Vector Calculus

May 24, 2023

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Introduction

We will learn to differentiate and integrate functions (or maps) of the form

$$f: \underbrace{\mathbb{R}^m}_{\text{domain}} \to \underbrace{\mathbb{R}^n}_{\text{codomain}}$$

An element of $\mathbb{R}^m$ or $\mathbb{R}^n$ is a vector so this subject is called *vector calculus.*

Examples of Maps

(1) A function $f : \mathbb{R} \to \mathbb{R}^n$ defines a *curve* in $\mathbb{R}^n$. In physics, we might think of $\mathbb{R}$ as time and $\mathbb{R}^n$ as physical space and write this as

$$f : t \mapsto \mathbf{x}(t)$$

with $\mathbf{x} \in \mathbb{R}^n$. (Obviously we should take $n = 3$). Generalising, a map

$$f : \mathbb{R}^2 \to \mathbb{R}^n$$

defines a surface in $\mathbb{R}^n$, and so on.

(2) In other applications, the domain $\mathbb{R}^m$ might be viewed as physical space. For example, in physics a *scalar field* is a map

$$f : \mathbb{R}^3 \to \mathbb{R}$$

for example temperature $T(\mathbf{x})$ is a scalar field, as is the Higgs field.
A *vector field* is a map

$$f: \underbrace{\mathbb{R}^3}_{\text{physical space}} \to \underbrace{\mathbb{R}^3}_{\text{somethinge more abstract}}$$

for example the electric field $\mathbf{E}(\mathbf{x})$ and magnetic field $\mathbf{B}(\mathbf{x})$ are vector fields.

1 Curves

We consider maps of the form

$$f : \mathbb{R} \to \mathbb{R}^n$$

Assign a coordinate t to $\mathbb{R}$ and use Cartesian coordinates on $\mathbb{R}^n$.

$$\mathbf{x} = (x^1, \dots, x^n) = x^i \mathbf{e}_i$$

where $\mathbf{e}_i$ is an orthonormal basis such that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. Note that summation convention is used here. (For $\mathbb{R}^3$ we also use notation $\{\mathbf{e}_i\} = \{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$.)

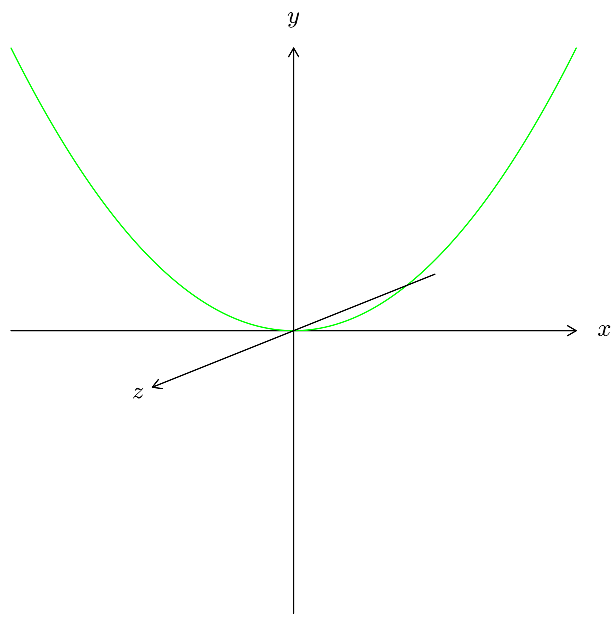
The image of of the function f is a *parametrised curve* $\mathbf{x}(t)$, with t the parameter.

Examples

(1) Consider the map $\mathbb{R} \to \mathbb{R}^3$ given by

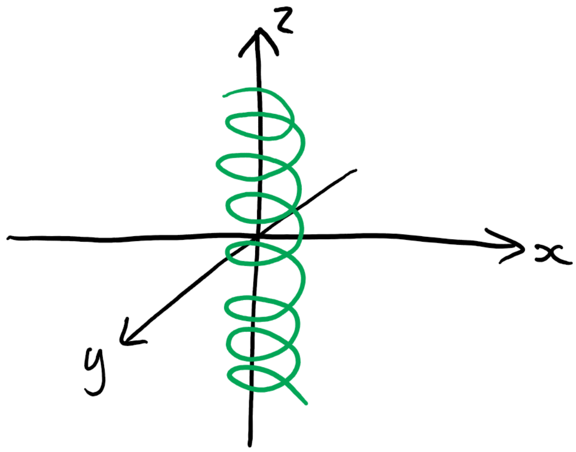
$$\mathbf{x}(t) = (at, bt^2, 0)$$

The curve C is the parabola $a^2 y = bx^2$ in the plane $z = 0$.



> **Note.** When plotting the curve, we lose information about the parameter t.

(2) Consider $\mathbf{x}(t) = (\cos t, \sin t, t)$



The curve C is a helix.

The choice of parametrisation is not unique, for example consider

$$\mathbf{x}(t) = (\cos\lambda t, \sin\lambda t, \lambda t).$$

This gives the same helix for all $\lambda \in \mathbb{R} \setminus \{0\}$.
Sometimes the choice of parametrisation matters, for example if t is time and $\mathbf{x}(t)$ is position, then the velocity is proportional to λ. But we will see that some questions are independent of the choice of parametrisation.

1.1 Differentiating the Curve

A vector function $\mathbf{x}(t)$ is differentiable of t if, as $\delta t \to 0$, we have

$$\mathbf{x}(t + \delta t) - \mathbf{x}(t) = \dot{\mathbf{x}(t)}\delta(t) + O(\delta t^2).$$

If $\dot{\mathbf{x}}(t)$ exists everywhere, the curve is said to be *smooth*.

Note. "Big O" notation $O(\delta t^2)$ means terms proportional to δt^2 or smaller.

In physics, dot is usually used for time derivatives, for example $\dot{\mathbf{x}}(t)$ and prime for spatial derivatives, for example $f'(x)$.
In maths, these are used interchangeably.

Some notation: we write

$$\delta\mathbf{x}(t) = \mathbf{x}(t + \delta t) - \mathbf{x}(t)$$

The derivative is then

$$\mathbf{x} \equiv \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} := \lim_{\delta t \to 0} \frac{\delta\mathbf{x}}{\delta t}.$$

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lecture 2

We will sometimes write $\mathrm{d}\mathbf{x} = \dot{\mathbf{x}}\mathrm{d}t$.
If we're in Cartesian coordinates then we just differentiate vector components

$$\mathbf{x}(t) = \mathbf{x}^i(t)\mathbf{e}_i \implies \dot{\mathbf{x}}(t) = \dot{x}^i(t)\mathbf{e}_i$$

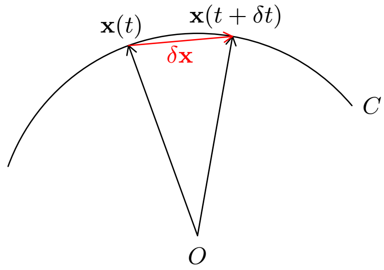
Note. If we have a function $f(t)$ and vectors $\mathbf{g}(t)$ and $\mathbf{h}(t)$ then the following identities hold

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{f}\mathbf{g}) = \dot{\mathbf{f}}\mathbf{g} + \mathbf{f}\dot{\mathbf{g}}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{g}\cdot\mathbf{h}) = \dot{\mathbf{g}}\cdot\mathbf{h} + \mathbf{g}\cdot\dot{\mathbf{h}}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{g}\times\mathbf{h}) = \dot{\mathbf{g}}\times\mathbf{h} + \mathbf{g}\times\dot{\mathbf{h}}$$

(just apply the product rule to the components.)

Tangent Vectors

The derivative $\dot{\mathbf{x}}(t)$ is the *tangent vector* to the curve



Note. The direction $\delta\mathbf{x}(t)$ is independent of the parametrisation (at least up to a sign), while the magnitude does depend on parametrisation.

For example, these two maps give the same curve C in $\mathbb{R}^2$

$$\mathbf{x}(t) = (t, t) \implies \dot{\mathbf{x}} = (1, 1)$$

$$\mathbf{x}(t) = (t^3, t^3) \implies \dot{\mathbf{x}} = 3t^2(1, 1)$$

C is just a line in $\mathbb{R}^2$. In the second case $\dot{\mathbf{x}} = 0$ at $t = 0$ but this is due to the parametrisation, not to C itself.
A parametrisation is *regular* if $\dot{\mathbf{x}}(t) \neq 0 \;\forall\, t$.
In what follows, we'll assume regular parametrisations.

The *arc length* is the distance along the curve. For nearby points

$$\begin{aligned} \delta s &= |\delta \mathbf{x}| + O(|\delta \mathbf{x}|^2) \\ &= |\dot{\mathbf{x}}\delta t| + O(\delta t^2) \\ \implies \frac{\mathrm{d}s}{\mathrm{d}t} &= \pm|\dot{\mathbf{x}}| \end{aligned}$$

($\pm$ depends on whether s increases or decreases as t increases.)
The arc length is defined by

$$s = \int_{t_0}^{t} \mathrm{d}t' |\dot{\mathbf{x}}(t')|$$

Note. For $t > t_0$, $s > 0$, and for $t < t_0$, $s < 0$.

Claim. s is independent of our choice of parametrisation.

Proof. Pick a different choice $\tau(t)$. Assume $\frac{\mathrm{d}\tau}{\mathrm{d}t} > 0$. Then

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}$$

and

$$\begin{aligned} s &= \int_{t_0}^{t} \mathrm{d}t' \left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t'}\right| \\ &= \int_{t_0}^{t} \mathrm{d}t' \frac{\mathrm{d}\tau'}{\mathrm{d}t'} \left|\frac{\mathbf{x}}{\tau'}\right| \\ &= \int_{\tau_0}^{\tau} \mathrm{d}\tau' \left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau'}\right| \end{aligned}$$

($\tau_0 = \tau(t_0)$) □

This means that s itself is a natural parametrisation of the curve. We can think of $\mathbf{x}(s)$.

Because $\frac{\mathrm{d}s}{\mathrm{d}t} = |\dot{\mathbf{x}}(t)|$, the associated tangent vector $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}$ has $\left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}\right| = 1$.

Curvature and Torsion

A curve C parametrised by the arc length s, has tangent vector

$$\mathbf{t} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}$$

Note. $\mathbf{t}$ is not the same thing as our previous parameter!

This has $|\mathbf{t}| = 1$.

The curvature $\kappa(S)$ is

$$\kappa(s) = \left|\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}s^2}\right| = \left|\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s}\right|$$

To get some intuition, consider a circle

$$\mathbf{x}(t) = (R\cos t, R\sin t)$$

Use $\frac{\mathrm{d}s}{\mathrm{d}t} = |\dot{\mathbf{x}}|$ to get $s = Rt$.

$$\implies \mathbf{x}(s) = \left(R\cos\left(\frac{s}{R}\right), R\sin\left(\frac{s}{R}\right)\right.$$

$$\implies \kappa(s) = \frac{1}{R}$$

(which is constant.)

Define the *(principle) normal*

$$\mathbf{n} = \frac{1}{\kappa}\frac{\mathrm{d}^2x}{\mathrm{d}s^2} = \frac{1}{\kappa}\frac{\mathrm{d}t}{\mathrm{d}s}$$

(when $\kappa(s) \neq 0$)

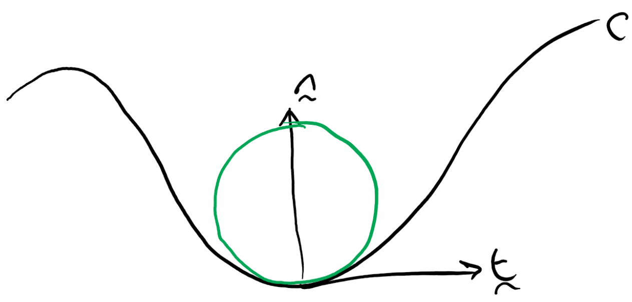
Note. $|\mathbf{n}| = 1$.

Claim. If $\kappa \neq 0$, then $\mathbf{n} \cdot \mathbf{t} = 0$.

Proof. $\mathbf{t} \cdot \mathbf{t} = 1 \implies \frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{t} \cdot \mathbf{t}) = 2\mathbf{t} \cdot \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = 0$. □

Hence $\mathbf{n}$ and $\mathbf{t}$ define the *oscillating plane.*

The curvature $\kappa(s)$ of a curve coincides with that of a circle touching C, at S, lying in the plane.



Note. Because $\mathbf{t} \cdot \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = 0$, for curves in $\mathbb{R}^3$, we can also compute the curvature as

$$\kappa = \left|\mathbf{t} \times \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s}\right| = \underbrace{|t|}_{=1} \left|\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s}\right|$$

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Example. Let C be the helix $\mathbf{x}(t) = (\cos t, \sin t, t)$. Then $\dot{\mathbf{x}}(t) = (-\sin t, \cos t, 1)$.

$$\implies \frac{\mathrm{d}s}{\mathrm{d}t} = \left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right| = \sqrt{2} \implies s = \sqrt{2}t.$$

The distance along the curve between $\mathbf{x}(0) = (1, 0, 0)$ and $\mathbf{x}(2\pi) = (1, 0, 2\pi)$ is

$$s = \int_0^{2\pi} \mathrm{d}t |\dot{\mathbf{x}}| = \sqrt{2} \times 2\pi = \sqrt{8}\pi$$

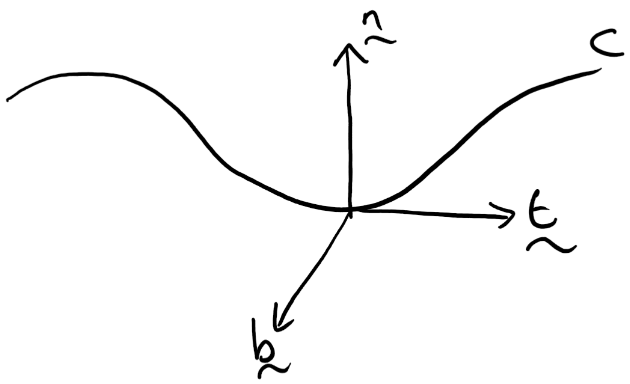
$$\mathbf{x}(s) = (\cos(s/\sqrt{2}), \sin(s/\sqrt{2}, s/\sqrt{2})$$

$$\mathbf{t} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \frac{1}{\sqrt{2}}\left(-\sin\left(\frac{s}{\sqrt{2}}\right), \cos\left(\frac{s}{\sqrt{2}}\right), 1\right)$$

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \underbrace{\frac{1}{2}}_{\kappa} \underbrace{\left(-\cos\left(\frac{s}{\sqrt{2}}\right), -\sin\left(\frac{s}{\sqrt{2}}\right), 0\right)}_{\mathbf{n}}.$$

For curves in $\mathbb{R}^3$, define the *binormal*

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}$$



Note. $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are an orthonormal basis for each s (at least with $\kappa(s) \neq 0$).

Because $|\mathbf{b}| = 1$ we have $\mathbf{b} \cdot \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = 0$. Moreover,

$$\mathbf{t} \cdot \mathbf{b} = 0 \implies \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} \cdot \mathbf{b} + \mathbf{t} \cdot \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = 0$$

$$\implies \kappa \mathbf{n} \cdot \mathbf{b} + \mathbf{t} \cdot \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = 0$$

$$\implies \mathbf{t} \cdot \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = 0$$

so $\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s}$ is $\perp$ to $\mathbf{b}$ and $\mathbf{t}$ hence $\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s}$ is parallel to $\mathbf{n}$.

Define the *torsion*, $\tau(s)$ as

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = -\tau(s)\mathbf{n}$$

The torsion measures how much the curve twists out of the plane. (It vanishes for planar curves.)

Note.

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \kappa(s)(\mathbf{b} \times \mathbf{t})$$

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = \tau(s)(\mathbf{t} \times \mathbf{b})$$

These are six first order DEs in six unknowns $\mathbf{t}$ and $\mathbf{b}$. For fixed $\kappa(s)$ and $\tau(s)$, there is a unique solution if we're given $\mathbf{t}(0)$ and $\mathbf{b}(0)$.
i.e. κ and τ specify C up to translations / rotations.

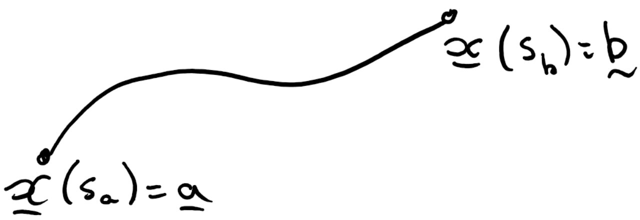
1.2 Line Integrals

A scalar field $\phi(\mathbf{x})$ is a map

$$\phi : \mathbb{R}^n \to R$$

We would like to integrate $\phi(\mathbf{x})$ along a curve C given by $\mathbf{x}(t)$ in a way that is *independent* of the parametrisation.

We work with the arc length. Let $\mathbf{x}(s)$ be a curve C that urns from $\mathbf{x} = \mathbf{a}$ to $\mathbf{x} = \mathbf{b}$.



