

$$\implies \lambda = 2 \pm \epsilon$$

General solution of detuned equation is

$$\begin{aligned} y &= Ae^{(2+\epsilon)x} + Be^{(2-\epsilon)x} \\ &= e^{2x}[Ae^{\epsilon x} + Be^{-\epsilon x}] \\ &= e^{2x}[(A + B) + \epsilon(A - B) + (A\epsilon^2) + O(B\epsilon^2)] \end{aligned}$$

Consider applying initial conditions

$$y(0) = C \qquad \text{and} \qquad y'(0) = D$$

(to original detuned equation) Have

$$A + B = C \qquad \text{and} \qquad 2(A + B) + \epsilon(A - B) = D$$

so

$$y \to e^{2x}[C + (D - 2C)x + O(A\epsilon^2) + O(B\epsilon^2)]$$

We have

$$\begin{aligned} A &= \frac{1}{2}\left(C + \frac{D - 2C}{\epsilon}\right) \\ \implies A\epsilon^2 &= \frac{1}{2}\left(C\epsilon^2 + (D - 2C)\epsilon\right) \\ \implies O(A\epsilon^2) &= O(\epsilon) \end{aligned}$$

It follows that the general solution of original equation is

$$y = e^{2x}[\alpha + \beta x]$$

for arbitrary constants α and β.

General rule: if $y_{c_1}(x)$ is a degenerate complimentary function of a linear ODE with constant coefficients, then

$$y_{c_2}(x) = x y_{c_1}(x)$$

is a second, linearly independent complimentary function.

2 Homogeneous 2nd-order ODEs with non-constant coefficients

General form

$$y'' + p(x)y' + q(x)y = 0 \qquad (*)$$

2.1 Second CF — reduction of order

Aim: given one solution of $(*)$, $y_2(x)$, find another, $y_2(x)$.
Look for a solution

$$y_2 = v(x)y_1(x)$$

So we have

$$\begin{aligned} y_2 &= v'y_2 + vy_1' \\ y_2'' &= v''y_1 + 2v'y_1' + vy_1'' \end{aligned}$$

If y_2 satisfies $(*)$ then

$$v''y_1 + v'(2y_1 + py_1) + v\underbrace{(y_1'' + py_1' + qy_1)}_{=0 \text{ by } y_1 \text{ in } (*)} = 0$$

Follows that

$$\begin{aligned} v''y_1 + v'(2y_1' + py_1) &= 0 \\ u'y_1 + u(2y_1' + py_1) &= 0 \qquad \text{for } u = v' \end{aligned}$$

(note that this is a first order ODE in u!) Separable:

$$\begin{aligned} \frac{u'}{u} &= -\frac{y_1'}{y_1} - p(x) \\ \implies \ln u &= -2\ln y_1 - \int_0^x p(w)\mathrm{d}w + \ln A \\ \implies u(x) &= \frac{A}{[y_1(x)]^2} \exp\left[-\int_0^x p(w)\mathrm{d}w\right] \end{aligned}$$

Integrate again to find $v(x)$.

Example.

$$y'' \underbrace{-4}_{p(x)} y' \underbrace{+4}_{q(x)} y = 0$$

Have a degenerate solution $y_1 = e^{2x}$. Reduction of order:

$$\frac{u'}{u} = -2 \times 2 + 4 = 0$$

$$\implies v' = \text{const} \implies v = Ax + B.$$

Have $y_2(x) = (Ax + B)e^{2x}$. So xe^{2x} is a second linearly independent solution.

2.2 Phase space

For n-th order linear ODE

$$p(x)y^{(n)} + q(x)y^{(n-1)} + \cdots + r(x)y = f(x)$$

From this equation, $y^{(n)}(x)$ is determined by $y(x)$, $y'(x)$, $\ldots$, $y^{(n-1)}(x)$. By differentiating the equation, we can also find that higher derivatives are determined. Hence we can compute all the derivatives at a point, and therefore construct the Taylor series about x_0 if $y(x_0)$, $y'(x_0$, $\ldots$, $y^{(n-1)}(x)$ are specified. So solution is determined by these initial conditions.

State of system fully specified (can predict subsequence evalutation) at any x by *solution vector*

$$\mathbf{Y}(x) = \begin{pmatrix} y(x) \\ y'(x) \\ \vdots \\ y^{(n-1)}(X) \end{pmatrix}$$

At any x, $\mathbf{Y}(x)$ defines a point in *nD phase space*.

Example.

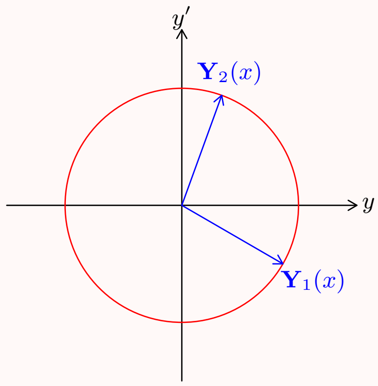
$$y'' = 4y = 0 \tag{SHM}$$

$$y_1 = \cos 2x; \qquad y_2 = \sin 2x$$

$$\mathbf{Y}_1(x) = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} \cos 2x \\ -2\sin 2x \end{pmatrix}$$

$$\mathbf{Y}_2(x) = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix} = \begin{pmatrix} \sin 2x \\ 2\cos 2x \end{pmatrix}$$

Phase space



Note. $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are linearly independent vectors $\forall x$ (in linear algebra sense).

Can use $\mathbf{Y}_1$ and $\mathbf{Y}_2$ at any x as a *basis* for phase space.

2.3 Wronskian and linear dependence

Recall, $y_i(x)$ $(i = 1, \ldots, n)$ are linearly depdendent if

$$\sum_{i=1}^{n} c_i y_i(x) = 0 \qquad \forall x$$

must hold for all x hence differentiable $n-1$ times:

$$\sum_{i=1}^{n} c_i \mathbf{Y}_i(x) = \mathbf{0}$$

See that

$$\{y_i(x)\} \text{ linearly dependent } \implies \{\mathbf{Y}_i\} \text{ linearly depdendent } \forall x$$

Fundamental matrix: $(\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n)$ has zero determinant if the $\{y_i\}$ are linearly dependent.

Definition (Wronskian)**.** Wronskian $W(x)$ of n functions $y_i(x)$ $(i = 1, \ldots, n)$ is

$$W(x) = |\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \ldots, \mathbf{Y}_n| = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2 & \cdots & y_n' \\ \vdots & \vdots & \ddots & \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{pmatrix}$$

We have

$$\{y_i(x)\} \text{ linearly dependent } \implies W(x) = 0 \ \forall x$$

(test for linear dependence of n functions)

Remark. $W(x) = 0$ does not necessarily imply linear dependence of the $y_i(x)$.

Example.

$$y'' + 4y = 0 \implies y_1 = \cos 2x, y_2 = \sin 2x$$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$W \neq 0 \implies y_1 \text{ and } y_2 \text{ are linearly independent.}$$

2.4 Abel's Theorem

Theorem (Abel's Theorem)**.** Given any two solutions of

$$y'' + p(x)y + q(x)y = 0$$

If $p(x)$ and $q(x)$ are continuous on an interval I, then either $W(x) = 0\ \forall x \in I$ or $W(x) \neq 0\ \forall x \in I$.

Proof (sketch).

$$\begin{aligned}
W &= y_1 y_2' - y_2 y_1' \\
\implies W' &= y_1 y_2'' - y_2 y_1'' \\
&= y_2(py_1' + qy_1) - y_1(py_2' + qy_2) \\
&= y_1 py_1' - y_1 py_2' \\
&= -p(x)W
\end{aligned}$$

Separable ODE for W:

$$W(x) = W(x_0) \underbrace{\exp\left(-\int_{x_0}^{x} p(u)\mathrm{d}u\right)}_{\text{never zero}}$$

(This is known as "Abel's identity") If $W(x)$ vanished at some point x_0, it is zero $\forall x$; else is is never zero. □

Note. If $p(x) = 0$ (no y' term in ODE), then the Wronskian is constant.

Example (Bessel's Equation)**.**

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

generally no closed form solutions (but for example for some half integer values there exists a closed form solution)

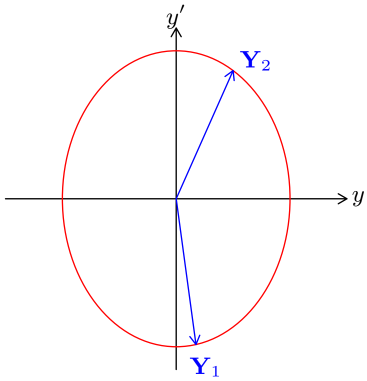
$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right) y = 0$$

Abel:

$$\begin{aligned}
W(x) &= W(x_0)\exp\left(-\int_{x_0}^{x} \frac{\mathrm{d}u}{u}\right) \\
&= W(x_0)\frac{x_0}{x}
\end{aligned}$$

Note: Abel's identity determines form of W without having to solve ODE.

Geometric interpretation: (phase space)



Solution vectors always collinear or never collinear as x varies. [Basis at some $x \to$ basis $\forall x$.]

Applications of Abel's Theorem

$$y_1 y_2' - y_2 y_1' = W(x_0) \exp\left(-\int_{x_0}^{x} p(w) \mathrm{d}w\right)$$

If y_1 known, can solve this ODE to find y_2:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y_2}{y_1}\right) = \frac{W(x_0)}{y_1^2(x)} \exp\left(-\int_{x_0}^{x} p(u) \mathrm{d}u\right)$$

Exactly as for "reduction of order".

Generalisation

Abel's Theorem also holds for solutions of nth-order linear, homogeneous ODEs.
Such an equation can always be written as a system of first-order equations:

$$\mathbf{Y}' + \underline{\mathbf{A}}\mathbf{Y} = \mathbf{0}$$

Example.

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0$$

$$\begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ r & q & p \end{pmatrix}}_{\underline{\mathbf{A}}} \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = \mathbf{0}$$

Can be shown (Q7, Ex 3) that

$$W' + \mathrm{Tr}(\underline{\mathbf{A}})W = 0$$
$$\implies W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} \mathrm{Tr}[\underline{\mathbf{A}}(u)]\mathrm{d}u\right)$$

so Abel's theorem holds.

2.5 Linear equidimensional ODEs

Definition (Equidimensional ODE)**.** A linear, second-order ODE is *equidimensional* if it has the following form

$$ax^2y'' + bxy' + cy = f(x) \qquad (*)$$

(where a, b and c are constants)

Why call this equidimensional?
Let $y(x)$ be a complimentary function of $(*)$. Consider

$$\phi(x) = y(\lambda x)$$

for some arbitrary real λ. Then

$$x\frac{\mathrm{d}\phi}{\mathrm{d}x} = x\lambda y'(\lambda x)$$
$$x^2\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = x^2\lambda^2 y''(\lambda x)$$
$$\implies ax^2\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + bx\frac{\mathrm{d}\phi}{\mathrm{d}x} + c\phi = \text{LHS of } (*) \text{ evaluated at } \lambda x = 0$$

Solving by Eigenfunctions

$y = x^k$ is an eigenfunction of $x\frac{\mathrm{d}}{\mathrm{d}x}$, with eigenvalue k:

$$x\frac{\mathrm{d}}{\mathrm{d}x}x^k = kx^k$$

Look for complimentary functions of $(*)$: $y_c = x^k$

$$ak(k-1) + bk + c = 0$$
$$\implies ak^2 + (b-a)k + c = 0$$

Solve quadratic equation to determine k_1 and k_2. If $k_1 \neq k_2$ (non-degenerate) general complimentary function:

$$y_c = Ax^{k_1} + Bx^{k_2}$$

Solving by substitution

Switch independent variable from x to

$$z \equiv \ln x$$

For $y(x) = y(e^z)$

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \underbrace{e^z y'(e^z)}_{xy'(x)}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = \underbrace{e^z y'(e^z)}_{x'(x)} + \underbrace{e^{2z} y''(e^z)}_{x^2 y''(x)}$$