We define the *line integral* from **a** to **b**

$$\int_C \phi \mathrm{d}s = \int_{s_a}^{s_b} \phi(\mathbf{x}(s))\mathrm{d}s$$

where we take $s_a < s_b$.

Note. This is defined so that $\int_C \mathrm{d}s$ is the length of C and is always positive. In other words, the line integral from **a** to **b** gives the same answer as **b** to **a**.

If you're given the curve $\mathbf{x}(t)$ using some other parameter, with $\mathbf{x}(t_a) = \mathbf{a}$ and $\mathbf{x}(t_b) = \mathbf{b}$ and $t_a < t_b$ then

$$\begin{aligned}\int_C \phi \mathrm{d}s &= \int_{b_a}^{b_b} \phi(\mathbf{x}(t))\frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t \\ &= \int_{t_a}^{t_b} \phi(\mathbf{x}(t))|\dot{\mathbf{x}}|\mathrm{d}t\end{aligned}$$

(using $\frac{\mathrm{d}s}{\mathrm{d}t} = |\dot{\mathbf{x}}|$. This factor $|\dot{\mathbf{x}}|$ ensures independence of parametrisation.)

A vector field $\mathbf{F}(\mathbf{x})$ is a map

$$\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n.$$

The *line integral* of a vector field $\mathbf{F}(\mathbf{x})$ along a curve C, parametrised by $\mathbf{x}(t)$, from $\mathbf{x}(t_a) = \mathbf{a}$ to $\mathbf{x}(t_b) = \mathbf{b}$ is

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_{t_a}^{t_b} \mathbf{F}(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t)\mathrm{d}t$$

This is the integral of **F** tangent to the curve (for example $\tau = \tau(t)$).

Note. This time the direction of the integral matters. The integral from **a** to **b** is the negative of the integral from **b** to **a**.

The choice of direction along C is called an *orientation*.

Again: the line integral of a scalar field does *not* depend on the orientation of C; the line integral of a vector field does depend on the orientation.

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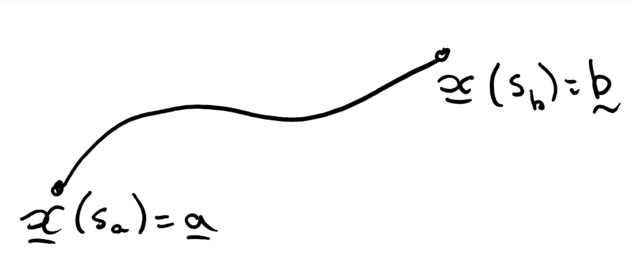
Example. Let $\mathbf{F}(\mathbf{x}) = (xe^y, z^2, xy)$. For C_1 let $\mathbf{x}(t) = (t,t,t)$, and for C_2 let $\mathbf{x}(t) = (t, t^2, t^3)$. We'll integrate from $(0,0,0)$ to $(1,1,1)$.



For C_1: $\mathbf{F}(t) = (te^{t^2}, t^6, t^3)$ and $\dot{\mathbf{x}}(t) = (1, 2t, 3t^2)$.

$$\implies \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_0^1 \mathrm{d}t (te^{t^2} + 2t^7 + 3t^5) = \frac{1}{4}(1+2e)$$

For C_2: $\mathbf{F}(t) = (te^t, t^2, t^z)$ and $\dot{\mathbf{x}}(t) = (1,1,1)$.

$$\implies \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_0^1 \mathrm{d}t (te^t + 2t^2) = \frac{5}{3}$$

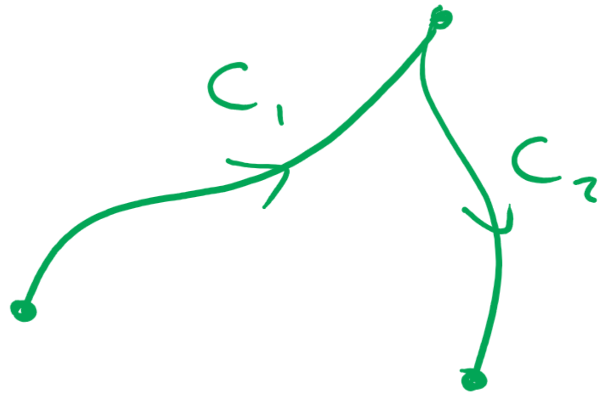
Note. Answer depends on C.

Sometimes we will integrate along a closed path C, with $\mathbf{a} = bf$. The line integral is the *circulation* of $\mathbf{F}$ around C, denoted as

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

Sometimes we will have a *piecewise smooth* curve $C = C_1 + C_2$, and then we define

$$\int_{C_1 + C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{x} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{x}.$$



The curve $-C$ is the curve C but with the opposite orientation, so

$$\int_{-C} \mathbf{F} \cdot d\mathbf{x} = -\int_C \mathbf{F} \cdot d\mathbf{x}$$

for example let $C = C_1 - C_2$ in our example. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{x} = \int_{C_1} Fbf \cdot d\mathbf{x} - \int_{C_2} \mathbf{F} \cdot d\mathbf{x} = \frac{1}{4}(1+2e) - \frac{5}{3}$$

1.3 Conservative Fields

Question: Do there exist $\mathbf{F}$ such that $\int_C \mathbf{F} \cdot d\mathbf{x}$ is independent of the path chosen between two fixed end points $\mathbf{a}$ and $\mathbf{b}$ i.e.

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{x} = \int_{C_2} \mathbf{F} \cdot d\mathbf{x}?$$

(for all C_1 and C_2 with the same end points).

Equivalently, considering $C = C_1 - C_2$, this would mean

$$\oint_C \mathbf{F} \cdot d\mathbf{x} = 0$$

for all closed paths C.

The gradient

Consider a scalar field $\phi : \mathbb{R}^n \to \mathbb{R}$. The *partial derivatives* are defined to be

$$\frac{\partial \phi}{\partial x^1} = \lim_{e \to 0} \frac{1}{e}[\phi(x^1 + e, x^2, \ldots, x^n) - \phi(x^1, x^2, \ldots, x^n)]$$

and similar for $\frac{\partial \phi}{\partial x^2}$ etc. The function is differentiable if all n partial derivatives exist. We write:

$$\partial_i \phi = \frac{\partial \phi}{\partial x^i} \qquad i = 1, \ldots, n$$

Also, it's not uncommon to stress which variables are held fixed by writing

$$\left(\frac{\partial \phi}{\partial x^1}\right)_{x^2, \ldots, x^n}$$

Let $\{\mathbf{e}_i\}$ be orthonormal basis of $\mathbb{R}^n$. Then the *gradient* of a scalar field is vector field, defined as

$$\nabla \phi = \frac{\partial \phi}{\partial x^i} \mathbf{e}_i$$

Note. Sometimes the ∇ is written with bold or underline.

If we want to compute how ϕ changes in some direction $\hat{\mathbf{n}}$ with $|\hat{\mathbf{n}}| = 1$, then we compute the *directional derivative* $\hat{\mathbf{n}} \cdot \nabla \phi$.

This is maximised at any point $\mathbf{x}$ by picking $\hat{\mathbf{n}} \parallel \nabla \phi$. But this means that $\nabla \phi(\mathbf{x})$ points in the direction in which $\phi(\mathbf{x})$ increases most quickly.

Back to Conservative Fields

A vector field $\mathbf{F}$ is called *conservative* if it can be written as

$$\mathbf{F} = \nabla \phi$$

for some ϕ called a *potential*.

Claim.

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = 0 \ \forall \ C$$

if and only if $\mathbf{F}$ is conservative.

Proof. If $\mathbf{F} = \nabla \phi$ then along any *open* curve C, parametrised by $\mathbf{x}(t)$, we have

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} &= \int_C \nabla \phi \cdot \mathrm{d}\mathbf{x} \\
&= \int_{t_a}^{t_b} \frac{\partial \phi}{\partial x^i} \frac{\mathrm{d}x^i}{\mathrm{d}t} \mathrm{d}t \\
&= \int_{t_a}^{t_b} \frac{\mathrm{d}}{\mathrm{d}t} \phi(\mathbf{x}(t)) \mathrm{d}t \\
&= \phi(\mathbf{x}(t_b)) - \phi(\mathbf{x}(t_a))
\end{aligned}$$

i.e. only depends on the end points. Conversely, suppose that

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = 0$$

Let $\phi(\mathbf{0}) = 0$ and define

$$\phi(\mathbf{y}) = \int_{C(\mathbf{y})} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

Then

$$\begin{aligned} \frac{\partial \phi}{\partial x^i}(\mathbf{y}) &= \lim_{e \to 0} \frac{1}{e} \left[\int_{C(\mathbf{y}+e\mathbf{e}_i} \mathbf{F} \cdot \mathrm{d}\mathbf{x} - \int_{C(\mathbf{y})} \right] \\ &= \lim_{e \to 0} \frac{1}{e} \int_{\mathbf{y}}^{\mathbf{y}+e\mathbf{e}_i} \mathbf{F} \cdot \mathrm{d}\mathbf{x} \\ &= \lim_{e \to 0} \frac{1}{e} (eF_i) \\ &= F_i \end{aligned}$$

□

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Question: Given **F**, how do we know if its conservative?

Answer: There is a check. If $F_i = \frac{\partial \phi}{\partial x^i}$ then

$$\frac{\partial F_i}{\partial x^i} = \frac{\partial^2 \phi}{\partial x^i \partial x^j} = \frac{\partial F_j}{\partial x^i} \qquad \forall\ i, j.$$

This is a necessary condition. We will later see that this is also a sufficient condition (if **F** is everywhere well defined).

Example.

$$\mathbf{F} = (3x^2 y \sin z, x^3 \sin z, x^3 y \cos z).$$

Check:

$$\begin{aligned} \partial_1 F_2 &= 3x^2 \sin z = \partial_2 F_1 \\ \partial_1 F_3 &= 3x^2 y \cos z = \partial_3 F_1 \\ \partial_2 F_3 &= x^3 \cos z = \partial_3 F_2. \end{aligned}$$

Indeed $\mathbf{F} = \nabla \phi$ with $\phi = x^3 y \sin z$, so $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x}$ depends only on the end points of C.

Exact Differentials

Given a function $\phi(\mathbf{x})$, the *differential* is

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x^i}\mathrm{d}x^i = \nabla\phi \cdot \mathrm{d}\mathbf{x}.$$

Given a vector field $\mathbf{F}$, the object $\mathbf{F}\cdot\mathrm{d}\mathbf{x}$ is *exact* if it can be written as

$$\mathbf{F}\cdot\mathrm{d}\mathbf{x} = \mathrm{d}\phi$$

An Application

The trajectory $\mathbf{x}(t)$ of a particle is governed by Newton's second law

$$m\ddot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x})$$

We define the kinetic energy

$$k = \frac{1}{2}m\dot{\mathbf{x}}^2$$

This changes over time as

$$\begin{aligned} k(t_2) - k(t_1) &= \int_{t_1}^{t_2} \frac{\mathrm{d}k}{\mathrm{d}t}\mathrm{d}t \\ &= \int_{t_1}^{t_2} m\dot{\mathbf{x}}\cdot\ddot{\mathbf{x}}\mathrm{d}t \\ &= \int_{t_1}^{t_2} \mathbf{F}\cdot\dot{\mathbf{x}}\mathrm{d}t \\ &= \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{x} \end{aligned}$$

This is called the *work done*. For conservative forces

$$\mathbf{F} = -\nabla V$$

Then

$$\begin{aligned} k(t_2) - k(t_1) = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{x} = -V(t_2) + V(t_1) \\ \implies \mathbf{F}(t) = k(t) + V(t) = \text{constant}. \end{aligned}$$

A Subtlety

Consider

$$\mathbf{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

Check

$$\partial_x F_y = \partial_y F_x = \frac{y^2 - x^2}{x^2+y^2}$$

and indeed $\mathbf{F} = \nabla\phi$ with $\phi = \tan^{-1}(y/x)$. Now integrate $\mathbf{F}$ around

$$\mathbf{x}(t) = (R\cos t, R\sin t) \qquad 0 \le t < 2\pi$$

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} &= \int_0^{2\pi} \mathbf{F} \ \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\mathrm{d}t \\
&= \int_0^{2\pi} \left(-\frac{\sin t}{R}(-R\sin t) + \frac{\cos t}{R}(R\cos t) \right) \mathrm{d}t \\
&= \int_0^{2\pi} \mathrm{d}t \\
&= 2\pi \\
&\neq 0
\end{aligned}$$

Why?!

It's because $\mathbf{F}$ isn't defined at the origin. Moreover, ϕ is discontinuous along the $\mathbf{x} = 0$ axis.

Our previous claim that $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = 0$ only holds when ϕ is a continuous function, or when $\mathbf{F}$ is defined inside C in $\mathbb{R}^2$.

2 Surfaces (and Volumes)

2.1 Multiple Integrals

Area Integrals

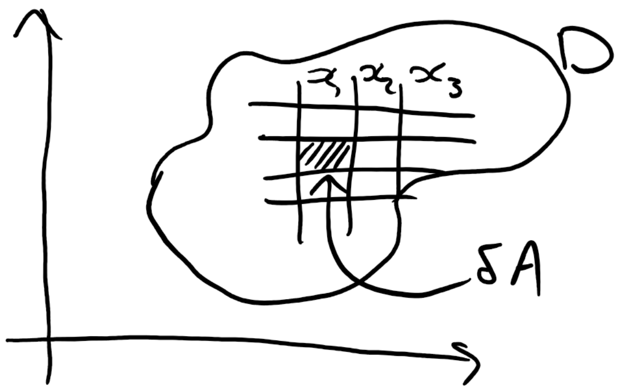
Consider a region $D \subset \mathbb{R}^2$. We want to integrate a scalar field $\phi(x, y)$ over D, i.e.

$$\int_D \phi \mathrm{d}A$$

($\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ is the *area element*).

Note. It's sometimes written $\iint_D \phi \mathrm{d}A$.

Basic idea:



$$\int_D \phi(x)\mathrm{d}A = \sum_n \phi(x_n)\delta A$$

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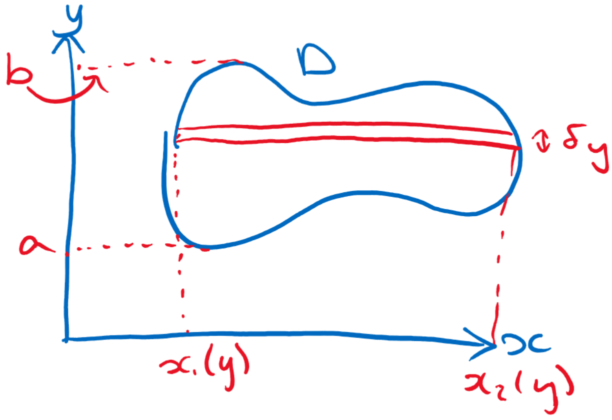
Note. For $\phi = 1$, $\int_D \mathrm{d}A$ is the area of D.

To evaluate the area integral, we split the region D into strips.



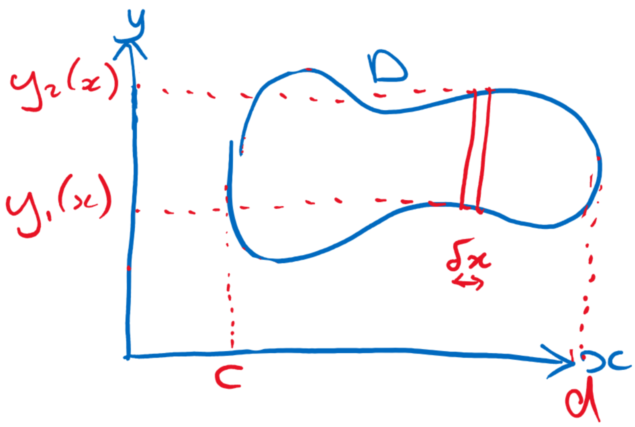
Do $\int \mathrm{d}x$ for some fixed y, and then do $\int \mathrm{d}y$.

$$\int_D \phi \mathrm{d}A = \int_a^b \mathrm{d}y \int_{x_1(y)}^{x_2(y)} \mathrm{d}x \phi(x, y)$$

($x_1(y)$ and $x_2(y)$ trace the outline of D).

Note. This is written as $\int \mathrm{d}x$(integrand) instead of $\int$(integrand)$\mathrm{d}x$. You do $\int \mathrm{d}x$ first, and then $\int \mathrm{d}y$.

Alternatively, we could divide D as

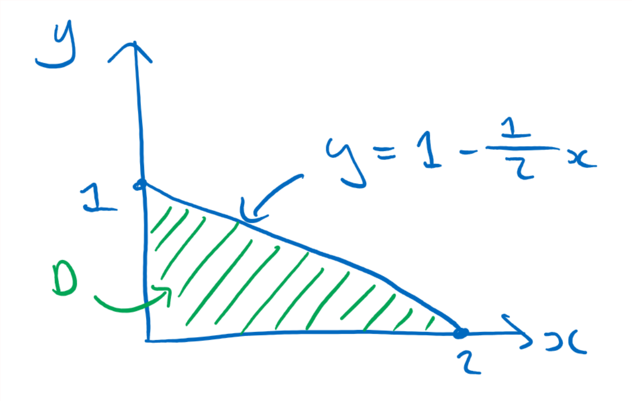


$$\int_D \phi \mathrm{d}A = \int_c^d \mathrm{d}x \int_{y_1(x)}^{y_2(x)} \mathrm{d}y \phi(x, y)$$

Now do $\int \mathrm{d}y$ first and then $\int \mathrm{d}x$.

For suitably well behaved ϕ and D, any way of splitting up $\int \mathrm{d}A$ gives the same result. (Fubri's theorem).

Example. Let $\phi(x, y) = x^2 y$ and D be the triangle



$$\begin{aligned}\int_D \phi \mathrm{d}A &= \int_0^1 \mathrm{d}y \int_0^{2-2y} \mathrm{d}x x^2 y \\ &= \int_0^1 \mathrm{d}y y \left[\frac{x^3}{3}\right]_0^{2-2y} \\ &= \frac{8}{3}\int_0^1 \mathrm{d}y y(1-y)^3 \\ &= \frac{2}{15}\end{aligned}$$

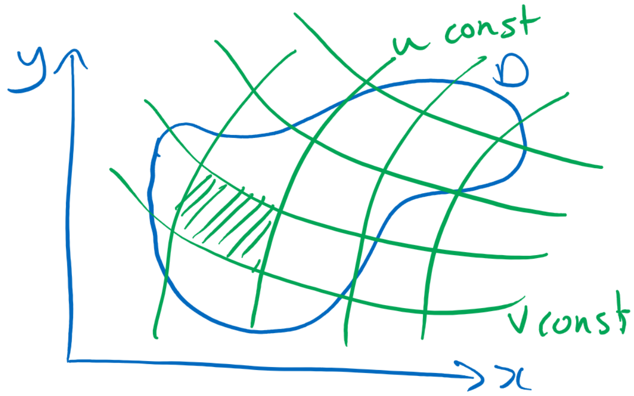
or

$$\begin{aligned}\int_D \phi \mathrm{d}A &= \int_0^2 \mathrm{d}x \int_0^{1-x^2/2} \mathrm{d}y x^2 y \\ &= \int_0^2 \mathrm{d}x x^2 \left[\frac{1}{2}y^2\right]_0^{1-x^2/2} \\ &= \frac{1}{2}\int_0^2 \mathrm{d}x x^2 \left(1 - \frac{1}{2}x\right)^2 \\ &= \frac{2}{15}\end{aligned}$$

It is often useful to evaluate integrals using something other than cartesian coordinates. Consider a change of variables

$$(x, y) \mapsto (u, v).$$

We assume that this map is smooth and invertible. We can then use (u, v) as coordinates on $\mathbb{R}^2$.



How do we do the integral in (u, v) coordinates?

Claim. The area integral can be written as

$$\int_D \phi \mathrm{d}A = \int_{D'} \mathrm{d}u\mathrm{d}v |J(u,v)| \phi(u,v)$$

Here the *Jacobian* is the modulus of the determinant

$$|J(u,v)| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

We will also write the matrix

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)}$$

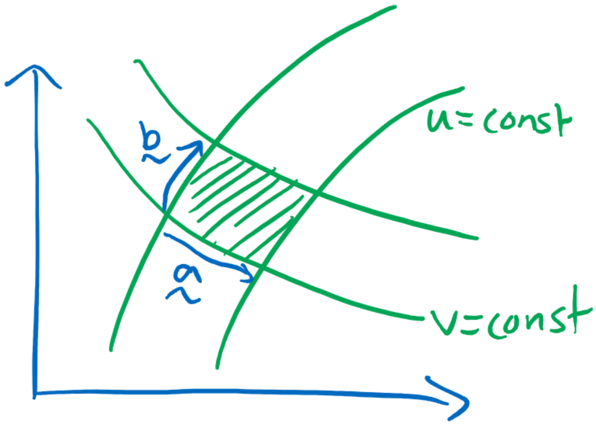
Proof. We sum over the small parallelograms sandwiched between $u, v =$ constant lines. Let $x = x(u,v)$ and $y = y(u,v)$.

$$\implies \delta x = \frac{\partial x}{\partial u}\delta u + \frac{\partial x}{\partial v}\delta v$$

and

$$\delta y = \frac{\partial y}{\partial u}\delta u + \frac{\partial y}{\partial v}\delta v$$

$$\implies \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \delta u \\ \delta v \end{pmatrix}$$



So

$$\mathbf{a} = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{pmatrix} \delta u$$

$$\mathbf{b} = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \end{pmatrix} \delta v$$

The area of parallelogram is

$$\delta A = |\mathbf{a} \times \mathbf{b}| = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \delta u \delta v = |J| \delta u \delta v.$$

□

An Example: 2D Polar Coordinates

Plane polar coordinates are defined by

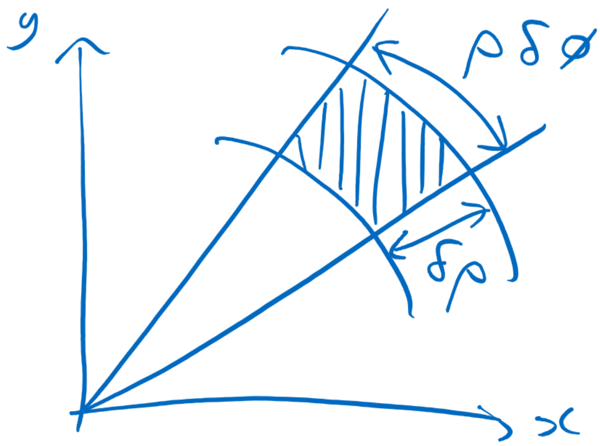
$$x = \rho \cos \phi \qquad y = \rho \sin \phi$$

with $\phi \in [0, \infty)$ and $\phi \in [0, 2\pi)$. Then

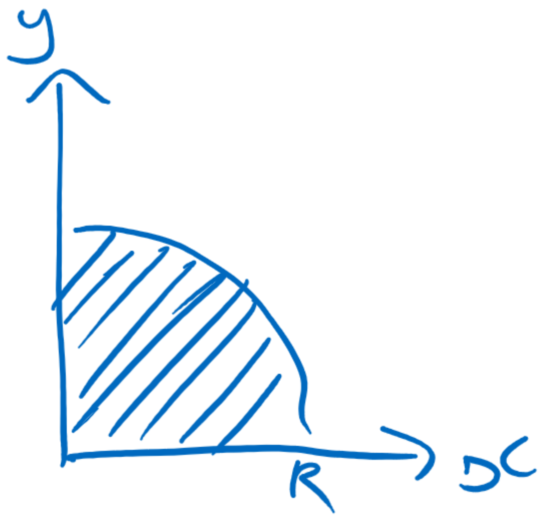
$$J = \frac{\partial(x, y)}{\partial(\rho, \phi)} = \begin{vmatrix} \cos \phi & -\rho \sin \phi \\ \sin \phi & \rho \cos \phi \end{vmatrix} = \rho$$

The area element is

$$\delta A = \rho \delta \rho \delta \phi$$



As an example, let D be the region $x, y \geq 0$ and $x^2 + y^2 \leq R^2$.



This is $0 \leq \rho \leq R$ and $0 \leq \phi \leq \frac{\pi}{2}$. We will integrate $f = e^{-(x^2+y^2)/2} = e^{-\rho^2/2}$.

$$\begin{aligned}\int_D f \mathrm{d}A &= \int_0^{\pi/2} \mathrm{d}\phi \int_0^R \mathrm{d}\rho \rho e^{-\rho^2/2} \\ &= \frac{\pi}{2}[-e^{-\rho^2/2}]_0^R \\ &= \frac{\pi}{2}(1 - e^{-R^2/2})\end{aligned}$$

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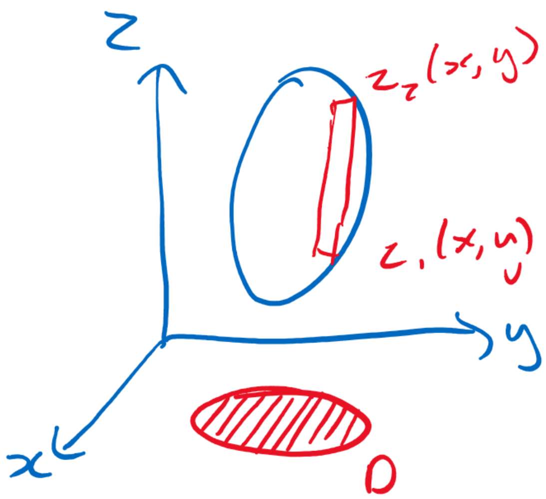
> **Note.** As $R \to \infty$, we integrate over the whole of $x, y \geq 0$ quadrant. In cartesian coordinates, we have
>
> $$\begin{aligned} \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y e^{-(x^2+y^2)/2} \\ &= \left(\int_0^\infty \mathrm{d}x e^{-x^2/2}\right)\left(\int_0^\infty \mathrm{d}y e^{-x^2/2}\right) \\ &= \left(\int_0^\infty \mathrm{d}x e^{-x^2/2}\right)^2 \\ &= \frac{\pi}{2} \\ \implies \int_0^\infty \mathrm{d}x e^{-x^2/2} &= \sqrt{\frac{\pi}{2}} \end{aligned}$$

Volume Integrals

We now generalise to integrals over a region $V \subset \mathbb{R}^3$. We have

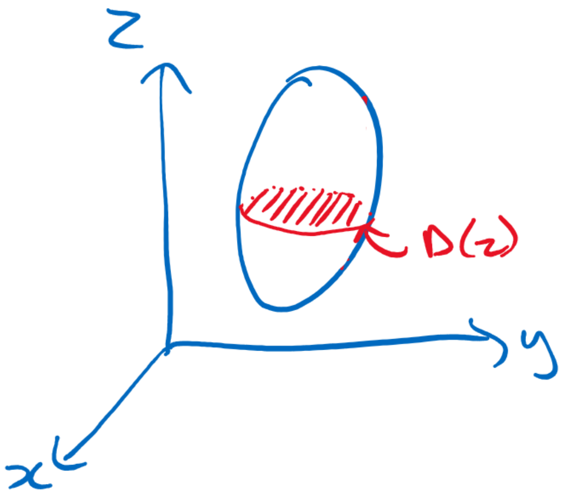
$$\int_V \phi(\mathbf{x}) \mathrm{d}V = \lim_{\delta V \to 0} \sum_n \phi(\mathbf{x}_n) \delta V$$

We again perform the integral one coordinate at a time. Again, the order doesn't matter.



$$\int_V \phi \mathrm{d}V = \int_D \mathrm{d}A \int_{z_1(x,y)}^{x_2(x,y)} \mathrm{d}z \phi(x, y, z)$$

or



$$\int_V \phi \mathrm{d}V = \int \mathrm{d}z \int_{D(z)} \mathrm{d}x\mathrm{d}y\phi(x,y,z)$$

Under an invertible, smooth change of coordinates

$$(x,y,z) \mapsto (u,v,w)$$

we have

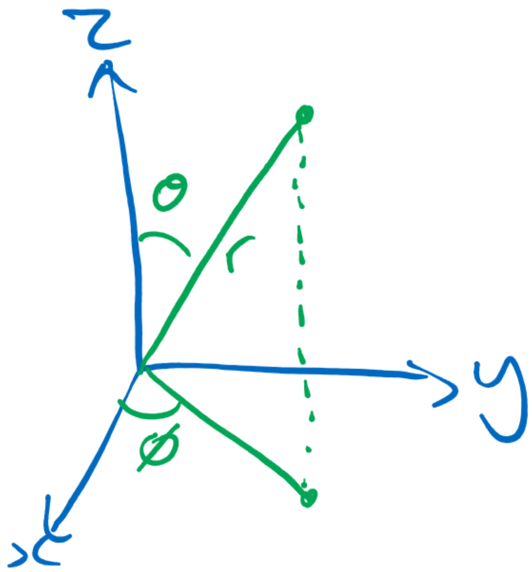
$$\mathrm{d}V = |J|\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

with

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

The proof is similar to before. For example, *spherical polar coordinates* are

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

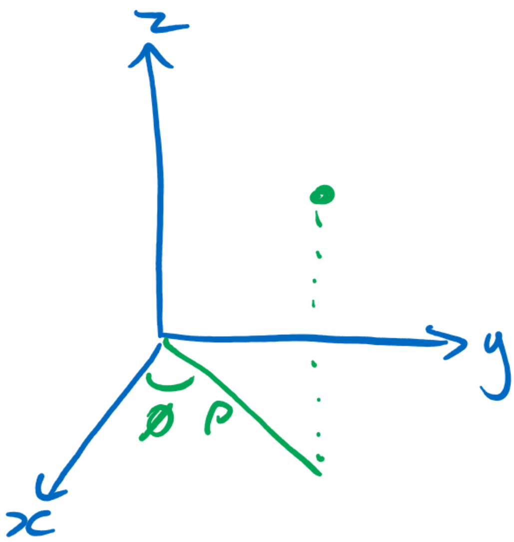


with $r \geq 0$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. We find $J = r^2 \sin\theta$

$$\implies \mathrm{d}V = r^2 \sin\theta \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi$$

Cylindrical polar coordinates are

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$



with $\rho \geq 0$ and $\phi \in [0, 2\pi)$. Now $J = \rho$ and

$$\mathrm{d}V = \rho \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}z.$$

Examples

(1) A spherically symmetric function $f(r)$ integrated over a ball of radius R

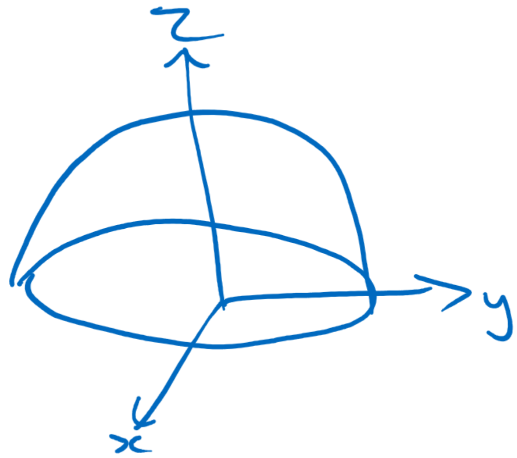
$$\begin{aligned}\int_V f \mathrm{d}V &= \int_0^R \mathrm{d}r \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \underbrace{r^2 \sin\theta}_{J} f(r) \\ &= 2\pi[-\cos\theta]_0^\pi \int_0^R \mathrm{d}r r^2 f(r) \\ &= 4\pi \int_0^R \mathrm{d}r r^2 f(r)\end{aligned}$$

If $f = 1 \implies V = \frac{4\pi R^3}{3}$ = volume of the ball.

(2) What is the volume of a ball of radius R with cylinder of radius $s < R$ removed from the middle? .image In cylindrical polars, V is $s \leq \rho \leq R$ and $-\sqrt{R^2 - \rho^2} \leq z \leq \sqrt{R^2 - \rho^2}$ and $0 \leq \phi > 2\pi$. So

$$\begin{aligned}\mathrm{Vol} &= \int_V \mathrm{d}V \\ &= \int_0^{2\pi} \mathrm{d}\phi \int_s^R \mathrm{d}\rho\rho \int_{-\sqrt{R^2-\rho^2}}^{+\sqrt{R^2-\rho^2}} \mathrm{d}z \\ &= 2\pi \int_s^R \mathrm{d}\rho 2\rho\sqrt{R^2 - \rho^2} \\ &= \frac{4\pi}{3}(R^2 - s^2)^{3/2}\end{aligned}$$

(3) A hemisphere H of radius R and $z \geq 0$ has charge density $f(z) = f_0 \frac{z}{r}$ with $f_0 =$ constant. What is the total charge?



Use spherical polars.

$$r \le R$$
$$0 \le \phi \le 2\pi$$
$$0 \le \theta \le \frac{\pi}{2}$$

$$\implies \int_H f \mathrm{d}V = \frac{f_0}{r} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi/2} \mathrm{d}\theta \int_0^R \mathrm{d}r \underbrace{r^2 \sin\theta}_{J} \underbrace{r\cos\theta}_{z}$$
$$= \frac{2\pi f_0}{R} \left[\frac{r^4}{4}\right]_0^R \left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}$$
$$= \frac{1}{4}\pi R^3 f_0$$

(4) To compute the centre of mass of an object, we need vector valued integrals. Let $\rho(\mathbf{x})$ be the density

$$\implies \text{mass } M = \int_V \rho(\mathbf{x})\mathrm{d}V$$

and *center of mass* is

$$\mathbf{X} = \frac{1}{M}\int_V \rho(\mathbf{x})\mathbf{x}\mathrm{d}V$$

For example for the solid hemisphere of constant density ρ

$$M = \int_H \rho \mathrm{d}V = \frac{2\pi}{3}\rho R^3$$

and $\mathbf{X} = (X, Y, Z)$.

$$X = \frac{\rho}{M}\int_0^{2\pi}\mathrm{d}\phi \int_0^R \mathrm{d}r \int_0^{\pi/2} \mathrm{d}\theta x \underbrace{r^2\sin\theta}_{J} = 0$$

Similarly $Y = 0$.

$$Z = \frac{\rho}{M} \int_0^{2\pi} \mathrm{d}\phi \int_0^R \mathrm{d}r \int_0^{\pi/2} \mathrm{d}\theta z r^2 \sin\theta = \frac{3R}{8}$$

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2.2 Surface Integrals

We define surfaces in $\mathbb{R}^3$ by

- A function $F(x, y, z) = 0$
- A *paramterised surface* is a map

$$\mathbf{x} : \mathbb{R}^2 \mapsto \mathbb{R}^3$$

At each point on the surface, the *normal vector* $\mathbf{n}$ points away in a perpendicular direction.