Follows that $(*)$ becomes

$$\underbrace{a\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + (b-a)\frac{\mathrm{d}y}{\mathrm{d}z} + cy}_{\text{constant coefficients}} = f(e^z)$$

Complimentary function $y_c \propto e^{\lambda z}$

$$a\lambda^2 + (b-a)\lambda + c = 0$$

(Some characteristic equation as for k_1 and k_2) General complimentary function ($k_1 \neq k_2$):

$$y_c = Ae^{k_1 z} + Be^{k_2 z} = Ax^{k_1} + Bx^{k_2}$$

Degenerate case ($k_1 = k_2 = k$)

$$y_c = Ae^{kz} + Bxe^{kz} = Ax^k + B\ln(x)x^k$$

3 Inhomogeneous (Forced) 2nd-order ODEs

3.1 Particular Integrals of equations with constant coefficients

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x)$$

$f(x)$	$y_p(x)$
e^{mx}	Ae^{mx}
$\sin kx$ or $\cos kx$	$A\sin kx + B\cos kx$
Polynomial $p_n(x)$ (n-th degree)	Polynomial $q_n(x)$ $(a_nx^n + \cdots + a_1x + a_0)$

Example.

$$y'' - 5y' + 6y = \underbrace{2x + e^{4x}}_{f(x)}$$

Try

$$\begin{aligned} y_p &= \overbrace{(Ax+B)}^{\text{for } 2x} + \overbrace{ce^{4x}}^{\text{for } e^{4x}} \\ y_p' &= A + 4ce^{4x} \\ y_p'' &= 16ce^{4x} \end{aligned}$$

$$\underbrace{(16C - 20C + 6C)}_{\to 1} e^{4x} + \underbrace{(6A)}_{\to 2} x + \underbrace{(-5A + 6B)}_{\to 0} = 2x + e^{4x}$$

$$C = \frac{1}{2}, \quad A = \frac{1}{3}, \quad B = \frac{5}{18}$$

Complementary function: $\alpha e^{3x} + \beta e^{2x}$. General solution:

$$\alpha e^{3x} + \beta e^{2x} + \frac{1}{2}e^{4x} + \frac{1}{3}x + \frac{5}{18}$$

Resonance

What if forcing term involves a complimentary function? $\to$ Detuning.

Example.
$$\ddot{y} + \omega_0^2 y = \sin\omega_0 t \qquad (*)$$

Complimentary function:

$$y_c(t) = A\cos\omega_0 t + B\sin\omega_0 t \qquad \text{(SHM)}$$

Oscillator driven "resonantly".
"Detuned equation":
$$\ddot{y} + \omega_0^2 y = \sin\omega t \qquad (\omega \neq \omega_0) \qquad (\dagger)$$

Try particular integral:
$$y_p(t) = C\sin\omega t$$

Substitute into (†):

$$C(\omega_0^2 - \omega^2) = 1$$
$$\implies C = \frac{1}{\omega_0^2 - \omega^2}$$

Limit as $\omega \to \omega_0$ does not exist. Add in a complimentary function to regularise the limit $\omega \to \omega_0$:
$$y_p(t) = \frac{1}{\omega_0^2 - \omega^2}\left(\sin\omega t - \underbrace{\sin\omega_0 t}_{\text{CF}}\right)$$

Evaluate indeterminate limit with L'Hôpital's rule:

$$\lim_{\omega\to\omega_0} y_p(t) = \lim_{\omega\to\omega_0}\left(\frac{t\cos\omega t}{-2\omega}\right) = -\frac{t\cos\omega_0 t}{2\omega_0}$$

Particular integral of original equation $(*)$ is

$$y_p(t) = -\frac{t}{2\omega_0}\cos\omega_0 t$$

General rule: if forcing is a linear combination of complimentary functions, particular integral is of the form
$$y_p(t) = t \times \underbrace{(\text{non-resonant PI})}_{\cos\omega_0 t \text{ above}}$$

Resonance in Equidimensional ODEs

Complimentary function for equidimensional ODE (non-degenerate):

$$y_c = Ax^{k_1} + Bx^{k_2}$$

If forcing $f(x) \propto x^{k_1}$ (or x^{k_2}), particular integral is

$$y_p(x) \propto \ln x \times x^{k_1}$$

Follows from transforming to $z = \ln x$:

$$y_p(z) \propto z e^{k_1 z} = \ln x \; x^{k_1}$$

3.2 Variation of Parameters

Systematic way of finding a particular integral given two linearly independent complimentary functions (y_1 and y_2).

Consider

$$y'' + p(x)y' + q(x)y = f(x) \tag{$*$}$$

with linearly independent complimentary functions y_1 and y_2. Solution vectors $\mathbf{Y}_1(x)$ and $\mathbf{Y}_2(x)$. We will use these as a *basis* in phase space at any argument x to write the solution vector of the particular integral as

$$\mathbf{Y}_p(x) = u(x)\mathbf{Y}_1(x) + v(x)\mathbf{Y}_2(x)$$

In components:

$$y_p = u(x)y_1 = v(x)y_2 \tag{$\dagger$}$$

$$y_p' = u(x)y_1' + v(x)y_2' \tag{$\ddagger$}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} y_p' = y_p'' \underbrace{=}_{(\ddagger)} u y_1'' + u' y_1' + v y_2'' = v' y_2'$$

Follows that

$$f(x) = u y_1'' + u' y_1' + v y_2'' + v' y_2' + p(x)[u y_1' + v y_2'] + q(x)[u y_1 + v y_2]$$

Now using the fact that y_1 and y_2 are complimentary functions, we get that

$$f(x) = u' y_1' + v' y_2'$$

However ($\ddagger$) has to be consistent with $\frac{\mathrm{d}}{\mathrm{d}x}(\dagger)$, so

$$\frac{\mathrm{d}}{\mathrm{d}x} y_p = u' y_1 + u y_1' + v' y_2 + v y_2'$$

Compare to ($\ddagger$):

$$u' y_1 + v' y_2 = 0$$

Combining with $u' y_1' + v' y_2' = f(x)$, we have

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$$\implies \begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{\underbrace{W}_{y_1 y_2' - y_2 y_1'}} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix}$$

Hence

$$u' = -\frac{y_2}{W} f; \quad v' = \frac{y_1}{W} f$$

Integrate to find

$$y_p(x) = y_2(x) \int^x \frac{1}{W(t)} f y_1(t) f(t) \mathrm{d}t - y_1(x) \int^x \frac{1}{W(t)} y_2(t) f(t) \mathrm{d}t$$

(changing the lower limit in these integrals just adds a multiple of a complimentary function).

Example.

$$y'' + 4y = \underbrace{\sin 2x}_{f(x)}$$

Complimentary functions $y_1 = \sin 2x$, $y_2 = \cos 2x$. (Note that forcing term is resonant).

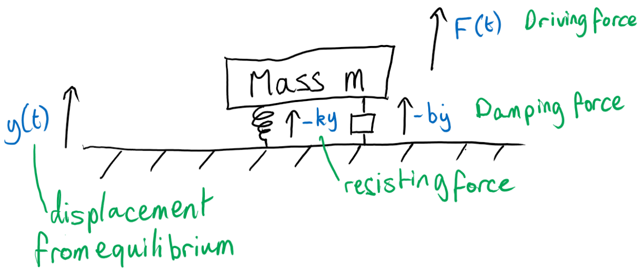
$$W(x) = -2$$

so

$$y_p(x) = \cos 2x \int^x -\frac{1}{2} \underbrace{\sin 2t \sin 2t}_{\frac{1}{2}(1-\cos 4t)} \mathrm{d}t - \sin 2x \int^x -\frac{1}{2} \underbrace{\cos 2t \sin 2t}_{\frac{1}{2}\sin 4t} \mathrm{d}t$$

$$\begin{aligned} y_p(x) &= -\frac{1}{4} \cos 2x \left[x - \frac{1}{4} \sin 4x \right] + \frac{1}{4} \sin 2x \left[-\frac{1}{4} \cos 4x \right] \\ &= \frac{1}{16} \sin 2x - \frac{1}{4} x \cos 2x \end{aligned}$$

4 Forced Oscillating Systems: Transients and Damping



Newton II:

$$\begin{aligned} m\ddot{y} &= \sum \text{forces} \\ &= -ky - b\dot{y} + F(t) \end{aligned}$$

$$m\ddot{y} + b\dot{y} + ky = F(t)$$

(b, k positive constants). For $b = 0$ and $F(T) = 0$, simple harmonic motion at

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Dimensionless t coordinate:

$$\tau = \omega_0 t \qquad \left(\frac{\mathrm{d}}{\mathrm{d}\tau} = 1\right)$$

Divide ODE by k:

$$y'' + \underbrace{2\kappa y'}_{\frac{b}{m\omega_0}} + y = \underbrace{f(\tau)}_{\frac{F(t)}{k}}$$

Unforced system described by one dimensionless parameter κ.

4.1 Free (unforced or natural) response

$f = 0$:

$$y'' + 2\kappa y' + y = 0$$

Look for solutions $y \propto e^{\lambda x}$:

$$\lambda^2 + \kappa\lambda + 1 = 0$$

(characteristic equation).

$$\lambda_1, \lambda_2 = -\kappa \pm \sqrt{\kappa^2 - 1}$$

Light damping (underdamping): $\kappa < 1$

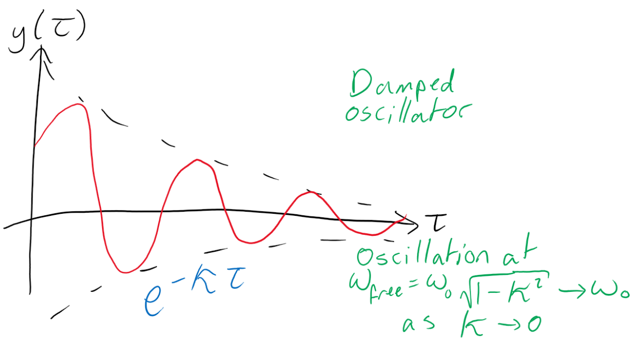
λ_1 and λ_2 complex:

$$\lambda_1, \lambda_2 = -\kappa \pm i\sqrt{1-\kappa^2}$$

General solution:

$$y(\tau) = e^{-\kappa\tau}[A\sin(\sqrt{1-\kappa^2}\tau)] + B\cos(\sqrt{1-\kappa^2}\tau)$$

(A and B constants)



Critical damping: $\kappa = 1$

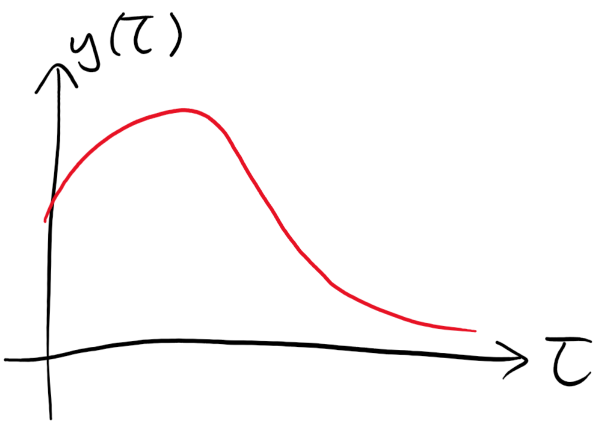
Degenerate case:

$$\lambda_1 = \lambda_2 = -\kappa$$

General solution:

$$y(\tau) = e^{-\kappa\tau}(A + B\tau)$$

(A and B constants)



Heavy damping (overdamping): $\kappa > 1$

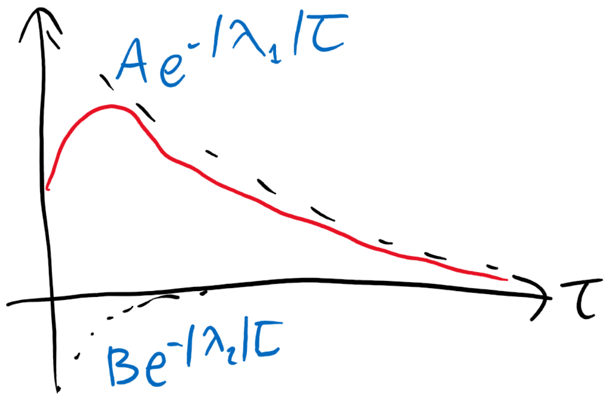
λ_1 and λ_2 are real: take $|\lambda_2| > |\lambda_1|$ without loss of generality.

$$\lambda_1 = \underbrace{-\kappa + \sqrt{\kappa^2 - 1}}_{\text{negative}} \qquad \lambda_2 = \underbrace{-\kappa - \sqrt{\kappa^2 - 1}}_{\text{positive}}$$

General solution:

$$y(\tau) = Ae^{-|\lambda_1|\tau} + Be^{-|\lambda_2|\tau}$$

(first term dominates long-term motion if present)



Note: unforced response decays eventually in all cases.