Claim. For the surface $F(\mathbf{x}) = 0$, $\mathbf{n} \parallel \nabla F$.

Proof. $\mathbf{m} \cdot \nabla F$ is the rate of change of F in the direction $\mathbf{m}$. There are two linearly independent vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ that lie tangent to the surface and obey $\mathbf{m}_i \cdot \nabla F = 0$, $i = 1, 2$. The normal vector $\mathbf{n}$ is perpendicular to $\mathbf{m}_1$ and $\mathbf{m}_2$ and so $\mathbf{n} \parallel \nabla F$. □

We usually define

$$\mathbf{n} = \pm \frac{1}{|\nabla F|} \nabla F$$

For a parametrised surface $\mathbf{x}(u, v)$ the tangent vectors are

$$\frac{\partial \mathbf{x}}{\partial u} \quad \text{and} \quad \frac{\partial \mathbf{x}}{\partial v}$$

The normal vector is $\mathbf{n} \parallel \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}$.

Definition. If $\mathbf{n} \neq 0$ at all points, the surface is *regular*.

Examples

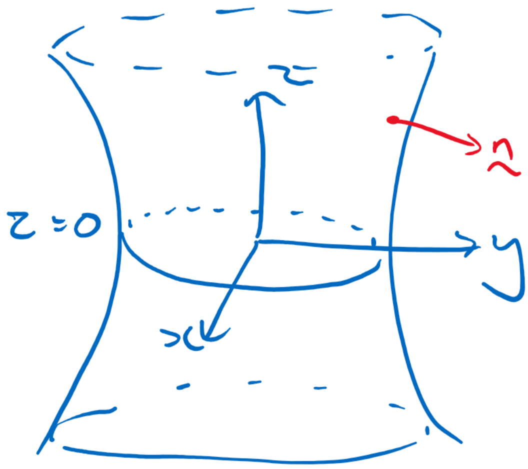
(1) $F(\mathbf{x}) = x^2 + y^2 + x^2 - R^2 = 0$ is a sphere of radius R. The normal vector is $\parallel$ to ∇F and is

$$2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

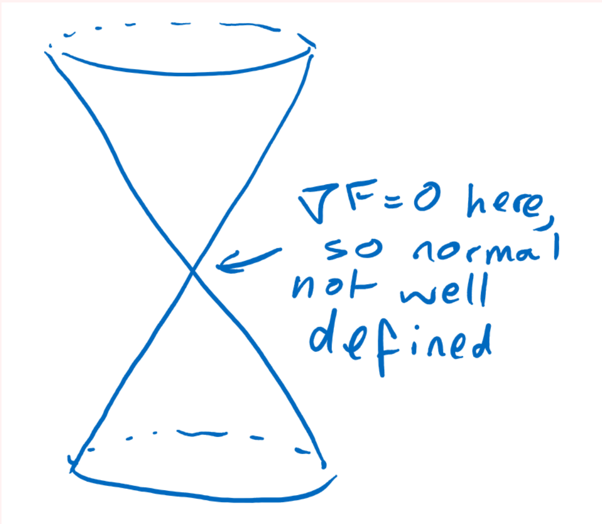
(2) A hyperboloid is defined by

$$F(\mathbf{x}) = x^2 + y^2 - z^2 - R^2 = 0$$

$$\nabla F = 2 \begin{pmatrix} x \\ y \\ -z \end{pmatrix}$$



Note. When $R = 0$ the surface is no longer regular:



A surface S can have a boundary. The boundary is a closed curve C, denoted as $C = \partial S$.

Deep Fact: The boundary curve C is closed, i.e. it has no end points.

Another Deep Fact: We denote the boundary of something by ∂. The fact that the boundary of a boundary vanishes is written

$$\partial C = \partial^2 S = 0$$

Definition. A surface is *bounded* / *unbounded* if it doesn't / does stretch to infinity. A bounded surface with no boundary is *closed*.

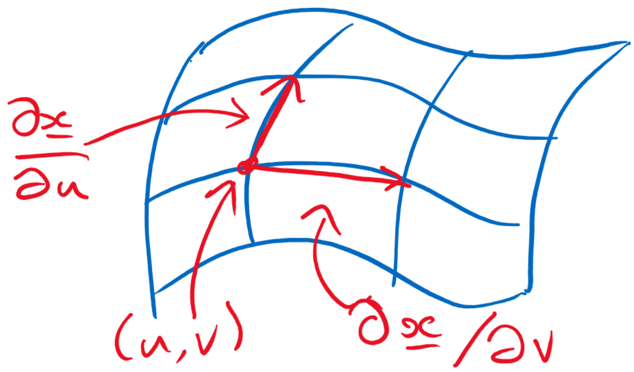
Note. There is no canonical way to fix the $\pm$ sign of $\mathbf{n}$. If there is a consistent choice over the surface S, then S is *orientable*. For example the sphere S^2 is orientable but the Möbius M with $\partial M = S^1$ (circle) is non-orientable. We will only work with orientable surfaces.

Integrating Scalar Fields

Consider a parametrised surface

$$\mathbf{x}(u, v)$$

sit at some point (u, v) and move a small amount δu or δv.



The parallelogram defined by $\frac{\partial \mathbf{x}}{\partial u}$ and $\frac{\partial \mathbf{x}}{\partial v}$ has scalar area

$$\delta S = \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| \delta u \delta v$$

The integral of a scalar field $\phi(\mathbf{x})$ over a parametrised surface is

$$\int_S \phi(\mathbf{x}) \mathrm{d}S = \int_D \mathrm{d}u\mathrm{d}v \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| \phi(\mathbf{x}(u, v))$$

> **Note.** This does not depend on the orientation of S. Also $\int_S \mathrm{d}S$ is the area of the surface. Also, the integral does not depend on the choice of parametrisation.

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To see that this integral is parametrisation invariant, suppose that $\mathbf{x}(\tilde{u}, \tilde{v})$ describes the same surface. Then

$$\frac{\partial \mathbf{x}}{\partial u} = \frac{\partial \mathbf{x}}{\partial \tilde{u}}\frac{\partial \tilde{u}}{\partial u} + \frac{\partial \mathbf{x}}{\partial \tilde{v}}\frac{\partial \tilde{v}}{\partial u}$$
$$\frac{\partial \mathbf{x}}{\partial v} = \frac{\partial \mathbf{x}}{\partial \tilde{u}}\frac{\partial \tilde{u}}{\partial v} + \frac{\partial \mathbf{x}}{\partial \tilde{v}}\frac{\partial \tilde{v}}{\partial v}$$
$$\implies \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} = \frac{\partial(\tilde{u}, \tilde{v})}{\partial(u, v)}\frac{\partial \mathbf{x}}{\partial \tilde{u}} \times \frac{\partial \mathbf{x}}{\partial \tilde{v}}$$

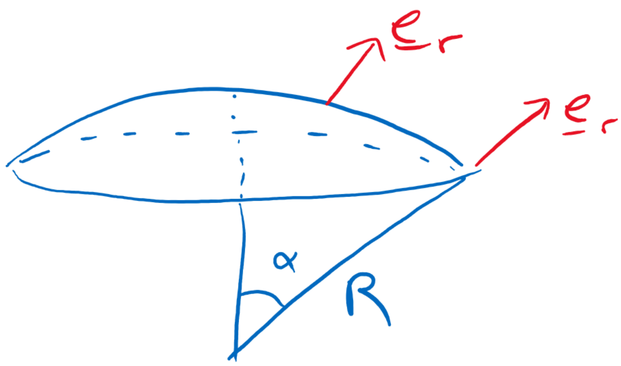
But from earlier,

$$\mathrm{d}\tilde{u}\mathrm{d}\tilde{v} = \frac{\partial(\tilde{u}, \tilde{v})}{\partial(u, v)}\mathrm{d}u\mathrm{d}v$$
$$\implies \mathrm{d}S = \left|\frac{\partial \mathbf{x}}{\partial \tilde{u}} \times \frac{\partial \mathbf{x}}{\partial \tilde{v}}\right|\mathrm{d}\tilde{u}\mathrm{d}\tilde{v}$$

and the integral takes the same form for (u, v) and $(\tilde{u}, \tilde{v})$.

An Example

Let s be the surface of a sphere of radius R subtended by angle α.



In spherical polars,

$$\begin{aligned}\mathbf{x}(\theta, \phi) &= R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ &:= R\mathbf{e}_r\end{aligned}$$

(with $\phi \in [0, 2\pi)$ and $\theta \in [0, \alpha]$). We also write $\mathbf{e}_r = \hat{\mathbf{r}}$. We have

$$\begin{aligned}
\frac{\partial \mathbf{x}}{\partial \theta} &= R(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \\
&:= R\mathbf{e}_\theta \\
\frac{\partial \mathbf{x}}{\partial \phi} &= R(-\sin\theta\sin\phi, \sin\theta\cos\phi, 0) \\
&:= R\sin\theta\mathbf{e}_\phi \\
\implies \frac{\partial \mathbf{x}}{\partial \theta} \times \frac{\partial \mathbf{x}}{\partial \phi} &= R^2 \sin\theta \mathbf{e}_r \\
\implies \mathrm{d}S &= R^2 \sin\theta \mathrm{d}\theta\mathrm{d}\phi
\end{aligned}$$

The area is now

$$\begin{aligned}
A &= \int_0^{2\pi} \mathrm{d}\phi \int_0^\alpha \mathrm{d}\theta R^2 \sin\theta \\
&= 2\pi R^2 (1 - \cos\alpha)
\end{aligned}$$

Integrating Vector Fields

It is often useful to integrate a vector field over a surface to yield a number. We do this by

$$\int_S \mathbf{F}(\mathbf{x}) \cdot \mathbf{n}\mathrm{d}S = \int_D \mathrm{d}u\mathrm{d}v \left(\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right) \cdot \mathbf{F}(\mathbf{x}(u, v))$$

($\mathbf{n}$ is unit normal to the surface). This is the *flux* of $\mathbf{F}$ through S. Again, it is reparametrisation invariant.

We define the *vector area element*

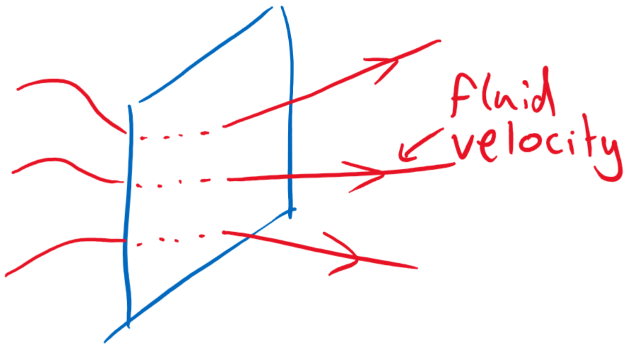
$$\mathbf{d}\boldsymbol{S} = \mathbf{n}\mathrm{d}S = \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\mathrm{d}u\mathrm{d}v$$

Clearly $|\mathbf{d}\boldsymbol{S}| = \mathrm{d}S$. Then the flux can be written as

$$\int_S \mathbf{F} \cdot \mathbf{d}\boldsymbol{S}$$

The flux depends on the orientation of S, i.e. on the sign of $\mathbf{n}$.

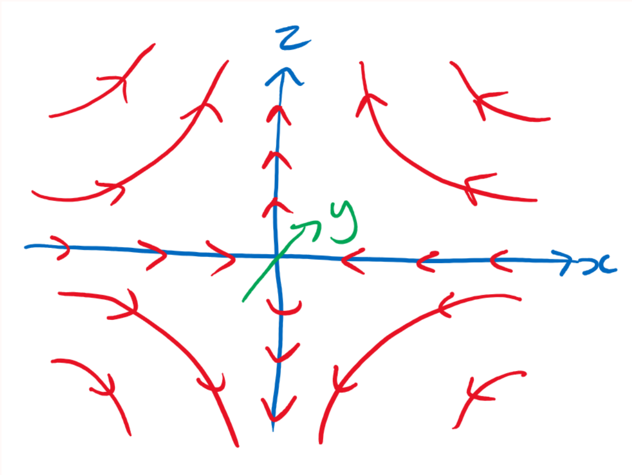
An Application: Consider a fluid with a velocity field $\mathbf{F}(\mathbf{x})$.



In a small δt, the amount of fluid that flows through S is

$$\text{Fluid flow} = \mathbf{F}\delta t \cdot \mathbf{n}\delta S$$
$$\text{Flow} = \int \mathbf{F} \cdot \mathbf{d}\boldsymbol{S} = \text{fluid crossing } S \text{ per unit time}$$

Example. Let $\mathbf{F} = (-x, 0, z)$.



We'll integrate this over the spherical cap $r = R$, $0 \leq \theta \leq \alpha$ and $0 \leq \phi < 2\pi$. We know that

$$\mathbf{d}\boldsymbol{S} = \mathbb{R}^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \mathbf{e}_r$$

$$\mathbf{e}_r \equiv \hat{\mathbf{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

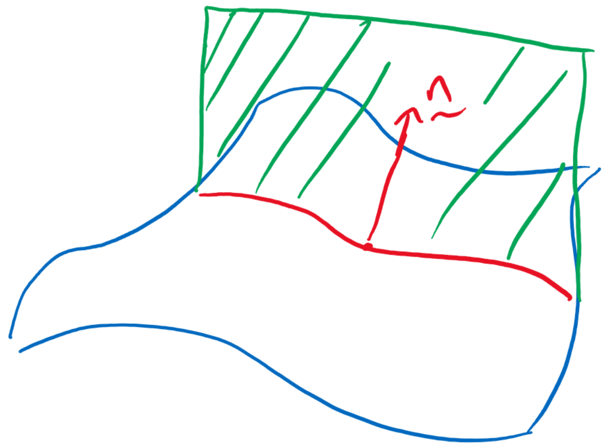
$$\begin{aligned}\mathbf{F} \cdot \mathbf{e}_r &= -x\sin\theta\cos\phi + z\cos\theta \\ &= R(-\sin^2\theta\cos^2\phi + \cos^2\theta)\end{aligned}$$

using x, z polar coordinates.

$$\begin{aligned}\int \mathbf{F} \cdot \mathbf{d}\boldsymbol{S} &= \int_0^\alpha \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi R^3 \sin\theta(-\sin^2\cos^2\phi + \cos^2\theta) \\ &= \pi R^3 \cos\alpha \sin^2\alpha\end{aligned}$$

The Gauss-Bonnet Theorem (non-examinable)

Consider a surface S with a *normal* $\mathbf{n}$ at some point.



Draw a plane containing $\mathbf{n}$. The intersection of the plane and S gives a curve C, with curvature κ at the point.

Now we rotate the plane about $\mathbf{n} \implies$ the curve and κ change. The *Gaussian curvature* of S at the point is

$$K = \kappa_{\text{min}} \kappa_{\text{max}}$$

Theorem (Gauss-Bonnet v1)**.** For a closed surface S,

$$\int_S \kappa \mathrm{d}S = 4\pi(1-g)$$

where g = genus = number of holes. For example, for a sphere $g = 0$, for a torus $g = 1$ and a double torus has $g = 2$.

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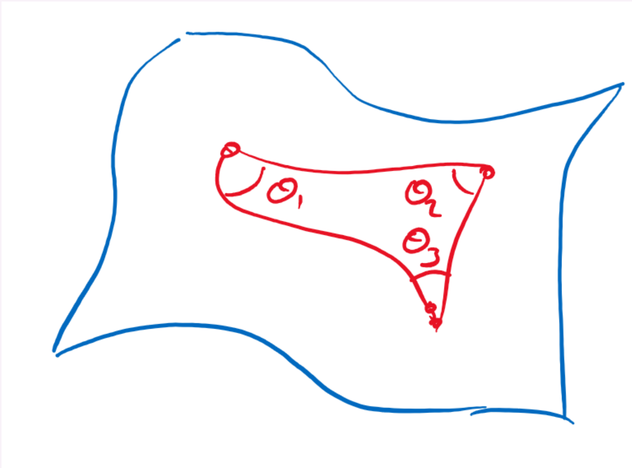
Theorem (Gauss-Bonnet v2)**.** Draw a geodesic triangle on a surface S.



The sides are *geodesics*, meaning curves with shortest arc length between two points.

$$\theta_1 + \theta_2 + \theta_3 = \phi + \int_{\triangle} \kappa \mathrm{d}S.$$

3 Grad, Div and Curl

We will consider different ways to differentiate.

3.1 The Gradient

Consider a scalar field $\phi : \mathbb{R}^n \to \mathbb{R}$. Then we define the *gradient* by

$$\phi(\mathbf{x}+\mathbf{h}) = \phi(\mathbf{x}) + \mathbf{h}\cdot\nabla\phi + \theta(|\mathbf{h}|^2)$$

In cartesian coordinates $\mathbf{x} = (x^1, \dots, x^n)$ with $\{\mathbf{e}_i\}$ the associated orthonormal basis of $\mathbb{R}^n$, we take

$$\mathbf{h} = \varepsilon \mathbf{e}_i$$

with $\varepsilon \ll 1$ and this reduces to our earlier definition

$$\nabla\phi = \frac{\partial\phi}{\partial x^i}\mathbf{e}_i$$

This is what we use practice.

Example. Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ with

$$\phi(\mathbf{x}) = -\frac{1}{r}$$

with $r = \sqrt{x^2+y^2+z^2}$. Then

$$\frac{\partial\phi}{\partial x} = \frac{x}{(x^2+y^2+z^2)^{3/2}} = \frac{x}{r^3}$$

and similarly for $\frac{\partial\phi}{\partial y}$ and $\frac{\partial\phi}{\partial z}$

$$\implies \nabla\phi = \frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}}{r^3} = \frac{\hat{\mathbf{r}}}{r^2}$$

where $\hat{\mathbf{r}}$ is the unit vector pointing radially (also called $\mathbf{e}_r$).

Application

Let $\mathbf{x}(t) : \mathbb{R} \to \mathbb{R}^n$ define a curve in $\mathbb{R}^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ be a scalar field. Then

$$\phi(\mathbf{x}(t)) : \mathbb{R} \to \mathbb{R}$$

is the value of ϕ along the curve. We can differentiate ϕ along the curve using the chain rule

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial\phi}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} = \nabla\phi\cdot\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$$

3.2 Div and Curl

We define the *gradient operator*

$$\nabla = \mathbf{e}_i \frac{\partial}{\partial x^i}$$

This is both a vector and a differential operator. These operators are waiting for some function to come along to be differentiated. ∇ is also called *nabla* or *del*.

Originally we introduced ∇ as acting on a scalar $\phi : \mathbb{R} \to \mathbb{R}$. But we can also ask how it might act on other fields.

Consider a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$. The *divergence* of $\mathbf{F}$ is a scalar field, defined by

$$\begin{aligned} \nabla \cdot \mathbf{F} = \left(\mathbf{e}_i \frac{\partial}{\partial x^i}\right) \cdot (\mathbf{e}_j F_j) \\ &= (\mathbf{e}_i \cdot \mathbf{e}_j) \frac{\partial F_j}{\partial x^i} \\ &= \frac{\partial F_i}{\partial x^i} \end{aligned}$$

but $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. For example in $\mathbb{R}^3$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

with $\mathbf{F} = (F_1, F_2, F_3)$.

We'll later see that $\nabla \cdot \mathbf{F}$ measures the net flow of $\mathbf{F}$ into / out of a point $\mathbf{x}$.

For vector fields $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$, we can also define the *curl*

$$\begin{aligned} \nabla \times \mathbf{F} &= \left(\mathbf{e}_i \frac{\partial}{\partial x^i}\right) \times (\mathbf{e}_j F_j) \\ &= \varepsilon_{ijk} \frac{\partial F_j}{\partial x^i} \mathbf{e}_k \end{aligned}$$

Equivalently

$$\nabla \times \mathbf{F} = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix}$$

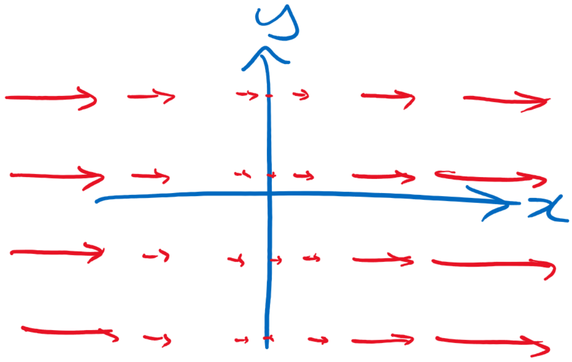
where $\partial_i \equiv \frac{\partial}{\partial x^i}$. Alternatively

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix}$$

We'll later see that $\nabla \times \mathbf{F}$ measures the rotation of $\mathbf{F}$.

Examples in $\mathbb{R}^3$

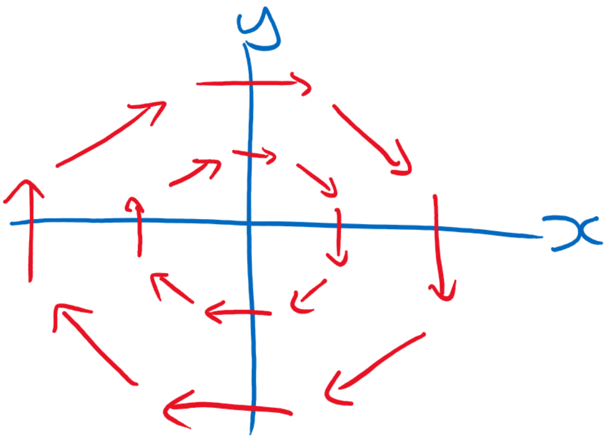
(1) Consider $\mathbf{F} = (x^2, 0, 0)$



$$\implies \nabla \cdot \mathbf{F} = 2x \implies \text{more out than in at any point}$$

$$\nabla \times \mathbf{F} = \mathbf{0} \implies \text{no rotation}$$

(2) Consider $\mathbf{F} = (y, -x, 0)$



$$\nabla \cdot \mathbf{F} = 0 \implies \text{no build up at any point}$$

$$\nabla \times \mathbf{F} = (0, 0, -z) \implies \text{rotation in } x - y \text{ plane}$$

(3) $\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2}$. You can check that $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = \mathbf{0}$. Except there's a subtlety at $r = 0$ where $\mathbf{F}$ is singular. It turns out that

$$\nabla \times \mathbf{F} = \mathbf{0}$$

$$\nabla \cdot \mathbf{F} = 4\pi \delta^3(\mathbf{x})$$

Where

$$\delta^3(\mathbf{x}) = \delta(x)\delta(y)\delta(z)$$

(δ is the Dirac delta function).

Start of lecture 11

Note. Here are some useful tips when evaluating derivatives of radial fields. We use

$$r^2 = x^i x^i \implies 2r\frac{\partial r}{\partial x^i} = 2x^i$$
$$\implies \frac{\partial r}{\partial x^i} = \frac{x^i}{r}$$

Then we have

$$\begin{aligned}\nabla r^p &= \mathbf{e}_i \frac{\partial r^p}{\partial x^i} \\ &= pr^{p-1}\frac{x_i}{r}\mathbf{e}_i \\ &= pr^{p-1}\hat{\mathbf{r}}\end{aligned}$$

The vector $\mathbf{x} = x^i\mathbf{e}_i$ can also be written as $\mathbf{r} = r\hat{\mathbf{r}}$ to highlight that it points outwards. We have

$$\nabla \cdot \mathbf{r} = \frac{\partial x^i}{\partial x^i} = \delta_i i = n$$

(in $\mathbb{R}^n$).
Also in $\mathbb{R}^3$,

$$\nabla \times \mathbf{r} = \varepsilon_{ijk}\frac{\partial x^j}{\partial x^i}\mathbf{e}_k = 0$$

Some Basic Properties

For constant α, scalar fields ϕ and ψ, and vector fields $\mathbf{F}$ and $\mathbf{G}$, we have

$$\begin{aligned}\nabla(\alpha\phi + \psi) &= \alpha\nabla\phi + \nabla\psi \\ \nabla \cdot (\alpha\mathbf{F} + Gbg) &= \alpha\nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G} \\ \nabla \times (\alpha\mathbf{F} + \mathbf{G}) &= \alpha\nabla \times \mathbf{F} + \nabla \times \mathbf{G}\end{aligned}$$

This is the statement that ∇ is a *linear operator*.

∇ obeys a generalised product rule (known as Leibniz property):

$$\begin{aligned}\nabla(\phi\psi) &= \phi\nabla\psi + \psi\nabla\phi \\ \nabla \cdot (\phi\mathbf{F}) &= (\nabla\phi) \cdot \mathbf{F} + \phi(\nabla \cdot \mathbf{F}) \\ \nabla \times (\phi\mathbf{F}) &= (\nabla\phi) \times \mathbf{F} + \phi(\nabla \times \mathbf{F})\end{aligned}$$

The proofs of these follow from the definitions. For example

$$\begin{aligned}\nabla\cdot(\phi\mathbf{F}) &= \frac{\partial}{\partial x^i}(\phi F_i)\\ &= \frac{\partial\phi}{\partial x^i}F_i + \phi\frac{\partial F_i}{\partial x^i}\\ &= \nabla\phi\cdot\mathbf{F} + \phi\nabla\cdot\mathbf{F}\end{aligned}$$

There are some further properties

$$\begin{aligned}\nabla\cdot(\mathbf{F}\times\mathbf{G}) &= (\nabla\times\mathbf{F})\cdot\mathbf{G} - \mathbf{F}\cdot(\nabla\times\mathbf{G})\\ \nabla(\mathbf{F}\cdot\mathbf{G}) &= \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F}) + (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F}\\ \nabla\times(\mathbf{F}\times\mathbf{G}) &= (\nabla\cdot\mathbf{G})\mathbf{F} - (\nabla\cdot\mathbf{F})\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}\end{aligned}$$

All of these are proven using index notation. IN the last two identities, we have introduced the notation

$$\mathbf{F}\cdot\nabla = F_i\frac{\partial}{\partial x^i}$$

Definitions

- A vector field $\mathbf{F}$ is *conservative* if it can be written

  $$\mathbf{F} = \nabla\phi$$

  for some scalar ϕ.
- A vector field is called *irrotational* if

  $$\nabla\times\mathbf{F} = 0$$

- A vector field is *divergence free* or *solenoidal* if

  $$\nabla\cdot\mathbf{F} = 0$$

Theorem (A baby version of the Poincaré lemma)**.** For fields defined everywhere on $\mathbb{R}^3$, conservative $\iff$ irrotational, i.e.

$$\nabla\times\mathbf{F} = 0 \iff \mathbf{F} = \nabla\phi$$

(sketch). If $F_i = \frac{\partial\phi}{\partial x^i}$ then

$$(\nabla\times\mathbf{F})_k = \varepsilon_{ijk}\frac{\partial^2\phi}{\partial x^i\partial x^j} = 0$$

by symmetry. We will show $\nabla\times\mathbf{F} = 0 \implies \mathbf{F} = \nabla\phi$ when we prove Stokes' theorem in section 4. □

Theorem. For $\mathbf{F}$ defined everywhere on $\mathbb{R}^3$,

$$\nabla \cdot \mathbf{F} = 0 \iff \mathbf{F} = \nabla \times \mathbf{A}$$

for some vector field $\mathbf{A}$.

(sketch). If $F_i = \varepsilon_{ijk}\partial_j A_k$

$$\implies \nabla \cdot \mathbf{F} = \partial_i(\varepsilon_{ijk}\partial_j A_k) = 0$$

by symmetry. The other way is an optional question on Example Sheet 2. □

Definition. The *Laplacian* is a second order differential operator

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial\partial^2}{\partial\partial x^i \partial x^i}$$

for example in $\mathbb{R}^3$, we have

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Acting on a scalar field ϕ, ∇^2 gives back another scalar field $\nabla^2\phi$.
It acts component-wise on a vector field $\mathbf{F}$ to give another vector field $\nabla^2\mathbf{F}$.

Claim. $\nabla^2\mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F})$

Proof. Use triple product formula for $\nabla \times (\nabla \times \mathbf{F})$. □

The Laplacian occurs in many places in maths and physics. For example, the *heat equation*

$$\frac{\partial T}{\partial t} = D\nabla^2 T$$

and tells us how temperature $T(\mathbf{x}, t)$ evolve in time. (D is called the diffusion constant).

The linear operator ∇ also appears in many laws of physics. For example, the electric field $\mathbf{E}(\mathbf{x}, t)$ and magnetic field $\mathbf{B}(\mathbf{x}, t)$ are governed by the *Maxwell equations*

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0\frac{\partial \mathbf{E}}{\partial t}\right)$$

where $\rho(\mathbf{x}, t)$ is the electric charge density and$\mathbf{J}(\mathbf{x}, t)$ is the electric current density, and μ_0 and ε_0 are constants of nature.

Start of lecture 12

3.3 Orthogonal Curvilinear Coordinates

We want to find expressions for ∇ in different coordinates systems.

Introduce coordinates u, v, w so

$$\mathbf{x} = \mathbf{x}(u, v, w)$$

A change of (u, v, w) changes the point $\mathbf{x}$ to $\mathbf{x} + \mathrm{d}\mathbf{x}$

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial u}\mathrm{d}u + \frac{\partial \mathbf{x}}{\partial v}\mathrm{d}v + \frac{\partial \mathbf{x}}{\partial w}\mathrm{d}w$$

(where $\frac{\partial \mathbf{x}}{\partial u}\mathrm{d}u$ is the tangent vector to $v, w = \text{constant}$) These are good coordinates at a point provided

$$\frac{\partial \mathbf{x}}{\partial u} \cdot \left(\frac{\partial \mathbf{x}}{\partial v} \times \frac{\partial \mathbf{x}}{\partial w} \right) \neq 0$$

If the three tangent vectors are mutually orthogonal then (u, v, w) are said to be *orthogonal curvilinear.*

For such coordinates, we introduce normalised tangent vectors, i.e.

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial u} &= h_u \mathbf{e}_u \\ \frac{\partial \mathbf{x}}{\partial v} &= h_v \mathbf{e}_v \\ \frac{\partial \mathbf{x}}{\partial w} &= h_w \mathbf{e}_w \end{aligned}$$

with $h_u, h_v, h_w > 0$ and $\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w$ are a right-handed orthonormal basis

$$\mathbf{e}_u \times \mathbf{e}_v = \mathbf{e}_w$$

so

$$\begin{aligned} \mathrm{d}\mathbf{x} &= h_u \mathbf{e}_u \mathrm{d}u + h_v \mathbf{e}_v \mathrm{d}v + h_w \mathbf{e}_w \mathrm{d}w \\ \implies \mathrm{d}\mathbf{x}^2 &= h_u^2 \mathrm{d}u^2 + h_v^2 \mathrm{d}v^2 + h_w^2 \mathrm{d}w^2 \end{aligned}$$

(scale factors tell us the change in length)

Examples

(1) Cartesian coordinates $\mathbf{x} = (x, y, z)$ with $h_y = h_v = h_w = 1$ and

$$\mathbf{e}_x = \hat{\mathbf{x}}, \qquad \mathbf{e}_y = \hat{\mathbf{y}}, \qquad \mathbf{e}_z = \hat{\mathbf{z}}$$

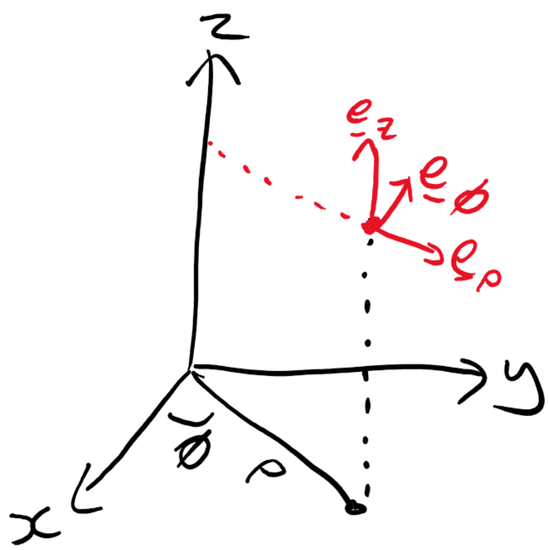
(2) Cylindrical polar coordinates have

$$\mathbf{x} = (\rho\cos\phi, \rho\sin\phi, z)$$

($\rho \geq 0$, $\phi \in [0, 2\pi)$) or $\rho = \sqrt{x^2 + y^2}$ and $\tan\phi = \frac{y}{x}$.

$$\begin{aligned}
\mathbf{e}_\rho &= \hat{\boldsymbol{\rho}} = (\cos\phi, \sin\phi, 0)\\
\mathbf{e}_\phi &= \hat{\boldsymbol{\phi}} = (-\sin\phi, \cos\phi, 0)\\
\mathbf{e}_z &= \hat{\mathbf{z}} = (0, 0, 1)
\end{aligned}$$

and $h_\rho = h_z = 1$ and $h_\phi = \rho$.