4.2 Forced response

Initial "transient" response from PI + CF, but CF decays leaving "steady-state" response (PI).

Example.

$$\ddot{y} + \mu\dot{y} + \omega_0^2 y = \frac{F_0}{m}\sin\omega t \qquad (*)$$

Assuming light damping ($\mu < 2\omega_0$). Complimentary function:

$$y_c(t) = e^{-\frac{\mu t}{2}}[A\sin\omega_{\text{free}}t + B\cos\omega_{\text{free}}t]$$

($\kappa = \frac{\mu}{2\omega_0}$) where

$$\omega_{\text{free}} = \sqrt{\omega_0^2 - \frac{\mu^2}{4}}$$

For PI: try

$$y_p(t) = \frac{F_0}{m}(C\sin\omega t + D\cos\omega t)$$

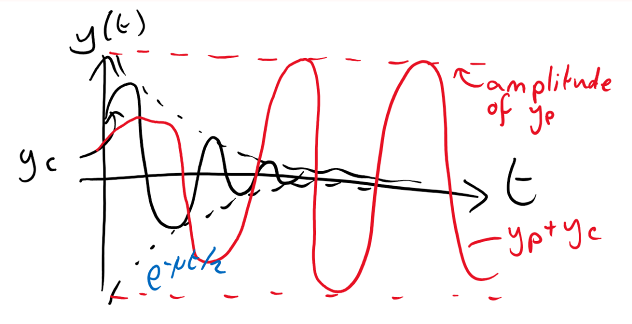
Substitute into $(*)$:

$$\sin\omega t : \qquad -\omega^2 C - \mu\omega D + \omega_0^2 C = 1$$

$$\cos\omega t : \qquad \omega^2 D + \mu\omega C + \omega_0^2 D = 0$$

Gives

$$y_p(t) = \frac{F_0/m}{(\omega_0^2 - \omega^2)^2 + (\mu\omega)^2}[(\omega_0^2 - \omega^2)\sin\omega t - \mu\omega\cos\omega t]$$



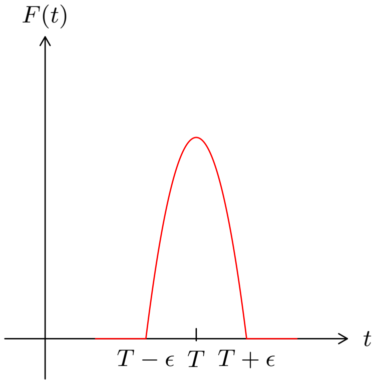
Amplitude of y_p:

$$\frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\mu\omega)^2}}$$

($\frac{F_0}{m}\sqrt{c^2 + p^2}$)

5 Impulses and Point Forces

Consider sudden force $F(t)$ applied between $t = T - \epsilon$ and $T + \epsilon$.



For example strike on oscillator.
As $\epsilon \to 0$, only the *impulse*

$$I = \int_{T-\epsilon}^{T+\epsilon} F(t)\mathrm{d}t$$

matters for subsequent motion $\to$ (mathematically) convenient to consider limit of a sudden impulse.

Forced, damped oscillator

$$m\ddot{y} + b\dot{Y} + ky = F(t)$$

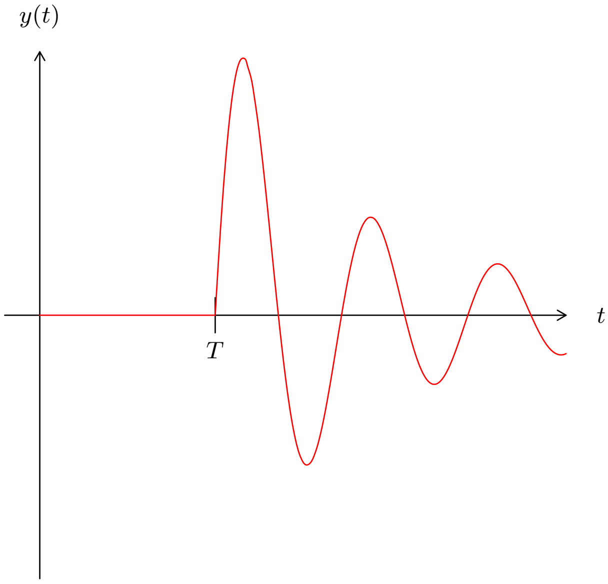
Integrate ODE from $T - \epsilon$ to $T + \epsilon$

$$\lim_{\epsilon \to 0} \left(m[\dot{y}]_{T-\epsilon}^{T+\epsilon} + \underbrace{b[y]_{T-\epsilon}^{T+\epsilon}}_{0 \text{ if } y \text{ continuous}} + \underbrace{k \int_{T-\epsilon}^{T+\epsilon} y\mathrm{d}t}_{0 \text{ if } y \text{ remains finite}} \right) = I$$

here I denotes impulse. Follows that

$$\lim_{\epsilon \to 0} m[\hat{y}]_{T-\epsilon}^{T+\epsilon} = I$$

velocity is discontinuous.



5.1 Dirac delta function

Consider a family of functions $D(t;\epsilon)$ with

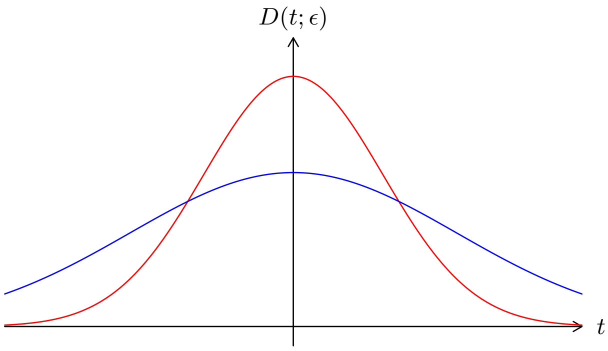
1. $\lim_{\epsilon\to 0} D(t;\epsilon) = 0 \qquad \forall t \neq 0.$

2. $\int_{-\infty}^{\infty} D(t,\epsilon)\mathrm{d}t.$

Impulse force considered scalar $F(t) = ID(t-T,\epsilon)$.
Example family:

$$D(t;\epsilon) = \frac{e^{-t^2/\epsilon^2}}{\epsilon\sqrt{\pi}}$$

(Q14 on example sheet 1)



Family not unique, but for any such family limit $\epsilon \to 0$ yields *Dirac delta function*.

Definition (Dirac delta function)**.** The Dirac delta function is defined by

$$\delta(t) \equiv \lim_{\epsilon \to 0} D(t; \epsilon).$$

Should only use inside integrals.
Key properties:

1. $\delta(x) = 0 \qquad \forall x \neq 0.$
2. $\int_{-\infty}^{\infty} \delta(x) \mathrm{d}x = 1.$
3. For all $g(x)$ continuous at $x = 0$

$$\begin{aligned}\int_{-\infty}^{\infty} g(x)\delta(x)\mathrm{d}x \\ &= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} g(x) D(x; \epsilon)\mathrm{d}x \\ &= g(0)\end{aligned}$$

Generalises to $(b > a)$

$$\int_a^b g(x)\delta(x - x_0)\mathrm{d}x = \begin{cases} g(x_0) & \text{if } a < x_0 < b \\ 0 & \text{otherwise} \end{cases}$$

5.2 Delta-function forming

Consider

$$y'' + p(x)y' + q(x)y = \delta(x)$$

For $x < 0$ or $x > 0$ we have

$$y'' + p(x)y' + q(x)y = 0$$

However, discontinuity in y' at $x = 0$: integrate ODE from $-\epsilon$ to $+\epsilon$

$$\lim_{\epsilon \to 0} [y']_{-\epsilon}^{\epsilon} + p(0) \underbrace{\lim_{\epsilon \to 0} [y]_{-\epsilon}^{\epsilon}}_{0 \text{ if } y \text{ continuous}} + \underbrace{\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} q(x)y\mathrm{d}x}_{0 \text{ if } y \text{ is finite}} = 1$$

$$\implies \lim_{\epsilon \to 0} [y']_{-\epsilon}^{\epsilon} = 1$$

"Jump condition"

Note. Continuity of y at $x = 0$ is required to avoid $y' \sim \delta(x)$ around $x = 0$ and y'' even worse behaved.

General rule: highest-order derivative term in the ODE addresses the delta-function forcing.

Example.

$$y'' - y = 3\delta\left(x - \frac{\pi}{2}\right)$$

boundary conditions $t = 0$ at $x = 0$ and $y = 0$ at $x = \pi$.
For $0 \leq x < \frac{\pi}{2}$: $y'' - y = 0$

$$y = A\sinh(x) \qquad [y(0) = 0]$$

For $\frac{\pi}{2} < x \leq \pi$: $y'' - y = 0$

$$y = C\sinh(\pi - x) \qquad [y(\pi) = 0]$$

Join up at $x = \frac{\pi}{2}$ with
We have y continuous $\implies A = C$.

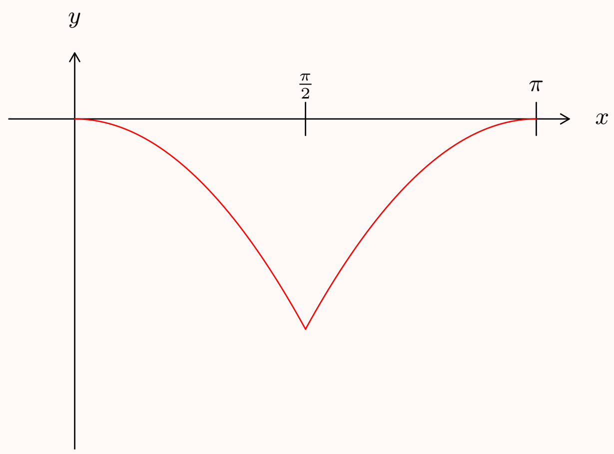
2. Now "Jump condition":

$$\lim_{\epsilon \to 0} [y']_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon} = 3$$

$$\implies -C\cosh\left(\frac{\pi}{2}\right) - A\cosh\left(\frac{\pi}{2}\right) = 3$$

hence

$$A = C = -\frac{3}{2\cosh\left(\frac{\pi}{2}\right)}$$



(discontinuity in y' at the "spike")

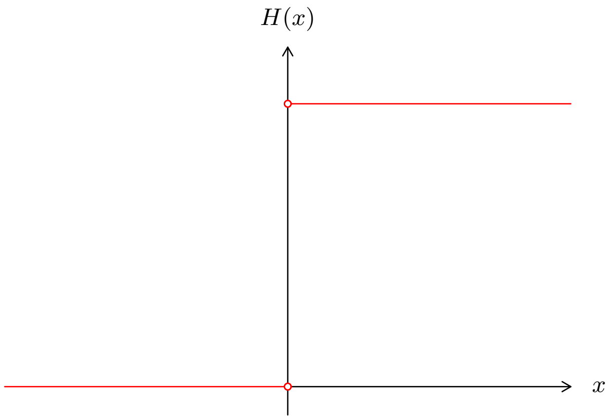
5.3 Heaviside Step Function

Definition (Heaviside function)**.** The *Heaviside step function* is

$$H(x) = \int_{-\infty}^{x} \delta(t)\mathrm{d}t$$

Follows that

$$t(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \\ \text{undefined} & \text{for } x = 0 \end{cases}$$



Fundamental theorem of calculus implies

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \delta(x)$$

Forcing with $H(x)$

Consider

$$y'' + p(x)y' + q(x)y = H(x)$$

Have

$$y'' + p(x)y' + q(x)y = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}$$

Follows that

$$\lim_{\epsilon\to 0}[y'']_{-\epsilon}^{\epsilon} + p(0)\lim_{\epsilon\to 0}[y']_{-\epsilon}^{\epsilon} + q(0)[y]_{-\epsilon}^{\epsilon} = 1 \tag{$*$}$$

If $y'' \sim H(x)$ around $x = 0$, y' and y are continuous and $(*)$ is satisfied.

$$\lim_{\epsilon\to 0}[y']_{-\epsilon}^{\epsilon} = 0 \quad \text{and} \quad \lim_{\epsilon\to 0}[y]_{-\epsilon}^{\epsilon} = 0$$

"Jump" conditions for Heaviside forcing.