(3) Spherical polar coordinates have

$$\mathbf{x} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$$

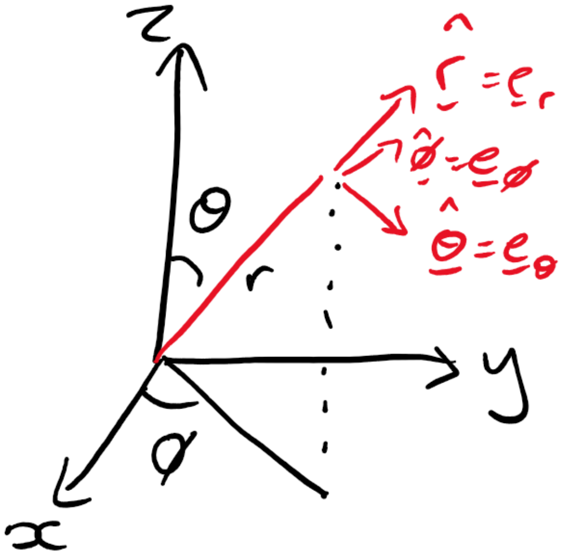
with $r \geq 0$, $\theta \in [0, \pi]$, $\theta \in [0, 2\pi)$.

$$\implies r = \sqrt{x^2 + y^2 + z^2}, \qquad \tan\theta = \frac{\sqrt{x^2 + y^2}}{z}, \qquad \tan\phi = \frac{y}{x}$$

We have

$$\begin{aligned}
\mathbf{e}_r &= \hat{\mathbf{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\\
\mathbf{e}_\theta &= \hat{\boldsymbol{\theta}} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)\\
\mathbf{e}_\phi &= \hat{\boldsymbol{\phi}} = (-\sin\phi, \cos\phi, 0)
\end{aligned}$$

with $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$.



Grad

If we shift $\mathbf{x} \to \mathbf{x} + \mathrm{d}\mathbf{x}$ then a scalar field $f(\mathbf{x})$ changes as

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\mathbf{x}$$

In a general coordinate system

$$\begin{aligned} \mathrm{d}f &= \frac{\partial f}{\partial u}\mathrm{d}u + \frac{\partial f}{\partial v}\mathrm{d}v + \frac{\partial f}{\partial w}\mathrm{d}w \\ &= \nabla f \cdot (h_u \mathbf{e}_u \mathrm{d}u + h_v \mathbf{e}_v \mathrm{d}v + h_w \mathbf{e}_w \mathrm{d}w) \\ \implies \nabla f &= \frac{1}{h_u}\frac{\partial f}{\partial u}\mathbf{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\mathbf{e}_v + \frac{\partial 1}{\partial h_w}\frac{\partial f}{\partial w}\mathbf{e}_w \end{aligned}$$

using $\mathbf{e}_u \cdot \mathbf{e}_v = 0$, etc.
For example in cylindrical polar

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

In spherical polar

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Div and Curl

In general coordinates we have

$$\nabla = \frac{1}{h_u}\mathbf{e}_u\frac{\partial}{\partial u} + \frac{1}{h_v}\mathbf{e}_v\frac{\partial}{\partial v} + \frac{1}{h_w}\mathbf{e}_w\frac{\partial}{\partial w}$$

This now acts on vector fields

$$\mathbf{F} = F_u \mathbf{e}_u + F_v \mathbf{e}_v + F_w \mathbf{e}_w$$

But now $\{\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w\}$ depend on (u, v, w) and are hit by derivatives $\implies$ a little messy.

Claim.

$$\nabla \cdot \mathbf{F} = \frac{1}{h_u h_v h_w} \left(\frac{\partial}{\partial u}(h_v h_w F_u) + \frac{\partial}{\partial v}(h_u h_w F_v) + \frac{\partial}{\partial w}(h_u h_v F_w) \right)$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} h_u \mathbf{e}_u & h_v \mathbf{e}_v & h_w \mathbf{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix} \times \frac{1}{h_u h_v h_w}$$

Proof. Nope! □

For cylindrical polar

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho} \frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

$$\nabla \times \mathbf{F} = \left(\frac{1}{\rho} \frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z} \right) \hat{\boldsymbol{\rho}} + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho} \right) \hat{\boldsymbol{\phi}} + \frac{1}{\rho} \left(\frac{\partial (p F_\phi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi} \right) \hat{\mathbf{z}}$$

$$\begin{aligned} \nabla^2 f &= \nabla \cdot \nabla f \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \end{aligned}$$

For spherical polar

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial (r^2 F_r)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial (\sin\theta F_\theta)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial F_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{F} = \frac{1}{r \sin\theta} \left(\frac{\partial (\sin\theta F_\phi)}{\partial \theta} - \frac{\partial F_\theta}{\partial \phi} \right) \hat{\mathbf{r}} + \frac{1}{r} \left(\frac{1}{\sin\theta} \frac{\partial F_r}{\partial \phi} - \frac{\partial (r F_\phi)}{\partial r} \right) \hat{\boldsymbol{\theta}} + \frac{1}{r} \left(\frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \hat{\boldsymbol{\theta}}$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2}$$

Start of lecture 13

4 Integral Theorems

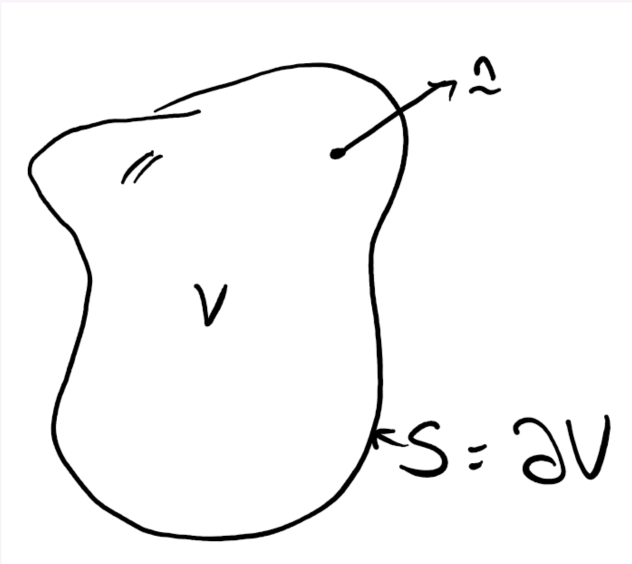
4.1 The Divergence Theorem

(Also known as Gauss' theorem).

Theorem (Gauss' Theorem)**.** Given a smooth vector field $\mathbf{F}(\mathbf{x})$ over $\mathbb{R}^3$

$$\int_V \nabla \cdot \mathbf{F} \mathrm{d}V = \int_S \mathbf{F} \cdot \mathbf{d}\boldsymbol{S}$$

with $S = \partial V$ and $\mathbf{d}\boldsymbol{S} = \mathrm{d}S\mathbf{n}$ is pointing outwards.



Note. The divergence theorem gives intuition for the meaning of $\nabla \cdot \mathbf{F}$.

In a suitably small volume, over which $\nabla \cdot \mathbf{F} \approx$ constant,

$$\int_V \nabla \cdot \mathbf{F} \mathrm{d}V = V \nabla \cdot \mathbf{F}(\mathbf{x})$$

$$\implies \nabla \cdot \mathbf{F} = \lim_{V \to 0} \frac{1}{V} \int_{\partial V} \mathbf{F} \cdot \mathbf{d}\boldsymbol{S}$$

$$\implies \text{divergence} \ = \ \text{net flow into/out of region } V$$

$$\nabla \cdot \mathbf{F} > 0 \implies \text{net flow out}$$

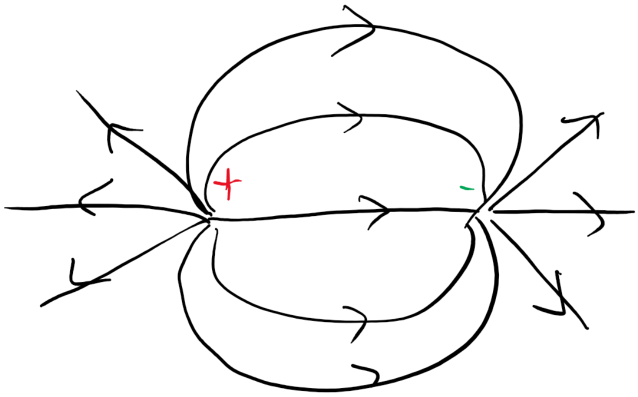
$$\nabla \cdot \mathbf{F} < 0 \implies \text{net flow int}$$

For example, MAxwell's equations tell us

$$\nabla \cdot \mathbf{B} = 0 \implies \text{magnetic field lines are continuous}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \implies \text{electric field lines are continuous when electric charge density } \rho(\mathbf{x}) = 0$$

But when $\rho(\mathbf{x}) \neq 0$, the electric field can begin and end.

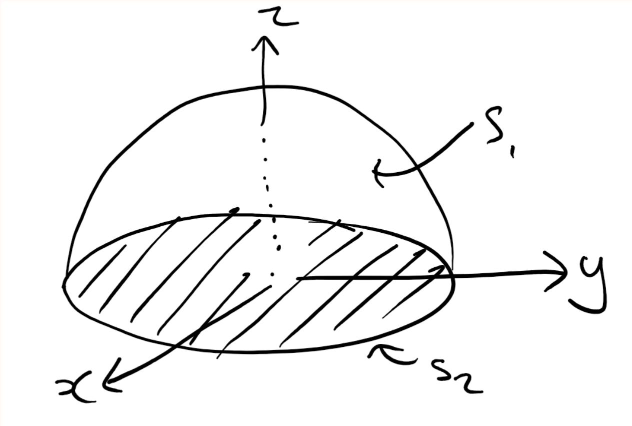


Example. Let $V =$ solid hemisphere.

$$x^2 + y^2 + z^2 \leq R^2$$

and $z \geq 0$.

$$\partial V = S_1 + S_2$$



We'll integrate $\mathbf{F} = (0, 0, z + R)$

$$\implies \nabla \cdot \mathbf{F} = 1$$

$$\implies \int_V \nabla \cdot \mathbf{F} \mathrm{d}V = \int_V \mathrm{d}V = \frac{2}{3}\pi R^3$$

On S^1:

$$\mathbf{n} = \frac{1}{R}(x, y, z)$$

$$\implies \mathbf{F} \cdot \mathbf{n} = \frac{z(z + R)}{R} = R\cos\theta(\cos\theta + 1)$$

(where $z = R\cos\theta$)

$$\implies \int_{S_1} \mathbf{F} \cdot \mathbf{d}\boldsymbol{S} = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi/2} \mathrm{d}\theta(R^2 \sin\theta) \times R\cos\theta(\cos\theta + 1) = \frac{5}{3}\pi R^3$$

On S_2,

$$\mathbf{n} = (0, 0, -1)$$

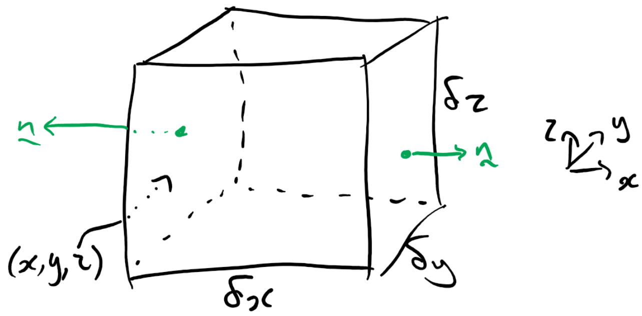
$$\implies \mathbf{F} \cdot \mathbf{n} = -R$$

on S_2

$$\implies \int_{S_2} \mathbf{F} \cdot \mathbf{d}\boldsymbol{S} = (-R) \times \pi R^2 = -\pi R^3$$

$$\implies \int_{S_1 + S_2} \mathbf{F} \cdot \mathbf{d}\boldsymbol{S} = \frac{2}{3}\pi R^3$$

of Divergence Theorem. First, a simple proof. Divide V into cubes



Flow of $\mathbf{F}$ through the (y, z) plane is roughly

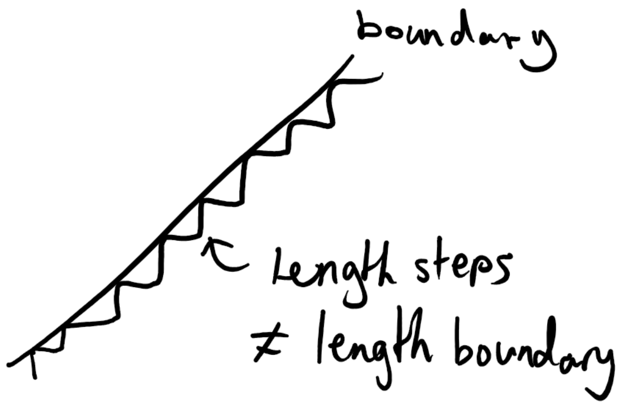
$$[F_x(x+\delta x, y, z) - F_x(x, y, z)]\delta y \delta z \approx \frac{\partial F_x}{\partial x}\delta x \delta y \delta z$$

Do the same for the other sides

$$\implies \int_S \mathbf{F} \cdot \mathbf{d}\boldsymbol{S} = \int_V \nabla \cdot \mathbf{F}\mathbf{d}\boldsymbol{V}$$

□

A concern: can we approximate the boundary with cubes? For example in 2d:



(better proof). First note that

$$\int_V \nabla \cdot \mathbf{F} \mathrm{d}V = \int_\partial V\mathbf{F} \cdot \mathbf{d}\boldsymbol{S}$$

holds in and $\mathbb{R}^n$. We start by proving the following:

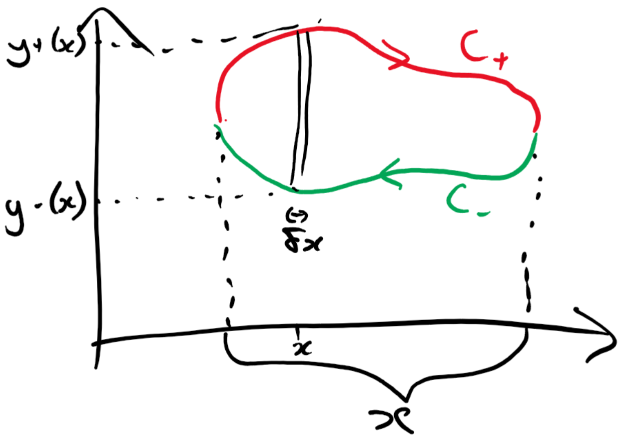
Lemma. 2d divergence theorem:

$$\int_D \nabla \cdot \mathbf{F} \mathrm{d}A = \int_C \mathbf{F} \cdot \mathbf{n} \mathrm{d}\boldsymbol{S}$$

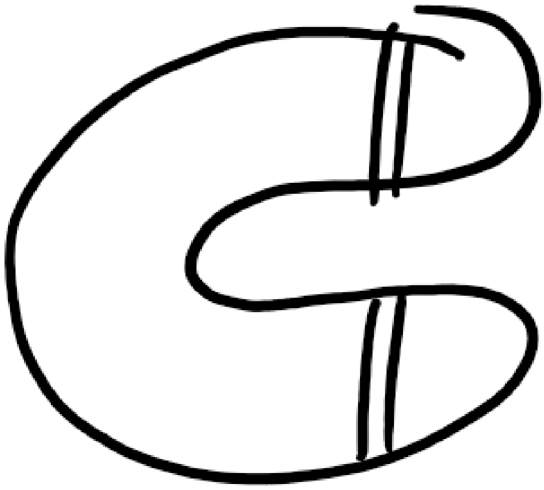
with $C = \partial D$.

of lemma.

$$\int_D \nabla \cdot \mathbf{F} \mathrm{d}A = \int_X \mathrm{d}x \int_{y_-(x)}^{y_+(x)} \mathrm{d}y \frac{\partial F}{\partial y}$$



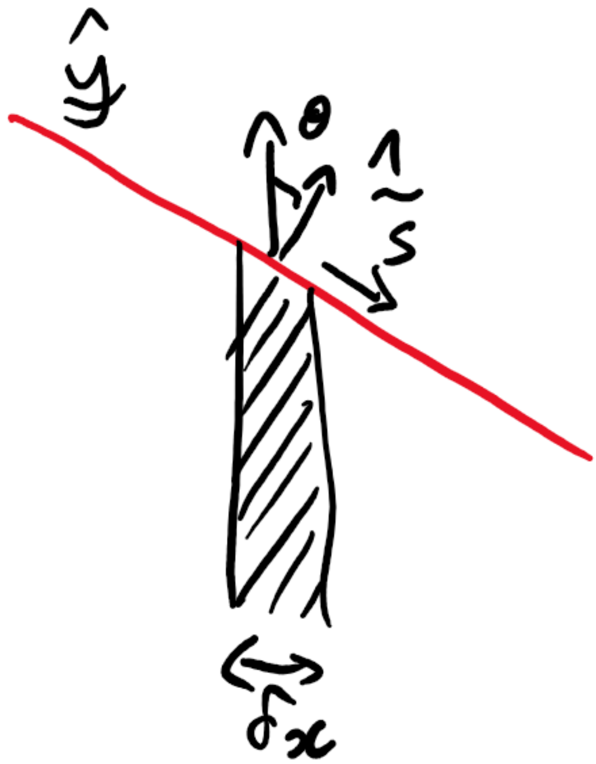
For now assume D is convex so that the $\int \mathrm{d}y$ is over a single interval, rather than



Then

$$\int_D \nabla \cdot \mathbf{F} \mathrm{d}A = \int_X \mathrm{d}x(F(x, y_+(x)) - F(x, y_-(x)))$$

Next we change $\int \mathrm{d}x$ to $\int \ dds$, where $s =$ arc length. Zoom in to the top curve C_+



The normal makes an angle

$$\cos\theta = \hat{\mathbf{y}}\cdot\mathbf{n}$$

and

$$\delta x = \cos\theta\delta s = \hat{\mathbf{y}}\cdot\mathbf{n}\delta s$$

along C_+. Similarly, along C_-,

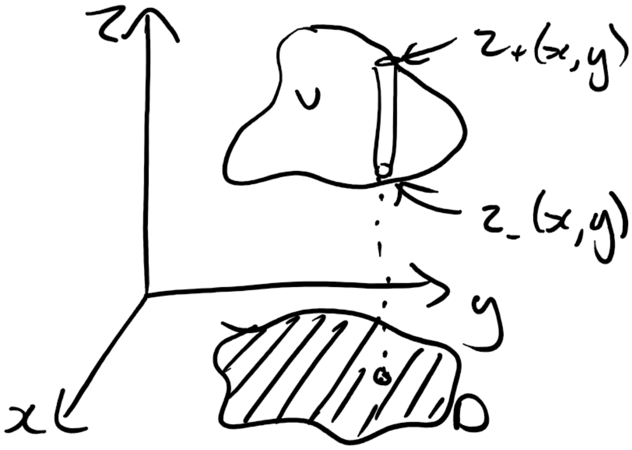
$$\delta x = \hat{\mathbf{y}}\cdot\mathbf{n}\delta s$$

$$\begin{aligned}\int_D \nabla\cdot\mathbf{F}\mathrm{d}A &= \int_X \mathrm{d}s\,(\mathbf{n}\cdot\mathbf{F}(x,y_+(x)) + \mathbf{n}\cdot\mathbf{F}(x,y_-(x)))) \\ &= \int_{C_+}\mathbf{F}\cdot\mathbf{n}\mathrm{d}s + \int_{C_-}\mathbf{F}\cdot\mathbf{n}\mathrm{d}s \\ &= \int_C \mathbf{F}\cdot\mathbf{n}\mathrm{d}s\end{aligned}$$

with $C = C_+ + C_-$. Finally if D is not convex, then just decompose C into more pieces. □

Back to 3D theorem.
We use the same strategy. Take $\mathbf{F} = F(\mathbf{x})\hat{\mathbf{z}}$. Then

$$\int_V \nabla\cdot\mathbf{F}\mathrm{d}V = \int_D \mathrm{d}A\int_{z_-(x,y)}^{z_+(x,y)}\mathrm{d}z\frac{\partial F}{\partial z}$$



$$\int_V \nabla \cdot \mathbf{F} \mathrm{d}V = \int_D \mathrm{d}A[F(x,y,z_+(c,y)) - F(x,y,z_-(x,y))]$$

with limits $z_\pm$ the upper / lower surface of V.
Now convert $\int \mathrm{d}A$ into the surface integral over $S = \partial V$. This again includes an angle $\cos\theta = \pm \mathbf{n} \cdot \hat{\mathbf{z}}$ with $\mathbf{n}$ normal to S. This gives the result. □

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Corollary. For a scalar field ϕ

$$\int_V \nabla\phi \mathrm{d}V = \int_{\partial V} \phi \mathbf{d}\boldsymbol{s}$$

Proof. Use divergence theorem with

$$\mathbf{F} = \phi\mathbf{a}$$

with $\mathbf{a}$ a constant.

$$\int_V \nabla \cdot (\phi\mathbf{a}) \mathrm{d}V = \int_{\partial V} \phi\mathbf{a} \cdot \mathbf{d}\boldsymbol{s}$$

$$\implies \mathbf{a} \cdot \left[\int_V \nabla\phi \mathrm{d}V - \int_{\partial V} \phi \mathbf{d}\boldsymbol{s}\right] = 0$$

But true $\forall$ $\mathbf{a}$ implies the result. □

4.2 An Application: Conservation Laws

Many things are conserved, for example energy, momentum, angular momentum, electric charge. Importantly, all of these are conserved *locally.* This means that stuff moves continuously to nearby points.

Let $\rho(\mathbf{x}, t)$ be the density of the conserved object, for example electric charge. Then

$$Q = \int_V \rho \mathrm{d}V$$

is the charge in a region V. Conservation of Q means that there exists a vector $\mathbf{J}(\mathbf{x}, t)$, known as a *current density* such that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

This is the *continuity equation.*

The change of Q in some fixed region V is

$$\begin{aligned} \frac{\mathrm{d}Q}{\mathrm{d}t} &= \int_V \frac{\partial \rho}{\partial t} \mathrm{d}V \\ &= -\int_V \nabla \cdot \mathbf{J} \mathrm{d}V \\ &= -\int_S \mathbf{J} \cdot \mathbf{d}\boldsymbol{s} \end{aligned}$$

($\mathbf{J}$ is current flowing in/out of V). If $\mathbf{J}(\mathbf{x}) = 0$ on S then $\dot{Q} = 0$. Often consider $V = \mathbb{R}^3$ and $\dot{Q} = 0$ provided that $J(\mathbf{x}) \to 0$ suitably quickly as $|\mathbf{x}| \to \infty$.

Example. A fluid has mass density $\rho(\mathbf{x}, t)$ and mass current $\mathbf{J} = \rho \mathbf{u}$ with $\mathbf{u}(\mathbf{x}, t)$ the velocity field.
Mass is conserved

$$\implies \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

But many fluids are incompressible with $\rho =$ constant. Then continuity equation gives

$$\nabla \cdot \mathbf{u} = 0$$

Example (Diffusion)**.** Consider a gas with energy density $\varepsilon(\mathbf{x}, t)$. Energy is conserved

$$\implies \frac{\partial \varepsilon}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

($\mathbf{J}$ is *heat current*)
Fact 1:

$$\varepsilon(\mathbf{x}, t) = c_v T(\mathbf{x}, t)$$

where c_v is heat capacity and T is temperature.
Fact 2: (Fick's law) Heat current is due to temperature differences

$$\mathbf{J} = -\kappa \nabla T$$

(κ is thermal conductivity)

$$\implies \frac{\partial T}{\partial t} = D \nabla^2 T$$

i.e. the heat equation with $D = \frac{\kappa}{c_v}$.

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4.3 Green's Theorem in the Plane

Theorem (Green's Theorem)**.** Let $P(x, y)$ and $Q(x, y)$ be smooth functions on $\mathbb{R}^2$. Then

$$\int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A = \oint_C P \mathrm{d}x + Q \mathrm{d}y$$

with $C = \partial A$ traversed anti-clockwise.

Proof. Let $\mathbf{F} = (Q, -P)$ so

$$\int_A \nabla \cdot \mathbf{F} \mathrm{d}A = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A = \oint_C \mathbf{F} \cdot \mathbf{n} \mathrm{d}s$$

by 2D divergence theorem.
Parametrise C by

$$\mathbf{x}(s) = (x(s), y(s))$$

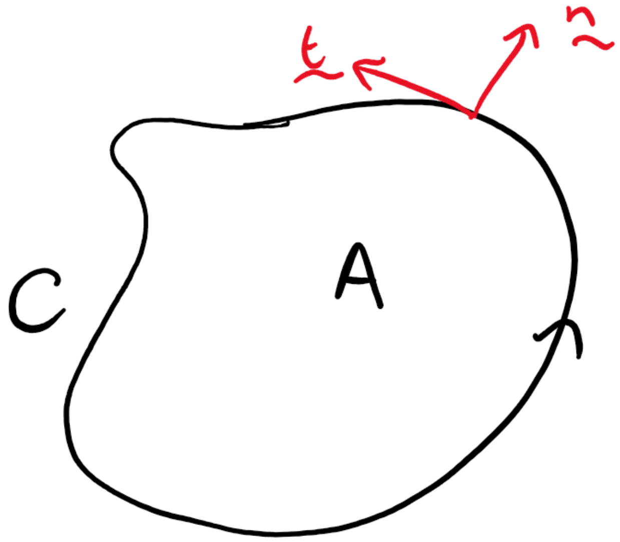
so the tangent vector is

$$\mathbf{t} = (x'(s), y'(s))$$

and the normal is

$$\mathbf{n} = (y'(s), -x'(s))$$

so that $\mathbf{n} \cdot \mathbf{t} = 0$.



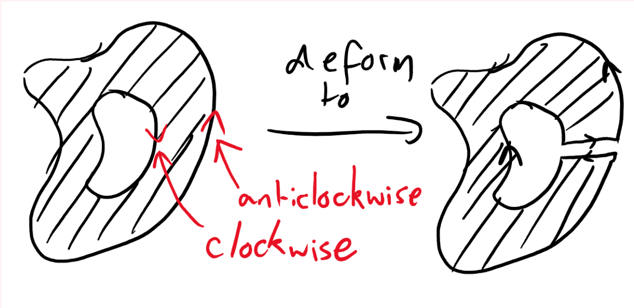
If s increases in an anti-clockwise direction, then $\mathbf{n}$ is outward pointing normal.

$$\mathbf{F} \cdot \mathbf{n} = Q\frac{\mathrm{d}y}{\mathrm{d}s} + P\frac{\mathrm{d}x}{\mathrm{d}s}$$

$$\implies \oint_C \mathbf{F} \cdot \mathbf{n} \mathrm{d}s = \oint_C P\mathrm{d}x + Q\mathrm{d}y$$

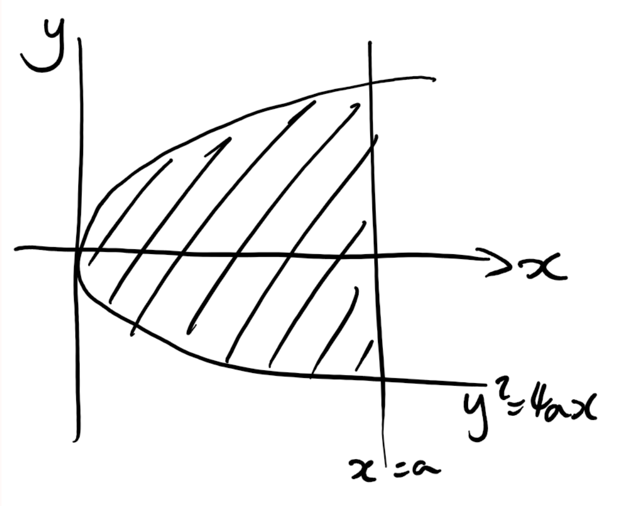
$\square$

Note. We can also use this theorem when $C = \partial A$ has disconnected components.



The two line integrals across the gap cancel.

Example. $P = x^2y$ and $Q = xy^2$. Integrate over



Claim.

$$\int_A (y^2 - x^2)\mathrm{d}A = \oint_C x^2y\mathrm{d}x + xy^2\mathrm{d}y$$

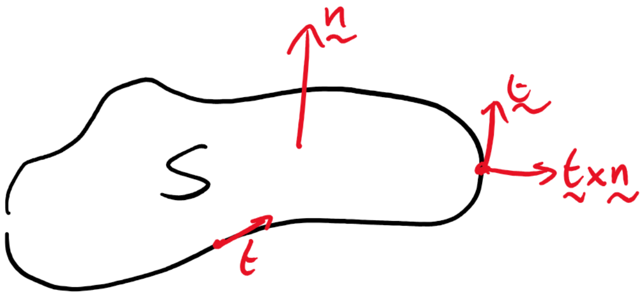
Proof. This is Example Sheet 1, Question 9 (both sides are equal to $\frac{104}{105}a^4$). ☐

4.4 Stokes Theorem

Theorem (Stokes Theorem)**.** Let S be a smooth surface in $\mathbb{R}^3$ with boundary $C = \partial S$. Then for a smooth vector field $\mathbf{F}(\mathbf{x})$

$$\int_S \nabla \times \mathbf{F} \cdot \mathbf{d}\boldsymbol{s} = \oint_C \mathbf{F} \cdot \mathbf{d}\boldsymbol{x}$$

To fix the orientation, if $\mathbf{n}$ is a normal to S and $\mathbf{t}$ is tangent to C then $\mathbf{t} \times \mathbf{n}$ should point out of S.



If **n** points towards you, then orientation of C is anti-clockwise.

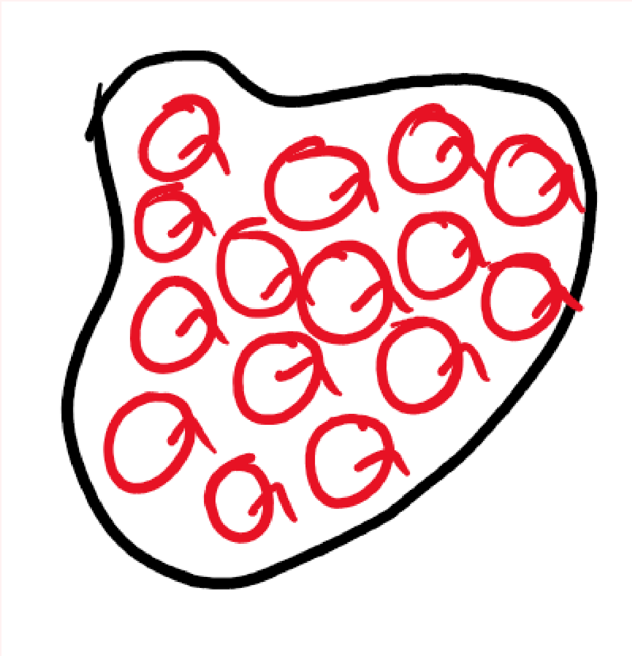
Note. Stokes theorem gives a meaning to curl. For suitably small S, $\nabla \times \mathbf{F} \approx$ constant and

$$\int_S \nabla \times \mathbf{F} \cdot \mathbf{d}\boldsymbol{s} \approx A\mathbf{n} \cdot (\nabla \times \mathbf{F})$$

where A is the area of S and $\mathbf{n}$ is the normal to S.

$$\implies \mathbf{n} \cdot (\nabla \times \mathbf{F}) = \lim_{A \to 0} \frac{1}{A} \int_C \mathbf{F} \cdot \mathbf{d}\boldsymbol{x}$$

i.e. curl in direction $\mathbf{n}$ is circulation in plane normal to $\mathbf{n}$. This also gives some intuition for Stokes' theorem:



At each point in S, $\nabla \times \mathbf{F}$ is circulation. But this cancels in the interior, leaving only the boundary contribution.

Corollary. Irrotational $\implies$ conservative:

$$\nabla \cdot \mathbf{F} = 0 \implies \oint_C \mathbf{F} \cdot \mathbf{d}\boldsymbol{x} = 0 \qquad \forall \text{ closed } C$$

$$\implies \mathbf{F} = \nabla \phi$$

from Section 1.

Example. S is a hemispherical cap of radius R with $0 \leq \theta \leq \alpha$.

$$\mathbf{F} = (0, xz, 0)$$
$$\implies \nabla \times \mathbf{F} = (-x, 0, z)$$

and

$$\int_S \nabla \times \mathbf{F} \cdot \mathbf{d}\boldsymbol{s} = \pi R^3 \cos\alpha \sin^2\alpha$$

from Section 2.
For the line integral, let

$$\mathbf{x}(\phi) = R(\sin\alpha\cos\phi, \sin\alpha\sin\phi, \cos\alpha)$$

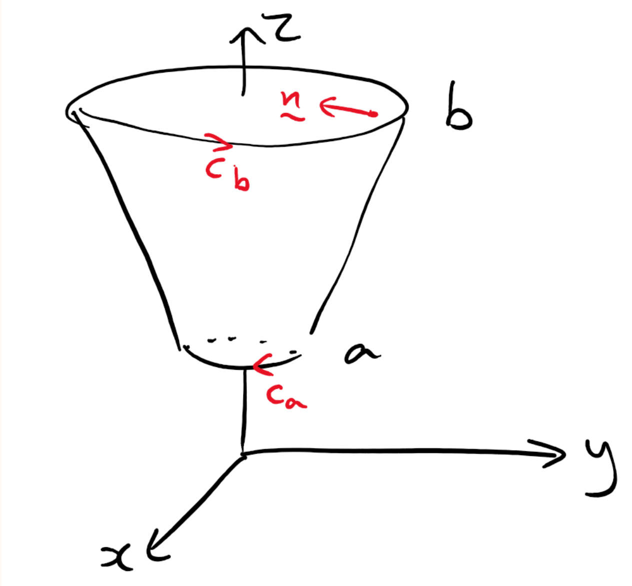
where $\phi \in [0, 2\pi)$.

$$\implies \mathbf{d}\boldsymbol{x} = R(-\sin\alpha\sin\phi, \sin\alpha\cos\phi, 0)\mathrm{d}\phi$$

and

$$\begin{aligned}\oint_C \mathbf{F} \cdot \mathbf{d}\boldsymbol{x} &= \int_0^{2\pi} \mathrm{d}\phi R x z \sin\alpha\cos\phi \\ &= \int_0^{2\pi} \mathrm{d}\phi R^3 \sin^2\alpha\cos\alpha\cos^2\phi \\ &= \pi R^3 \sin^2\alpha\cos\alpha\end{aligned}$$

Example. Cone $z^2 = x^2 + y^2$ and $a \le z \le b$, $a, b > 0$.



surface is

$$\mathbf{x}(\rho, \phi) = (\rho\cos\phi, \rho\sin\phi, \rho)$$

where $0 \le \phi < 2\pi$ and $a \le \rho \le b$. Tangent vectors:

$$\frac{\partial \mathbf{x}}{\partial \rho} = (\cos\phi, \sin\phi, 1)$$

$$\frac{\partial \mathbf{x}}{\partial \phi} = \rho(-\sin\phi, \cos\phi, 0)$$

$$\implies \mathbf{n} = \frac{\partial \mathbf{x}}{\partial \rho} \times \frac{\partial \mathbf{x}}{\partial \phi} = (-\rho\cos\phi, -\rho\sin\phi, \rho)$$

(points inwards).

$$\implies \mathbf{d}\boldsymbol{s} = \rho(-\cos\phi, -\sin\phi, 1)\mathrm{d}\rho\mathrm{d}\phi$$

Again integrate $\mathbf{F} = (0, xz, 0)$

$$\begin{aligned} \implies \nabla \times \mathbf{F} \cdot \mathbf{d}\boldsymbol{s} &= (x\cos\phi + z)\rho\mathrm{d}\rho\mathrm{d}\phi \\ &= \rho^2(1 + \cos^2\phi)\mathrm{d}\rho\mathrm{d}\phi \\ \implies \int_S \nabla \times \mathbf{F} \cdot \mathbf{d}\boldsymbol{s} &= \int_a^b \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi \rho^2(1 + \cos^2\phi) \\ &= \pi(b^3 - a^3) \end{aligned}$$

Compare to line integral over $\partial S = C_b - C_a$.