6 Higher Order Discrete Equations

Consider a linear, discrete second-order equation

$$ay_{n+2} + by_{n+1} + cy_n = f_n$$

where a, b and c are constants and f_n is a forcing sequence.

Note. Might arise from discretising a second order ODE with constant coefficients.

$$\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x_n} \simeq \frac{\overbrace{y(x_n+h)}^{y_{n+1}} + \overbrace{y(x_n-h)}^{y_{n-1}} - \overbrace{2y(x_n)}^{y_n}}{h^2}$$

Solve with similar methods to ODEs:

$$y_n = y_n^{(c)} + y_n^{(p)}$$

Complementary function satisfies

$$ay_{n+2}^{(c)} + by_{n+1}^{(c)} + cy_n^{(c)} = 0 \qquad (*)$$

For ODE with constant coefficients, y or $e^{\lambda x} \to e^{\lambda nk} \o k^n$.
Try $y_n^{(c)} \propto k^n$ in $(*)$:

$$\begin{aligned} ak^{n+2} + bk^{n+1} + ck^n &= 0 \\ \implies ak^2 + bk + c &= 0 \qquad k \neq 0 \end{aligned}$$

"Characteristic equation".
Two roots of characteristic equation in general, $k = k_1$ and $k = k_2$. So general complimentary function:

$$y_n^{(c)} = \begin{cases} Ak_1^n + Bk_2^n & \text{if } k_1 \neq k_2 \\ (A+Bn)k_1^n & \text{if } k_1 = k_2 \end{cases}$$

(Note that this is similar to differential equations, where we get $A + Bx$).

Now we will look at common forcing functions, and suitable particular integrals to try in these cases.

f_n	$y_n^{(c)}$
k^n	$Ak^n \text{if } k \notin \{k_1, k_2\}$
k_1^n	Ank_1^n
n^p(p a non-negative integer)	$An^p + Bn^{p-1} + \cdots + Cn + D$

Example (Fibonacci sequence).

$$y_n = y_{n-1} + y_{n-2}$$

with initial conditions $y_0 = 1$ and $y_1 = 1$. So the first few terms are $1, 1, 2, 3, 5, 8$. We have

$$y_{n+2} - y_{n+1} - y_n = 0 \qquad (*)$$

Try $y_n = k^n$:

$$\implies k^2 - k - 1 = 0$$

$$\implies k = \frac{1 \pm \sqrt{5}}{2}$$

"Golden ratio" $\phi_1 = \frac{1+\sqrt{5}}{2}$ and $\frac{-1}{\phi_1} = \frac{1-\sqrt{5}}{2}$. General solution of $(*)$ is

$$y_n = A\phi_1^n + B\phi_2^n$$

for some constants A and B, which are determined by the initial conditions:

$$y_0 = 1 \implies A + B = 1$$

$$y_1 = 1 \implies A\phi_1 + B\phi_2 = 1$$

$$\implies A = \frac{\phi_1}{\sqrt{5}}, B = \frac{-\phi_2}{\sqrt{5}} = \frac{1}{\sqrt{5}\phi_1}$$

Follows that

$$y_n = \frac{\phi_1^{n+1} - \phi_2^{n+1}}{\sqrt{5}} = \frac{\phi_1^{n+1} - (-1/\phi_1)^{n+1}}{\sqrt{5}}$$

Note that y_n is a sequence of integers, but ϕ_1 is irrational, so it's almost magical that the formula always gives an integer.
Since $\phi_1 > 1$, $y_n \to \phi_1^{n+1}/\sqrt{5}$ as $n \to \infty$, so

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \phi_1.$$

7 Series Solutions

Series solutions for linear, homogeneous, second-order ODEs.

$$p(x)y'' + q(x)y' + r(x)y = 0 \tag{$*$}$$

Seek a series solution around x_0. Feasibility depends on nature of $p(x)$, $q(x)$ and $r(x)$ around x_0.

7.1 Classification of Singular Points

Definition (Ordinary and Singular Points)**.** Point x_0 is an *ordinary point* if both $q(x)/p(x)$ *and* $r(x)/p(x)$ have Taylor series around x_0. ("analytic" there). Otherwise, x_0 is a *singular point.*

Types of singular point: if x_0 is a singular point of $(*)$, but $(*)$ can be rewritten as

$$\underbrace{P(x)(x-x_0)^2}_{p(x)} y'' + \underbrace{Q(x)(x-x_0)}_{q(x)} y' + \underbrace{R(x)}_{r(x)} y = 0.$$

with $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$ analytic at x_0, then x_0 is a *regular singular point.* Otherwise x_0 is an *irregular singular point.*
Equivalently: if

$$(x-x_0)\frac{q(x)}{p(x)} \left[= \frac{Q}{P}\right]$$

and

$$(x-x_0)^2\frac{r(x)}{p(x)} \left[= \frac{R}{P}\right]$$

are analytic at x_0, then x_0 is a regular singular point.
"No non-singular and equidimensional equation."

Example.

$$\underbrace{(1-x^2)}_{p} y'' - \underbrace{2x}_{q} y' + \underbrace{2}_{r} y = 0$$

$$\frac{q(x)}{p(x)} = \frac{-2x}{1-x^2} = \frac{2x}{(x-1)(x+1)}$$

$$\frac{r(x)}{p(x)} = \frac{2}{1-x^2} = -\frac{2}{(x-1)(x+1)}$$

$x = \pm 1$ are *singular* points. They are *regular* since, for example

$$(x-1)\frac{q(x)}{p(x)} = \frac{2x}{x+1}$$

analytic at $x = 0$.

Example.

$$\underbrace{\sin x}_{p}\, y'' + \underbrace{\cos x}_{q}\, y' + \underbrace{2}_{r}\, y = 0$$

We have

$$\frac{q}{p} = \frac{\cos x}{\sin x}$$

$$\frac{r}{p} = \frac{2}{\sin x}$$

$x = n\pi$ are singular points (n integer). However

$$\begin{aligned} \underbrace{(x - n\pi)}_{\varepsilon}\frac{q}{p} &= \varepsilon \frac{\omega s(n\pi + \varepsilon)}{\sin(n\pi + \varepsilon)} \\ &= \varepsilon \frac{\cos \varepsilon}{\sin \varepsilon} \end{aligned}$$

(Has a Taylor series about $\varepsilon = 0$). Also

$$(x - n\pi)^2 \frac{r}{p} = \frac{2\varepsilon^2}{(-1)^n \sin \varepsilon}$$

which has a Taylor series about $\varepsilon = 0$. Hence $x = n\pi$ are *regular singular points*.

Example.

$$\underbrace{(1 + \sqrt{x})}_{p}\, y'' \underbrace{-2}_{q}\, y' + \underbrace{2}_{r}\, y = 0$$

$$x\frac{q}{p} = -\frac{2x^2}{1 + \sqrt{x}}$$

does not have a Taylor series about $x = 0$, so $x = 0$ is an *irregular singular point*.

7.2 Method of Frobenius

$$p(x)y'' + q(x)y' + r(x)y = 0 \tag{$*$}$$

Theorem (Fuch's Theorem)**.**

1. If $x = x_0$ is an ordinary point of $(*)$, there are two linearly independent solutions of the form
$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$
(Taylor series convergent in some neighbourhood of x_0).

2. If x_0 is a regular singular point of $(*)$ there is at least one solution of the form
$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\sigma} \\ &= (x - x_0)^\sigma \sum_{n=0}^{\infty} a_n (x - x_0)^n \end{aligned}$$
for some real σ with $a_0 \neq 0$ (this is known as a "Frobenius series"). No guarantee that we obtain two linearly independent solutions in this case.

Method may fail completely for irregular singular points.

Example of ordinary point

$$(1 - x^2)y'' - 2xy' + 2y = 0$$

Expand around $x = 0$ (ordinary point). [$x = \pm 1$ are regular singular points]. Try

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

Convenient to multiply $(*)$ by x^2:

$$(1 - x^2)(x^2 y'') - 2x^2(xy') + 2x^2(y) = 0$$

Substituting:

$$\sum_{n=2}^{\infty} a_n [(1 - x)^2 n(n-1)] x^n - 2 \sum_{n=1}^{\infty} a_n x^2 n x^n + 2 \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

which can be rewritten as

$$\sum_{n=2}^{\infty} a_n[(1-x^2)n(n-1)]x^n - 2\sum_{n=2}^{\infty} a_{n-2}(n-2)x^n + 2\sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Equate coefficients of x^n for $n \geq 2$:

$$a_n n(n-1) - a_{n-2}(n-2)(n-3) - 2a_{n-2}(n-2) + 2a_{n-2} = 0 \qquad (n \geq 2)$$

$$\implies n(n-1)a_n = n(n-3)a_{n-2}$$

$$\implies a_n = \frac{n-3}{n-1}a_{n-2}$$

(recursive relation). a_0 and a_1 are not fixed (arbitrary constants). Since $a_3 = 0 \implies a_5 = 0 \implies a_7 = 0$ etc, there is one odd solution:

$$y = a_1 x$$

other, even solution, for n even:

$$\begin{aligned} a_n &= \frac{n-3}{n-1}a_{n-2} \\ &= \left(\frac{n-3}{n-1}\right)\left(\frac{n-5}{n-3}\right)a_{n-4} \\ &= \cdots \\ &= -\frac{1}{n-1}a_0 \end{aligned}$$

$$\implies y = a_0\left[1 - x^2 - \frac{x^4}{3} - \frac{x^4}{5} - \cdots\right]$$

Note:

$$\ln(1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - \cdots$$

so

$$y = a_0\left[1 - \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)\right]$$

Closed-form solution - note behaviour near $x = \pm 1$ (regular singular points); see below.

Example of Regular Singular Point

$$4xy'' + (1-x^2)y' - xy = 0 \qquad (*)$$

$x = 0$: regular singular point
Try

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\sigma} \qquad (a_0 \neq 0)$$

Multiply $(*)$ by x:

$$4(x^2y'') + 2(1-x^2)(xy') + x^2y = 0$$

Hence,

$$\sum_{n=0}^{\infty} a_n x^{n+\sigma}(4(n+\sigma)(n+\sigma-1) + 2(1-x^2)(n+\sigma) - x^2) = 0$$

σ determined by lowest power of x.
$\underline{x^\sigma}$: $a_0(4\sigma(\sigma-1)+2\sigma) = 0$ (note $a_0 \neq 0$ by assumption). Hence $2\sigma(2\sigma-1) = 0$ so $\sigma = 0$ or $\sigma = \frac{1}{2}$.
Next lowest power of x:
$\underline{x^{\sigma+1}}$: $a_1(4(\sigma+1)\sigma + 2(1+\sigma)) = 0$ hence $a_1(\sigma+1)(2\sigma+1) = 0$. Since we know that $\sigma = 0$ or $\frac{1}{2}$, we must have $a_1 = 0$.
$\underline{x^{n+\sigma}}$ $(n \geq 2)$: $a_n(4(n+\sigma)(n+\sigma-1) + 2(1+\sigma)) - 2a_{n-2}(n+\sigma-2) - a_{n-2} = 0$ so

$$2(n+\sigma)(2n+2\sigma-1)a_n = (2n+2\sigma-3)a_{n-2} \qquad (\dagger)$$

Consider roots separately:
$\underline{\sigma = 0}$: $(\dagger)$ gives $2n(2n-1)a_n = (2n-3)a_{n-2}$

$$\implies a_n = \frac{2n-3}{2n(2n-1)}a_{n-2}$$

$$a_2 = \frac{1}{4\times 3}a_0; \qquad a_4 = \frac{5}{8\times 7}a_2 = \frac{5\times 1}{8\times 7\times 4\times 3}a_0$$

($a_1 = 0$, so $a_3 = 0$ etc)
Solution:

$$y = a_0\left(1 + \frac{x^2}{4\times 3} + \frac{5x^4}{8\times 7\times 4\times 3} + \cdots\right)$$

(Note that this is a Taylor series since the root σ was an integer)
$\underline{\sigma = \frac{1}{2}}$ $(\dagger)$ gives $(2n+1)2na_0 = (2n-2)a_{n-2}$ for $n \geq 2$, hence

$$a_n = \frac{n-1}{n(2n+1)}a_{n-2}$$

$$a_2 = \frac{1}{2\times 5}; \qquad a_4 = \frac{3\times 1}{4\times 9\times 2\times 5}$$

Solution:

$$y = b_0 x^{\frac{1}{2}}\left(1 + \frac{x^2}{2\times 5} + \frac{3x^4}{4\times 9\times 2\times 5} + \cdots\right)$$

(Frobenius series).
Note that we have determined 2 linearly independent solutions, but this is not generally the case.

7.3 Second solutions

Expansion around a regular singular point guarantees one Frobenius solution. Whether we get a second depends on roots σ_1 and $\sigma_"$ of indicial equation.

1. $\sigma_1 - \sigma_2$ *not* an integer, then we get two linearly independent solutions:

$$y_1 = (x - x_0)^{\sigma_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$y_2 = (x - x_0)^{\sigma_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

   Note: as $x \to x_0$, general solution $\sim a_0 (x - x_0)^{\sigma_1}$ if $\sigma_1 < \sigma_2$.

2. $\sigma_1 - \sigma_2$ is a non-zero integer. One series solution involving *larger* root (say, σ_2).

$$y_1 = (x - x)6)^{\sigma_2} \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

   Second solution of ODE is generally of the form

$$y_2 = (x - x_0)^{\sigma_1} \sum_{n=0}^{\infty} b_n (x - x_0)^n + c y_1(x) \ln(x - x_0)$$

   (c may or may not be zero)
   Constant c is determined by a_1 (in y_1) and b_0 $\to$ general solution depends on two arbitrary constants as required.

3. $\sigma_1 = \sigma_2 = \sigma$ log term is *always* required, $c \neq 0$.

$$y_1 = (x - x_0)^{\sigma} \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$y_2 = (x - x_0)^{\sigma} \sum_{n=0}^{\infty} b_n (x - x_0)^n + c y_1(x) \ln(x - x_0)$$

Example (case 2).

$$x^2y'' - xy = 0 \qquad (*)$$

$x = 0$ is a regular singular point.
Look for

$$y = \sum_{n=0}^{\infty} a_n x^{n+\sigma} \qquad (a_0 \neq 0)$$

$(*)$ implies

$$\sum_{n=0}^{\infty} (a_n(n+\sigma)(n+\sigma-1)x^{n+\sigma} - a_n x^{n+\sigma+1}) = 0$$

Lowest power is x^σ: $a_0\sigma(\sigma-1) = 0$ hence $\sigma_1 = 0$ and $\sigma_2 = 1$ (since $a_0 \neq 0$).
Since $\sigma_1 - \sigma_2$ is a non-zero integer, this is case 2.
For $n \geq 1$.

$$a_n(n+\sigma)(n+\sigma-1) = a_{n-1}$$

$\underline{\sigma = 1}$ (larger root):

$$a_n = \frac{a_{n-1}}{n(n+1)} \implies a_n = \frac{a_0}{n!(n+1)!}$$

Solution

$$y_1 = a_0 x \left(1 + \frac{x}{2} + \frac{x^2}{12} + \cdots\right)$$

$\underline{\sigma = 0}$ (smaller root):

$$n(n-1)a_n = a_{n-1}$$

but $n = 1$ gives $a_0 = 0$, which contradicts initial assumption. So we need a log term:

$$y_2(x) = \underbrace{\sum_{n=0}^{\infty} b_n x^n}_{\sigma_1 = 0 \text{ series}} + c y_1(x) \ln x$$

Determine $\{b_n\}$ and c by direct substitution in ODE or by method of reduction or order. (See non-examinable section of lecture notes)

TOPIC V: MULTIVARIATE FUNCTIONS - APPLICATIONS

- Directional derivatives
- Extrema
- Coupled systems of first-order ODEs
- Partial differential equations

1 Directional Derivative

Consider function $f(x, y)$ and vector displacement $\mathrm{d}\mathbf{s} = (\mathrm{d}x, \mathrm{d}y)$. Infinitesimal change in f:

$$\begin{aligned}\mathrm{d}f &= \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y \\ &= (\mathrm{d}x, \mathrm{d}y) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \\ &= \mathrm{d}\mathbf{s} \cdot \nabla f\end{aligned}$$

∇f denotes the "gradient of f".

Definition (Gradient of f)**.**

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \tag{$*$}$$

Remark. This is vector valued, and sometimes instead denoted grad f.

If we write

$$\mathrm{d}\mathbf{s} = \mathrm{d}s\hat{\mathbf{s}}$$

where $\mathrm{d}s$ is distance displacement and $\hat{\mathbf{s}}$ is a unit vector ($|\hat{\mathbf{s}}| = 1$).

Definition (Directional derivative)**.** The *directional derivative* of f in direction $\hat{\mathbf{s}}$ is

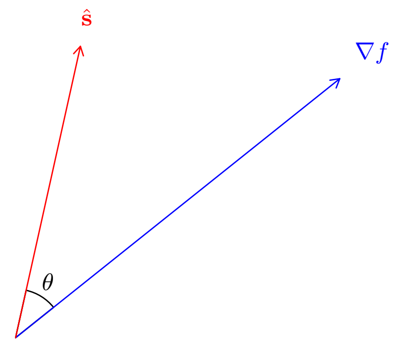
$$\frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot (\nabla f)$$

Rate of change of f with distance s along direction $\hat{\mathbf{s}}$.

Can then *define* gradient vector ∇f geometrically as that vector such that

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot (\nabla t)$$

Properties of gradient vector



$$\frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot \nabla f = \cos\theta |\nabla f|$$

1. Direction ∇f is that in which f *increases* most rapidly.
2. Magnitude of ∇f is maximum rate of change of f:

$$|\nabla f| = \max\left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)$$

3. If $\hat{\mathbf{s}}$ is parallel to contours of f, then

$$0 = \frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot (\nabla f)$$

Hence, ∇f is $\perp$ contours of $f(x, y)$.

2 Stationary Points

Always at least one direction for which $\frac{\mathrm{d}f}{\mathrm{d}s} = 0$ (parallel to contours of f).
Stationary points

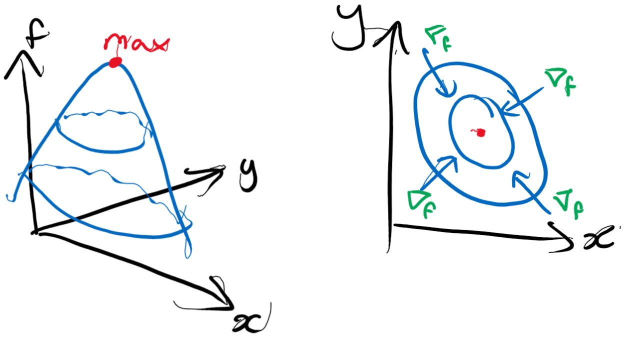
$$\frac{\mathrm{d}f}{\mathrm{d}s} = 0 \forall \hat{\mathbf{s}}$$

Since

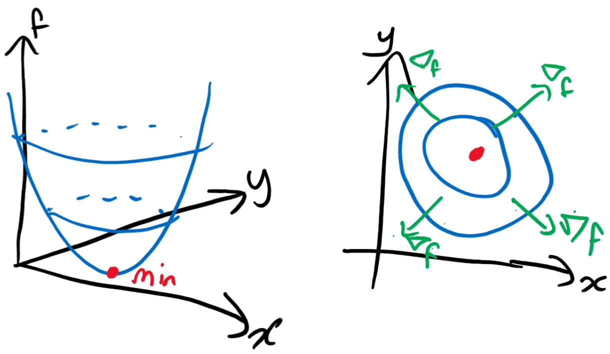
$$\frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot (\nabla f)$$

we must have ∇f at stationary points.

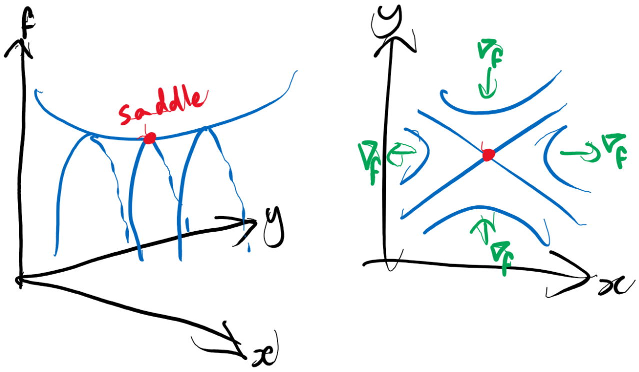
Types of stationary points
Local maxima:



Contours of f are locally elliptical.
Local minima:



Contours are locally elliptical.
Saddle points:

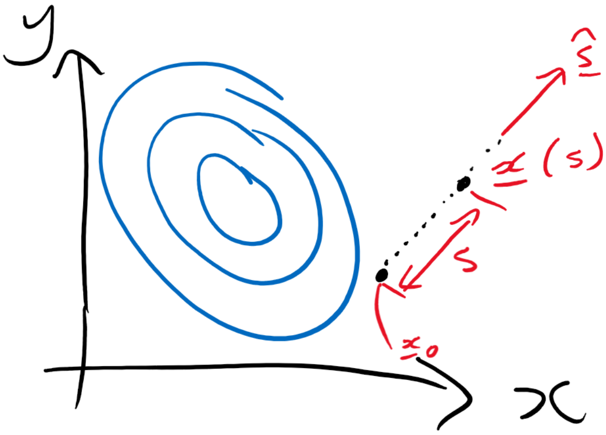


Contours locally hyperbolic. Contours cross at and only at saddle points.

3 Classification of Stationary Points

Consider how $f(x, y)$ varies along line

$$\mathbf{x}(s) = \mathbf{x}_0 + s\hat{\mathbf{s}}$$



Can think of $f(x(s), y(s))$ as a function of s so usual single-variable Taylor series holds:

$$\begin{aligned} f(\mathbf{x}_0 + s\hat{\mathbf{s}}) &= f(\mathbf{x}_0) + s\left.\frac{\mathrm{d}f}{\mathrm{d}s}\right|_{\mathbf{x}_0} + \frac{1}{2}s^2\left.\frac{\mathrm{d}^2 f}{\mathrm{d}s^2}\right|_{\mathbf{x}_0} + \cdots \\ &= f(\mathbf{x}_0) + \underbrace{s\hat{\mathbf{s}}\cdot\nabla f|_{\mathbf{x}_0}}_{1} + \underbrace{\frac{1}{2}s^2(\hat{\mathbf{s}}\cdot\nabla)(\hat{\mathbf{s}}\cdot\nabla f)}_{2} + \cdots \end{aligned}$$

Define vector displacement

$$\delta\mathbf{x} = s\hat{\mathbf{s}}$$

components

$$\delta x = x(s) - x_0, \qquad \delta y = y(s) - y_0$$

Now we focus on terms 1 and 2:

1. $$s\hat{\mathbf{s}}\cdot(\nabla f) = \delta\mathbf{x}\cdot\nabla = \delta x\frac{\mathrm{d}f}{\mathrm{d}x} + \delta y\frac{\mathrm{d}f}{\mathrm{d}y}$$

2. $$\begin{aligned} s^2(\hat{\mathbf{s}}\cdot\bar{\mathbf{v}})(\hat{\mathbf{s}}\cdot\nabla f) &= (\delta\mathbf{x}\cdot\nabla f) \\ &= \left(\delta x\frac{\partial}{\partial x} + \delta y\frac{\partial}{\partial y}\right)\left(\delta x\frac{\partial f}{\partial x} + \delta y\frac{\partial f}{\partial y}\right) \\ &= (\delta x)^2\frac{\partial^2 f}{\partial x^2} + \delta x\delta y\frac{\partial^2 f}{\partial x\partial y} + \delta y\delta x\frac{\partial^2 f}{\partial x\partial y} + (\delta y)^2\frac{\partial^2 f}{\partial y^2} \\ &= (\delta x, \delta y)\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{pmatrix}\begin{pmatrix}\delta x \\ \delta y\end{pmatrix} \end{aligned}$$

Definition (Hessian matrix)**.** The *Hessian matrix* is

$$\underline{\mathbf{H}} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \nabla\nabla f$$

Symmetric since $f_{xy} = f_{yx}$.

Multivariate Taylor series

$$f(x_0 + \delta x, y_0 + \delta y) = f(x_0, y_0) + \left(\delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y}\right)\Bigg|_{x_0, y_0} + \frac{1}{2}\left((\delta x)^2 \frac{\partial^2 f}{\partial x^2} + 2\delta x \delta y \frac{\partial^2}{\partial x \partial y} + (\delta y)^2 \frac{\partial^2 f}{\partial y^2}\right)\Bigg|_{x_0, y_0} + \cdots$$

Coordinate free form:

$$f(\mathbf{x}_0 + \delta\mathbf{x}) + f(\mathbf{x}_0) + \delta x \cdot \nabla f|_{x_0} + \frac{1}{2}\delta\mathbf{x}(\nabla\nabla f)|_{x_0}\delta\mathbf{x}^2 + \cdots$$

3.1 Nature of stationary points and the Hessian

Let $\mathbf{x}_0$ be a stationary point ($\nabla f \,|_{\mathbf{x}_0} = \mathbf{0}$). Around $\mathbf{x}_0$:

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \frac{1}{2} \underbrace{\delta\mathbf{x}}_{\mathbf{x}-\mathbf{x}_0} \underbrace{\underline{\mathbf{H}}(\mathbf{x}_0)}_{\text{Hessian}} \delta\mathbf{x}^\top$$

Nature of stationary point depends on properties of $\underline{\mathbf{H}}(\mathbf{x}_0)$.

Definition (Positive-definite and negative definite matrices)**.** A (real) symmetric matrix $\underline{\mathbf{H}}$ is positive definite if

$$\mathbf{x}\underline{\mathbf{H}}\mathbf{x}^\top > 0$$

for all real $\mathbf{x}$ ($\mathbf{x} \neq \mathbf{0}$). It is negative definite if

$$\mathbf{x}\underline{\mathbf{H}}\mathbf{x}^\top < 0$$

for all real $\mathbf{x}$ ($\mathbf{x} \neq \mathbf{0}$). Otherwise, indefinite.

If $\underline{\mathbf{H}}(\mathbf{x}_0)$ is positive definite, then $\delta\mathbf{x}\underline{\mathbf{H}}\delta\mathbf{x}^\top > 0$ for all $\delta\mathbf{x}$ so $f(\mathbf{x}) > f(\mathbf{x}_0)$ in vicinity of $\mathbf{x}_0$:

$$\underline{\mathbf{H}} \text{ positive definite} \implies \text{local minimum}$$

Similarly,

$$\underline{\mathbf{H}} \text{ negative definite} \implies \text{local maximum}$$

If indefinite, may be a maximum, minimum or a saddle.

Definiteness and eigenvalues

$\underline{\mathbf{H}}$ is real symmetric $\rightarrow$ diagonalise $\underline{\mathbf{H}}$ with an orthogonal transformation. In nD, use coordinates along principal axes.

$$\delta\mathbf{x}\underline{\mathbf{H}}\delta\mathbf{x}^\top = (\delta x_1, \delta x_2, \ldots, \delta x_n)\begin{pmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{pmatrix}\begin{pmatrix}\delta x_1\\ \delta x_2\\ \vdots\\ \delta x_n\end{pmatrix}$$

Follows that

$$\underline{\mathbf{H}} \text{ positive definite } \iff \text{ all } \lambda_i > 0$$

$$\underline{\mathbf{H}} \text{ negative definite } \iff \text{ all } \lambda_i < 0$$

If all λ_i are non-zero, but of mixed sign, have a saddle point.
If ant eigenvalues are 0, need higher terms in Taylor expansion.

Example.

$$f(x,y) = x^2 + y^4$$

Global *minimum* at $(0,0)$

$$\underline{\mathbf{H}} = \begin{pmatrix}2 & 0\\ 0 & 12y^2\end{pmatrix} = \begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix} \qquad \text{at } (0,0)$$

$$\implies \lambda_1 = 2, \lambda_2 = 0$$

Definiteness and signature of $\underline{\mathbf{H}}$

Alternative to establish if positive or negative definite.

Definition (Signature of $\underline{\mathbf{H}}$)**.** The *signature* of $\underline{\mathbf{H}}$ is pattern of signs of the ordered determinants of the leading principal minors of $\underline{\mathbf{H}}$: for example for $f(x_1, x_2, \ldots, x_n)$

$$\underbrace{|f_{x_1x_1}|}_{|\underline{\mathbf{H}}_1|}, \underbrace{\begin{vmatrix}f_{x_1x_1} & f_{x_1x_2}\\ f_{x_2x_1} & f_{x_2x_2}\end{vmatrix}}_{|\underline{\mathbf{H}}_2|}, \cdots, \underbrace{\begin{vmatrix}f_{x_1x_1} & \cdots & f_{x_1x_n}\\ \vdots & \ddots & \vdots\\ f_{x_nx_1} & \cdots & f_{x_nx_n}\end{vmatrix}}_{|\underline{\mathbf{H}}|=|\underline{\mathbf{H}}_n|}$$

Sylvestor's criterion

$$\underline{\mathbf{H}} \text{ is positive definite } \iff \text{ signature is } +,+,\cdots,+$$

$$\underline{\mathbf{H}} \text{ is negative definite} \iff \text{signature is } -, +, -, \cdots, (-)^n$$

$[\mathbf{x}\underline{\mathbf{H}}\mathbf{x}^\top$ with $\mathbf{x} = (x_1, x_2, 0, 0, \ldots, 0)]$

3.2 Contours near stationary points

Let $f(x, y)$ have a stationary point at (x_0, y_0). Adopt coordinates along principal axes of $\underline{\mathbf{H}}$ at $\mathbf{x}_0$:

$$\underline{\mathbf{H}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Write

$$\mathbf{x} = \mathbf{x}_0 + (\xi, \zeta)$$

then around $\mathbf{x}_0$

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \frac{1}{2}(\lambda_1 \xi^2 + \lambda_2 \zeta^2)$$

Contours of f close to $\mathbf{x}_0$ satisfy

$$\lambda_1 \xi^2 + \lambda_2 \zeta^2 = \text{constant}$$

Maximum or minimum: λ_1 and λ_2 same sign $\to$ elliptical contours
Saddle point: λ_1 and λ_2 opposite sign $\to$ hyperbolic contours.

Example.

$$f(x,y) = 4x^3 - 12xy + y^2 + 10y - 6$$

$$f_x = 12x^2 - 12y$$

$$f_y = -12x + 2y + 10$$

Stationary points:

$$f_x = 0 \implies x^2 = y$$

$$f_y = 0 \implies x^2 - 6x + 5 = 0 \implies x = 1, 5$$

Stationary points $(1, 1)$ and $(5, 25)$.

$$f_{xx} = 24x; f_{xy} = 12; f_{yy} = 2$$

$\underline{(1,1)}$:

$$\underline{\mathbf{H}} = \begin{pmatrix} 24 & -12 \\ -12 & 2 \end{pmatrix}$$

$$|\underline{\mathbf{H}}_1| = 24, |\underline{\mathbf{H}}_2| = -96$$

Signature is $+, -$ so *neither* positive definite nor negative definite. $|\underline{\mathbf{H}}| < 0 \implies \lambda_1$ and λ_2 are non-zero and opposite signs $\to$ *saddle point.*

$\underline{(5,25)}$:

$$\underline{\mathbf{H}} = \begin{pmatrix} 120 & -12 \\ -12 & 2 \end{pmatrix}$$

$|\underline{\mathbf{H}}_1| = 120$, $|\underline{\mathbf{H}}_2| = 96 \implies$ signature is $+, +$ so a *minimum*. Near saddle point, $(x, y) = (1, 1) + (\delta x, \delta y)$. Contours locally have

$$f_{xx}(\delta x)^2 + 2f_{xy}(\delta x)(\delta y) + f_{yy}(\delta y)^2 = \text{constant}$$

$$\implies 12(\delta x)^2 - 12\delta x \delta y + (\delta y)^2 = \text{constant}$$

Intersecting contours through $(1, 1)$ are asymptotes of hyperbola:

$$12(\delta x)^2 - 12\delta x \delta y + (\delta y)^2 = 0$$

$$\implies \delta y = (6 \pm 2\sqrt{6})\delta x.$$

4 Systems of Linear Equations

Consider two functions $y_1(t)$ and $y_2(t)$ with

$$\dot{y}_1 = ay_1 + by_2 + f_1(t)$$
$$\dot{y}_2 = cy_1 + dy_2 + f_2(t)$$

$(a, b, c, d$ constants). Vector form:

$$\dot{\mathbf{Y}} = \underbrace{\underline{\mathbf{M}}}_{\underline{\mathbf{M}}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \underbrace{\mathbf{Y}}_{\mathbf{Y}=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}} + \underbrace{\mathbf{F}}_{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}}$$

Two ways to solve.

(1) Convert to higher-order ODE in one variable

$$\begin{aligned} \ddot{y}_1 &= a\dot{y}_1 + b\dot{y}_2 + \dot{f}_1 \\ &= a\dot{y}_1 + b(cy_1 + dy_2 + f_2) + \dot{f}_1 \\ &= a\dot{y}_1 + bcy_1 + d(\dot{y}_1 - ay_2 - f_1) + bf_2 + \dot{f}_1 \end{aligned}$$

$$\implies \ddot{y}_1 - (a+d)\dot{y}_1 + (ad - bc)y_1 = bf_2 - df_1 + \dot{f}_1$$

(Linear, second-order ODE with constant coefficients)

(2) Solve first-order ODEs directly with matrix methods.
Sometimes convenient to deal with higher order ODEs as systems of first-order ODEs. For example

$$\ddot{y} + \alpha\dot{y} + \beta y = f$$

Define:

$$y_1 \equiv y \qquad \text{and} \qquad y_2 \equiv \dot{y}$$

so $\dot{y}_1 = y_2$ and $\dot{y}_2 = \ddot{y} = -\alpha y_2 - \beta y_1 + f$

$$\dot{\mathbf{Y}} = \begin{pmatrix} 0 & 1 \\ -\beta & -\alpha \end{pmatrix} \underbrace{\mathbf{Y}}_{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}} + \begin{pmatrix} 0 \\ f \end{pmatrix}$$

4.1 Matrix methods

To solve

$$\dot{\mathbf{Y}} = \underline{\mathbf{M}}\mathbf{Y} + \mathbf{F}(t)$$

(1) Write

$$\mathbf{Y} = \mathbf{Y}_c + \mathbf{Y}_p$$
$$\dot{\mathbf{Y}}_c = \underline{\mathbf{M}}\mathbf{Y}_c$$

(2) Look for $\mathbf{Y}_c$ in the form

$$\mathbf{Y}_c = \mathbf{v}e^{\lambda t}$$

Have

$$\dot{\mathbf{Y}}_c = \lambda \mathbf{Y}_c = \underline{\mathbf{M}}\mathbf{Y}_c$$

$$\implies \underline{\mathbf{M}}\mathbf{v} = \lambda \mathbf{v}$$

($\mathbf{v}$ must be an eigenvector of $\underline{\mathbf{M}}$ and λ is the eigenvalue) For systems of n equations with n distinct eigenvalues λ, n such complementary solutions which we can add to get general $\mathbf{Y}_c$.

(3) Find a particular solution $\mathbf{Y}_p$ that satisfies the full system of ODEs.

Example.

$$\dot{\mathbf{Y}} = \underbrace{\begin{pmatrix} -4 & 24 \\ 1 & -2 \end{pmatrix}}_{\underline{\mathbf{M}}} \mathbf{Y} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^t$$

Complementary solutions:

$$\mathbf{Y}_c = \mathbf{v} e^{\lambda t}$$

$$\implies \underline{\mathbf{M}}\mathbf{v} = \lambda \mathbf{v} \implies |\underline{\mathbf{M}} - \lambda \underline{\mathbf{I}}| = 0$$

$$\lambda_1 = 2 \qquad \text{or} \qquad \lambda_2 = -8$$

$$\mathbf{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$$

General complementary solution:

$$\mathbf{Y}_c = A \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + B \begin{pmatrix} -6 \\ 1 \end{pmatrix} e^{-8t}$$

For particular solution, try

$$\mathbf{Y}_p = \mathbf{u} e^t$$

Requires

$$\mathbf{u} = \underline{\mathbf{M}}\mathbf{u} + \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\implies \mathbf{u} = -(\underline{\mathbf{M}} - \underline{\mathbf{I}})^{-1} \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

(inverse exists since 1 is not an eigenvalue of $\underline{\mathbf{M}}$)

$$\mathbf{u} = -\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

General solution

$$\mathbf{Y} = A \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + B \begin{pmatrix} -6 \\ 1 \end{pmatrix} e^{-8t} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^t$$

(Note: if forcing $\propto e^{\lambda t}$ with λ an eigenvalue of $\underline{\mathbf{M}}$, try $\mathbf{Y}_p = \mathbf{u} t e^{\lambda t}$.)

4.2 Non-degenerate phase portraits

n first-order ODEs has phase space with points

$$\mathbf{Y} = (y_1, y_2, \ldots, y_n)^\top$$

Phase portrait: solution trajectories in phase space.
Consider

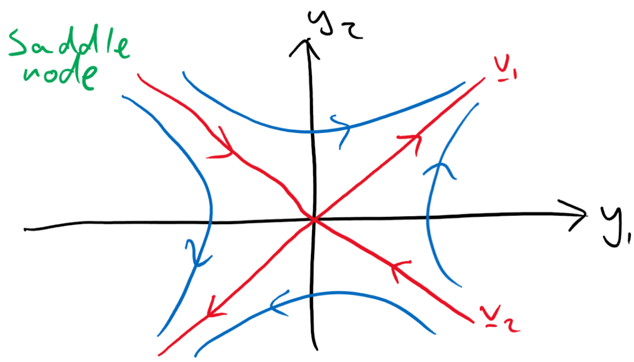
$$\dot{\mathbf{Y}} = \underline{\mathbf{M}}\mathbf{Y}$$

(homogeneous). Fixed point $\mathbf{Y} = \mathbf{0}$. For $n = 2$, and $\lambda_1 \neq \lambda_2$ (non-degenerate), general solution:

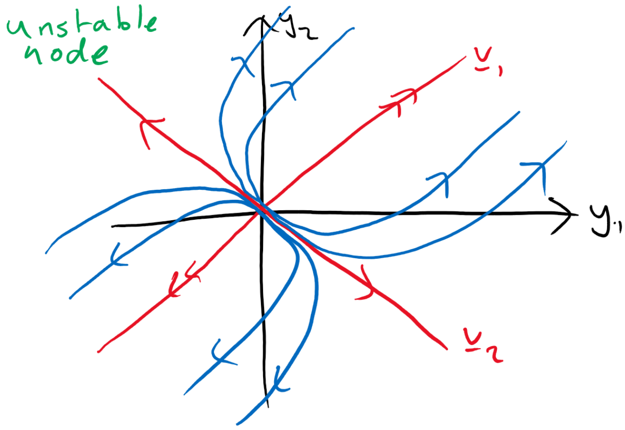
$$\mathbf{Y}(t) = \mathbf{v}_1 e^{\lambda_1 t} + \mathbf{v}_2 e^{\lambda_2 t}$$

(λ_1, λ_2 eigenvalues and $\mathbf{v}_1, \mathbf{v}_2$ eigenvectors). Consider phase portraits for $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ (and $\lambda_1 \neq \lambda_2$).

Case 1: λ_1 and λ_2 real and opposite signs. WLOG $\lambda_2 < 0 < \lambda_1$. $\mathbf{v}_1$ and $\mathbf{v}_2$ can be chosen to be real.



Case 2: λ_1 and λ_2 real and same sign. WLOG $|\lambda_1| > |\lambda_2|$ and $\mathbf{v}_1$ and $\mathbf{v}_2$ real.

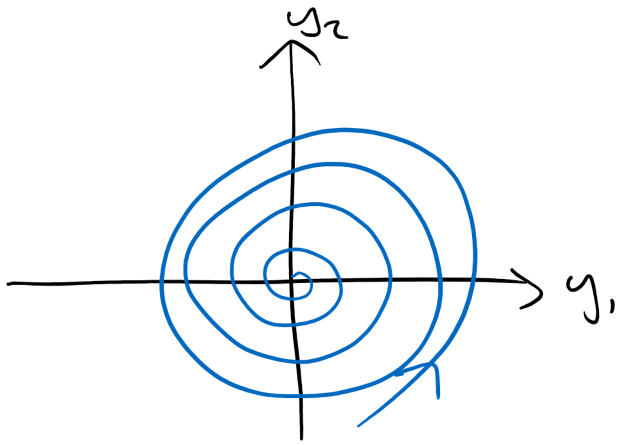


For $\lambda_1, \lambda_2 < 0$ - sable node (all arrows reversed).

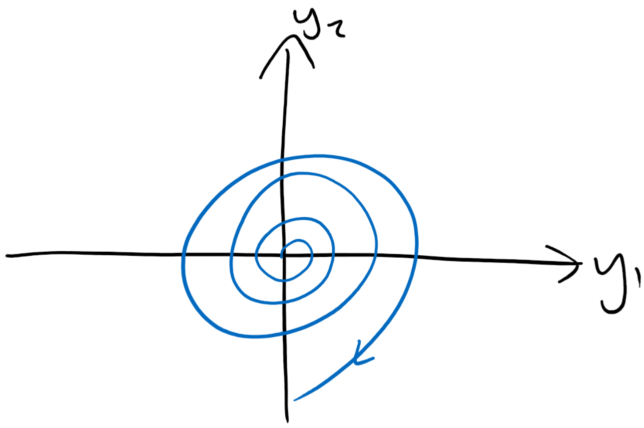
Case 3: λ_1 and λ_2 complex conjugate pairs, $\lambda_1 = \overline{\lambda_2}$, $\mathbf{v}_1 = \overline{\mathbf{v}_2}$

$$\begin{aligned}\mathbf{Y}(t) &= c\mathbf{v}_1 e^{\lambda_1 t} + c \cdot c_0 \\ &= c\mathbf{v}_1 e^{\mathrm{Re}(\lambda_1)t} e^{i\mathrm{Im}(\lambda_1)t} + c \cdot c_0\end{aligned}$$

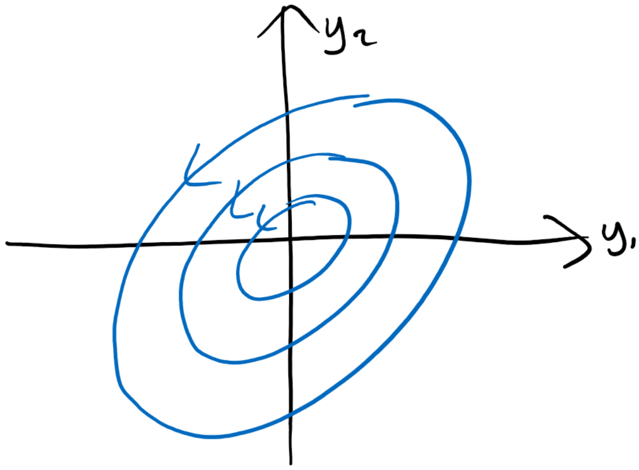
$\mathrm{Re}(\lambda_1) < 0 \implies$ stable spiral.



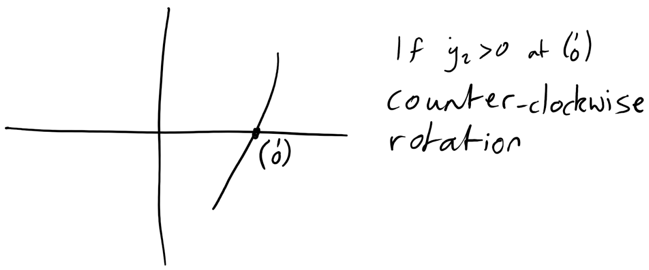
$\text{Re}(\lambda_1) > 0 \implies$ unstable spiral.



$\mathcal{R}\rceil(\lambda_1) = 0 \rightarrow$ *centre*
$\mathbf{Y}(t)$ is periodic $\rightarrow$ closed elliptical trajectories in phase space



Sense of rotation?
Determine $\dot{\mathbf{Y}}$ at one point:



5 Non-linear Dynamical Systems

Consider an *autonomous system* of two, non-linear, first-order ODEs:

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases} \tag{$*$}$$

(f, g general non-linear functions of x, y which are independent of t).

5.1 Equilibrium points

Definition (Equilibrium point)**.** An *equilibrium point* (fixed point) of system ($*$) is a point at which $\dot{x} = 0$ and $\dot{y} = 0$.

If (x_0, y_0) is a fixed point of ($*$):

$$f(x_0, y_0) = 0 = g(x_0, y_0)$$

Solve these to locate the fixed points. Stability via perturbation analysis:

$$x(T) = x_0 + \xi(t); y(t) = y_0 + \zeta(t)$$

(ξ, ζ small perturbations about (x_0, y_0))

$$\dot{x} = f(x,y) \implies \dot{\xi} = f(x_0 + \xi, y_0 + \zeta)$$

$$\implies \dot{\xi} \approx \underbrace{f(x_0, y_0)}_{=0 \text{ since fixed point}} + \xi \frac{\partial f}{\partial x}(x_0, y_0) + \zeta \frac{\partial f}{\partial y}(x_0, y_0)$$

Similarly,

$$\dot{\zeta} = \xi \frac{\partial g}{\partial x}(x_0, y_0) + \zeta \frac{\partial g}{\partial y}(x_0, y_0)$$

$$\implies \begin{pmatrix} \dot{\xi} \\ \dot{\zeta} \end{pmatrix} = \underbrace{\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}}_{\underline{\mathbf{M}}} \begin{pmatrix} \xi \\ \zeta \end{pmatrix}$$

Linear system of homogeneous, first-order ODEs $\to$ eigenvalues of $\underline{\mathbf{M}}$ determine stability of equilibrium points.

Example (predator-prey system).
$x(t)$: Number of prey at time t
$y(t)$: Number of predators at time t.
Dynamics of prey:

$$\dot{x} = \alpha x - \beta x^2 - \gamma xy$$

Where αx represents the excess births over natural deaths, βx^2 represents competition over scarce resource and γxy represents prey being killed - predators consume all prey they encounter.
Dynamics of predators:

$$\dot{y} = \varepsilon xy - \delta y$$

where εxy represents birth of predators and δy represents natural death rate.
Note $\alpha, \beta, \gamma, \varepsilon, \delta$ are all positive constants.
Specific case:

$$\dot{x} = \underbrace{8x - 2x^2 - 2xy}_{f(x,y)}$$

$$\dot{y} = \underbrace{xy - y}_{g(x,y)}$$

Equilibrium points:

$$2x(4 - x - y) = 0$$

$$y(x - 1) = 0$$

First of these requires: $x = 0 \ or \ x = 4 - y$.
Second requires: $y = 0 \ or \ y(3 - y) = 0$
Have equilibrium points:

$$(0,0), (1,3), (4,0)$$

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 8 - 4x - 2y & -2x \\ y & x - 1 \end{pmatrix}$$

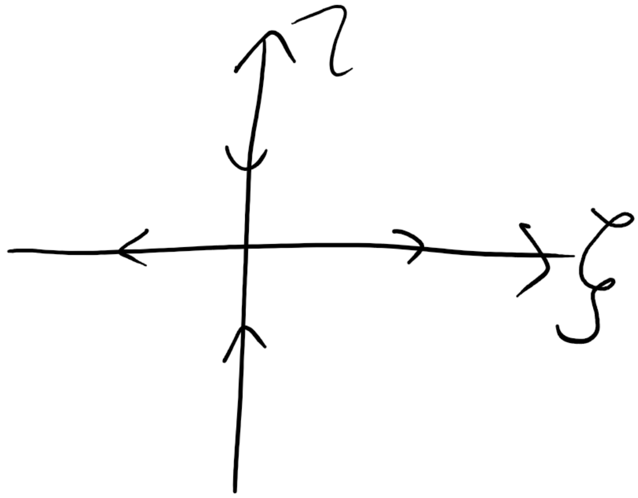
$\underline{(0,0)}$

$$\begin{pmatrix} \dot{\xi} \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix}$$

Eigenvalues / eigenvectors:

$$8, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad -1, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Saddle node



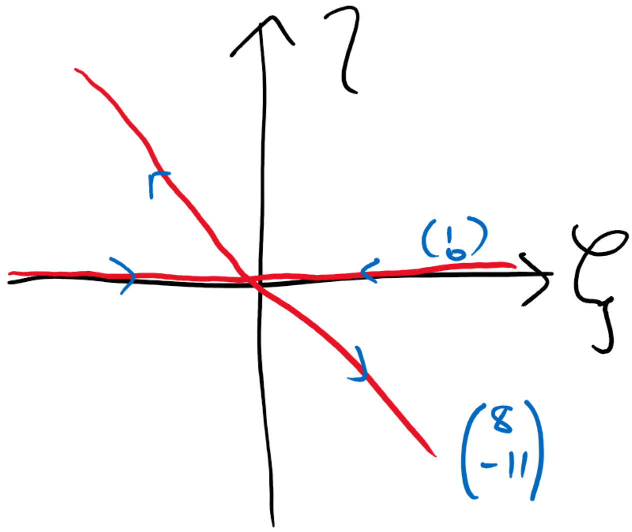
$\underline{(4,0)}$

$$\begin{pmatrix}\dot{\xi}\\ \dot{\zeta}\end{pmatrix} = \begin{pmatrix}-8 & -8\\ 0 & 3\end{pmatrix}\begin{pmatrix}\xi\\ \zeta\end{pmatrix}$$

Eigenvalues / eigenvectors:

$$-8, \begin{pmatrix}1\\0\end{pmatrix} \qquad 3, \begin{pmatrix}8\\-11\end{pmatrix}$$

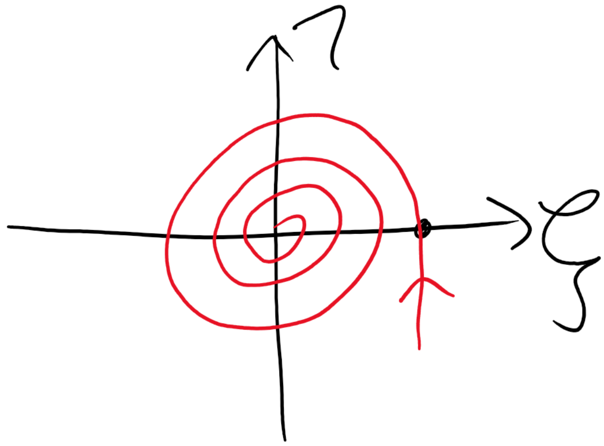
Saddle node



$\underline{(1,3)}$

$$\begin{pmatrix}\dot{\xi}\\ \dot{\zeta}\end{pmatrix} = \begin{pmatrix}-2 & -2\\ 3 & 0\end{pmatrix}\begin{pmatrix}\xi\\ \zeta\end{pmatrix}$$

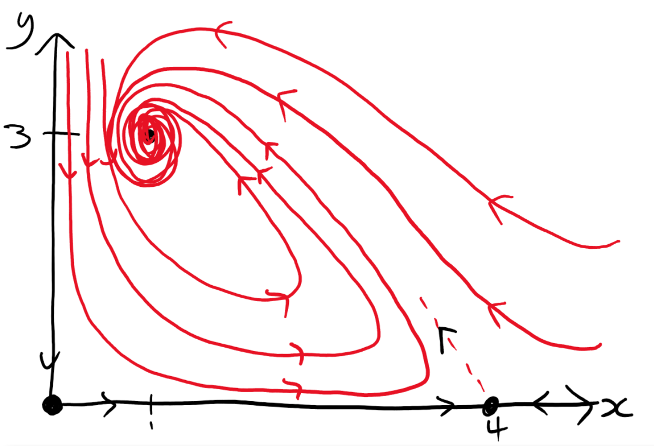
Eigenvalues: $-1 \pm i\sqrt{5}$ (note $\mathrm{Re}(\lambda) < 0$).
Stable spiral



At $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$\begin{pmatrix} \dot{\xi} \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

Note that the 3 in the last vector shows that $\dot{\zeta} > 0$ so the spiral must be anticlockwise.



6 Partial Differential Equations

PDEs - differential equations with multiple independent variables.

6.1 First-order wave equation

Consider $\psi(x,t)$ satisfying

$$\frac{\partial \psi}{\partial t} - c\frac{\partial \psi}{\partial x} = 0 \qquad (*)$$

(c is a constant with dimensions of speed, x is a 1 dimensional position and t is a time)
Solve with *method of characteristics*. How does ψ vary along path $x(t)$, so $\psi(x(t),t)$?

$$\begin{aligned} \frac{\mathrm{d}\psi}{\mathrm{d}t} &= \frac{\partial \psi}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial \psi}{\partial x} \\ &= \frac{\partial \psi}{\partial x}\left(c + \frac{\mathrm{d}x}{\mathrm{d}t}\right) && \text{using } (*) \end{aligned}$$

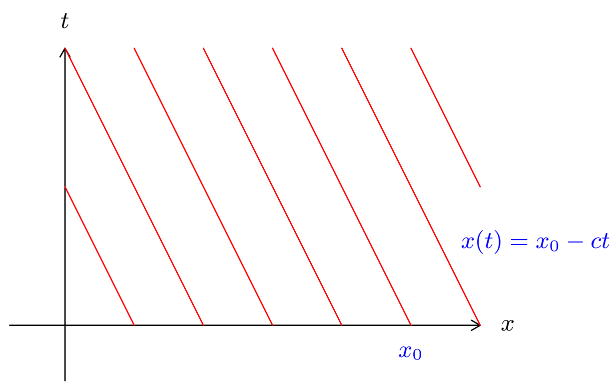
If choose $x(t)$ such that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -c$$
$$\implies x = x_0 - ct$$

where x_0 is a constant along the path (label paths). Then

$$\frac{\mathrm{d}\psi}{\mathrm{d}x} = 0 \implies \psi = \text{constant along the path}$$

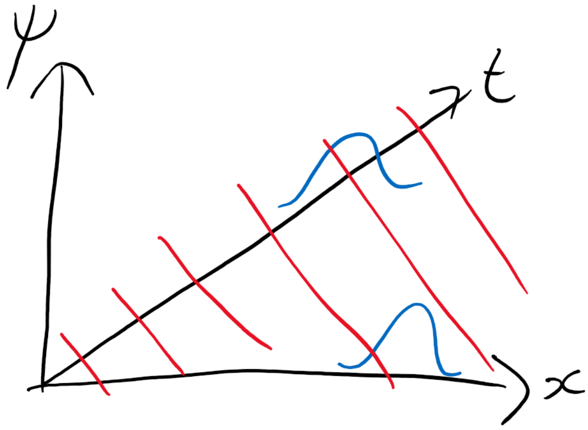
Paths $x = x_0 - ct$ are the characteristics of $(*)$.



As ψ is constant along characteristics, general solution of $(*)$ is

$$\psi(x,t) = f(x_0) = f(x+ct)$$

Simply translates the x-dependence of ψ at $t=0$ to left by ct at time t.



Left-moving wavelike solution.

Example (unforced wave equation)**.**

$$\frac{\partial \psi}{\partial t} - c\frac{\partial \psi}{\partial x} = 0$$

with $\psi(x,0) = x^2 - 3$. General solution: $\psi = f(x+ct)$. At $t=0$, $\psi(x,0) = f(x) = x^2 - 3$ so

$$\psi(x,t) = (x+ct)^2 - 3.$$

Example (forced wave equation)**.**

$$\frac{\partial \psi}{\partial t} + 5\frac{\partial \psi}{\partial x} = e^{-t}$$

with $\psi(x, 0) = e^{-x^2}$. Characteristics have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5 \implies x = x_0 + 5t$$

Along these

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = e^{-t}$$

$$\implies \psi = f(x_0) - e^{-t}$$

where f is an arbitrary function that is constant on each characteristic. General solution:

$$\psi(x, t) = f(x - 5t)$$

Initial conditions:

$$\psi(x, 0) = f(x) - \underbrace{e^{-0}}_{1} = e^{-x^2}$$

$$\implies f(x) = 1 + e^{-x^2}$$

Final solution:

$$\psi(x, t) = 1 + e^{-(x-5t)^2} - e^{-t}$$

6.2 Second-order wave equation

Allow propagation in both directions.

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} = 0$$

Can be factored as

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\psi = 0 \tag{$\dagger$}$$

(works because partial derivatives commute). The two first order operators commute so both

$$\underbrace{f(x + ct)}_{\substack{\text{nulled by} \\ \frac{\partial}{\partial t} - c\frac{\partial}{\partial x}}} \qquad \text{and} \qquad \underbrace{g(x - ct)}_{\substack{\text{nulled by} \\ \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}}}$$

are solutions. General solution:

$$\psi = f(x + ct) + g(x - ct)$$

for some arbitrary functions f and g.
Most general solution? Yes!
Let $\zeta = x + ct$ and $\eta = x - ct$.

$$\frac{\partial}{\partial x} = \underbrace{\frac{\partial \zeta}{\partial x}\bigg|_t}_{1} + \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial t} = c\frac{\partial}{\partial \zeta} - c\frac{\partial}{\partial \eta}$$

$$\implies \frac{\partial}{\partial t} - c\frac{\partial}{\partial x} = -2c\frac{\partial}{\partial \eta}$$

and

$$\frac{\partial}{\partial t} + c\frac{\partial}{\partial x} = 2x\frac{\partial}{\partial \zeta}$$

Wave equation (†) reduces to

$$-4c^2\frac{\partial^2 \psi}{\partial \eta \partial \zeta} = 0$$

$$\implies \psi = f(\zeta) + g(\eta)$$

(Integrate twice)

Example.

$$\frac{\partial^2\psi}{\partial t^2} - c^2\frac{\partial^2\psi}{\partial x^2} = 0$$

with

$$\psi(x,0) = \frac{1}{1+x^2} \qquad \text{and} \qquad \frac{\partial\psi}{\partial t}(x,0) = 0$$

General solution:

$$\psi = f(x+ct) + g(x-ct)$$

At $t = 0$:

$$\psi(x,0) = f(x) + g(x) = \frac{1}{1+x^2} \qquad (**)$$

$$\frac{\partial\psi}{\partial t}(x,0) = cf'(x) - cg'(x) = 0$$

$$\implies f(x) = g(x) + A$$

Combine with $(**)$:

$$f(x) = \frac{1}{2(1+x^2)} + \frac{A}{2}$$

$$g(x) = \frac{1}{2(1+x^2)} - \frac{A}{2}$$

Hence

$$\psi(x,t) = \frac{1}{2}\left[\frac{1}{1+(x+ct)^2} + \frac{1}{1+(x-ct)^2}\right]$$