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of Stokes' Theorem. First show that Stokes $\implies$ Green. Consider

$$\mathbf{F} = (P(x,y), Q(x,y), 0)$$

Take a flat surface S in $z = 0$ plane. Then

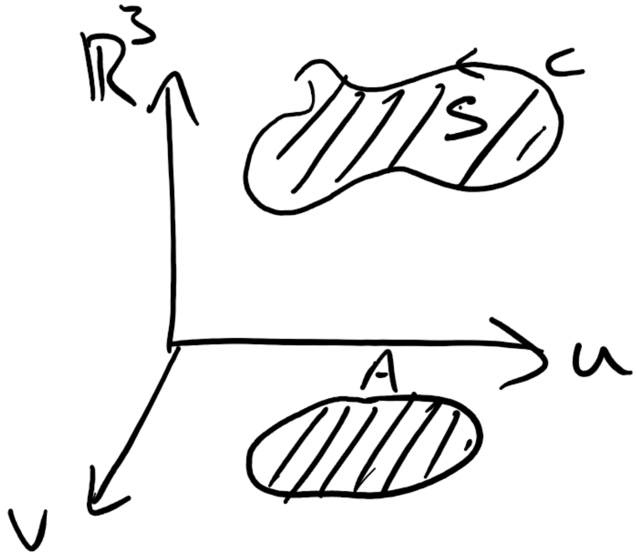
$$\int_S \nabla \times \mathbf{F} \cdot \mathbf{d}\boldsymbol{s} = \int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}s$$

and

$$\int_C \mathbf{F} \cdot \mathbf{d}\boldsymbol{x} = \int_C P\mathrm{d}x + Q\mathrm{d}y$$

This is Green's theorem in the plane.

Now we show that Green $\implies$ Stokes. Let $\mathbf{x}(u,v)$ be the parametrised surface S and $\mathbf{x}(u(t), v(t))$ be the parametrise boundary $C = \partial S$.



Let A be the associated area in (u,v) plane and ∂A the boundary. Now

$$\begin{aligned}\oint_C \mathbf{F} \cdot \mathbf{d}\boldsymbol{x} &= \int_C \mathbf{F} \cdot \left(\frac{\partial \mathbf{x}}{\partial u} \mathrm{d}u + \frac{\partial \mathbf{x}}{\partial v} \mathrm{d}v \right) \\ &= \int_{\partial A} F_u \mathrm{d}u + F_v \mathrm{d}v\end{aligned}$$

with $F_u = \mathbf{F} \cdot \frac{\partial \mathbf{x}}{\partial u}$ and $F_v = \mathbf{F} \cdot \frac{\partial \mathbf{x}}{\partial v}$.

$$\implies \oint_C \mathbf{F} \cdot \mathbf{d}\boldsymbol{x} = \int_A \left(\frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v} \right) \mathrm{d}A$$

by Green's Theorem. Now

$$\begin{aligned}
\frac{\partial F_v}{\partial u} &= \frac{\partial}{\partial u}\left(\mathbf{F}\cdot\frac{\partial \mathbf{x}}{\partial v}\right)\\
&= \frac{\partial}{\partial u}\left(F_i\frac{\partial x^i}{\partial v}\right)\\
&= \frac{\partial F_i}{\partial x^j}\frac{\partial x^j}{\partial u}\frac{\partial x^i}{\partial v} + F_i\frac{\partial^2 x^i}{\partial u\partial v}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial F_u}{\partial v} &= \frac{\partial F_i}{\partial x^j}\frac{\partial x^j}{\partial v}\frac{\partial x^i}{\partial u} + F_i\frac{\partial^2 x^i}{\partial u\partial v}\\
\implies \frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v} &= \frac{\partial x^j}{\partial u}\frac{\partial x^i}{\partial v}\left(\frac{\partial F_i}{\partial x^j} - \frac{\partial F_j}{\partial x^i}\right)\\
&= (\delta_{jk}\delta_{il} - \delta_{jl}\delta_{ik})\frac{\partial x^k}{\partial u}\frac{\partial x^l}{\partial v}\frac{\partial F_i}{\partial x^j}\\
&= \varepsilon_{jip}\varepsilon_{pkl}\frac{\partial x^k}{\partial u}\frac{\partial x^l}{\partial v}\frac{\partial F_i}{\partial x^j}\\
&= (\nabla\times\mathbf{F})\cdot\left(\frac{\partial \mathbf{x}}{\partial u}\times\frac{\partial \mathbf{x}}{\partial v}\right)\\
\implies \oint_C \mathbf{F}\cdot\mathbf{d}\boldsymbol{x} &= \int_A (\nabla\times\mathbf{F})\cdot\left(\frac{\partial \mathbf{x}}{\partial u}\times\frac{\partial \mathbf{x}}{\partial v}\right)\mathrm{d}u\mathrm{d}v\\
&= \int_S (\nabla\times\mathbf{F})\cdot\mathbf{d}\boldsymbol{s}
\end{aligned}$$

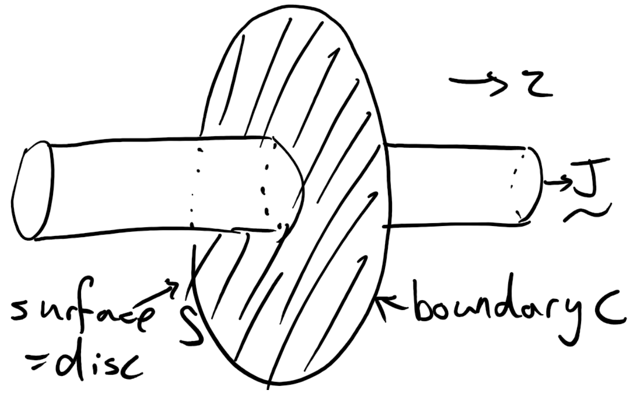
□

An Application: Magnetic Fields

One of the Maxwell equations reads

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}$$

where **B** is magnetic field and **J** is current density. This is known as Ampère's law.



$$
\begin{aligned}
\int_S \nabla \times \mathbf{B} \cdot \mathbf{d}\boldsymbol{s} &= \int_C \mathbf{B} \cdot \mathbf{d}\boldsymbol{x} \\
&= \int_S \mu_0 \mathbf{J} \cdot \mathbf{d}\boldsymbol{s} \\
&= \mu_0 I
\end{aligned}
$$

where I is total current. Parametrise $\mathbf{x} = \rho(\cos\phi, \sin\phi, 0)$ (ρ is radius of S, $\phi \in [0, 2\pi)$).

$$\mathbf{t} = \frac{\partial \mathbf{x}}{\partial \phi} = \rho(-\sin\phi, \cos\phi, 0)$$

Ansatz: $\mathbf{B}$ parallel to $\mathbf{t}$ everywhere, i.e. $\mathbf{B} = b(\rho)(-\sin\phi, \cos\phi, 0)$

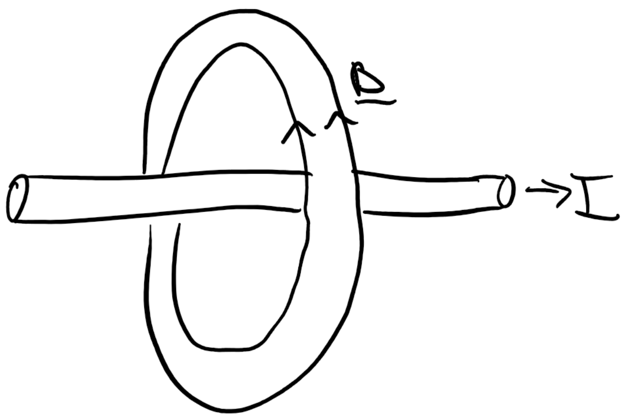
$$\implies \mathbf{B} \cdot \mathbf{t} = \rho b(\rho)$$

and Maxwell tells us that

$$\oint_C \mathbf{B} \cdot \mathbf{d}\boldsymbol{x} \int_0^{2\pi} \mathrm{d}\phi \rho b(\rho) = 2pi\rho b(\rho) = \mu_0 I$$

$$\implies b(\rho) = \frac{\mu_0 I}{2\pi\rho}$$

$$\implies B(\mathbf{x}) = \frac{\mu_0 I}{2\pi\rho}(-\sin\phi, \cos\phi, 0)$$



This is the magnetic field outside a current carrying wire.

5 Vector Calculus Equations

5.1 Gravity and Electrostatics

Two particles with mass m, M and charge q, Q, separated by a distance r, experience

- Newton's force: $\mathbf{F}(r) = -\frac{GMm}{r^2}\hat{\mathbf{r}}$
- Coulomb force: $\mathbf{F}(r) = \frac{Qq}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$

It's useful to think of one particle with mass m, charge q, moving in the background of the other.

Physically sensible if $m \ll M$, $q \ll Q$, write the force as

$$\mathbf{F}(\mathbf{x}) = m\mathbf{g}(\mathbf{x}) \qquad \text{and} \qquad \mathbf{F}(\mathbf{x}) = q\mathbf{E}(\mathbf{x})$$

where $\mathbf{g}$ and $\mathbf{E}$ are gravitational field and electric field respectively.

$$\mathbf{g}(\mathbf{x}) = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

$$\mathbf{E}(\mathbf{x}) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

These fields obey following equations:

$$\int_S \mathbf{g} \cdot \mathbf{d}\boldsymbol{s} = -4\pi GM$$

(M is the mass inside S)

$$\int_S \mathbf{E} \cdot \mathbf{d}\boldsymbol{s} = \frac{Q}{\varepsilon_0}$$

(Q is the charge inside S).

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Gauss' Law

Integrate these vector fields over a sphere S of radius r

$$\int_S \mathbf{g} \cdot \mathbf{d}\boldsymbol{s} = -4\pi GM$$

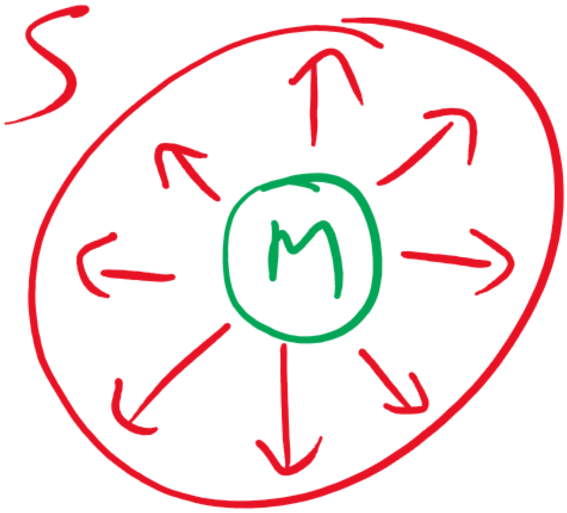
and

$$\int_S \mathbf{E} \cdot \mathbf{d}\boldsymbol{s} = \frac{Q}{\varepsilon_0}$$

Note. The result is independent of r! The flux of the field tells us the total mass / charge inside the sphere.

This is Gauss' law (in integrated form).

In fact, Gauss' law is equivalent to the force laws. Consider a sphere of radius R and mass M, with some spherically symmetric mass distribution.



$$\text{symmetry} \implies \mathbf{g}(\mathbf{x}) = g(f)\hat{\mathbf{r}}$$

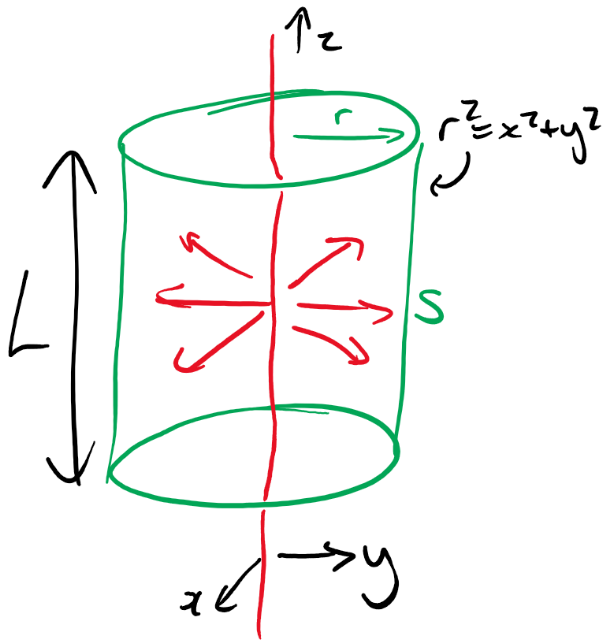
Consider a surface S which is a sphere of radius $r > R$.

$$\begin{aligned} \int_S \mathbf{g}(\mathbf{x}) \cdot \mathbf{d}\boldsymbol{s} &= \int_S g(r)\mathrm{d}s \\ &= 4\pi r^2 g(r) \\ &= -4\pi GM && \text{by Gauss} \\ \implies g(r) &= -\frac{GM}{r^2}\hat{\mathbf{r}} \end{aligned}$$

which is Newton's law.

Note. Don't need a point mass M. It holds for any spherically symmetric mass density.

This is known as the *Gauss flux method*.
We can also use Gauss' law in other situations. Consider an infinite wire with charge per unit length σ.



By symmetry $\mathbf{E}(r) = E(r)\hat{\mathbf{r}}$ (r cylindrical polar $r^2 = x^2 + y^2$)

$$\int_S \mathbf{E} \cdot \mathbf{d}\boldsymbol{s} = 2\pi r L E(r) = \frac{Q}{\varepsilon_0} = \frac{\sigma L}{\varepsilon_0}$$

Note. $\mathbf{E} \cdot \mathbf{n} = 0$ on end caps so no contribution.

$$\implies \mathbf{E}(r) = \frac{\sigma}{2\pi\varepsilon_0 r}\hat{\mathbf{r}}$$

is electric field due to wire.

Note. Now $\frac{1}{r}$ instead of $\frac{1}{r^2}$ as field spreads out in $\mathbb{R}^2$ instead of $\mathbb{R}^3$.

In $\mathbb{R}^n$, the electric / gravitational field would be $\mathbf{F} \sim \frac{\hat{\mathbf{r}}}{r^{n-1}}$.

There's a different way to write Gauss' law. If the mass density is $\rho(\mathbf{x})$ then from Gauss' theorem

$$\begin{aligned}
\int_V \nabla\cdot\mathbf{g}\mathrm{d}V &= \int_S \mathbf{g}\cdot\mathbf{d}\boldsymbol{s} && \text{by divergence theorem}\\
&= -4\pi GM && \text{by Gauss' law}\\
&= -4\pi G\int_V \rho(\mathbf{x})\mathrm{d}V
\end{aligned}$$

$$\implies \int_V (\nabla\cdot\mathbf{g} + 4\pi G\rho(\mathbf{x}))\mathrm{d}V = 0$$

But this is true for all volumes V

$$\implies \nabla\cdot\mathbf{g} = -4\pi G\rho(\mathbf{x}).$$

This is Gauss' law in differential form. It is the more sophisticated version of Newton's force law. Similarly

$$\nabla\cdot\mathbf{E} = \frac{\rho_e(\mathbf{x})}{\varepsilon_0}$$

(ρ_e is electric charge density). This is the grown-up version of Coulomb's law.

Potentials

It is a fact that the fields $\mathbf{g}$ and $\mathbf{E}$ are conservative, i.e.

$$\mathbf{g} = -\nabla\Phi \qquad \text{and} \qquad Ebf = -\nabla\phi$$

for potentials Φ and ϕ. We've seen some consequences of this. We have

$$\nabla\times\mathbf{g} = \nabla\times\mathbf{E} = 0$$

and

$$\oint_C \mathbf{g}\cdot\mathbf{d}\boldsymbol{x} = \oint_C \mathbf{E}\cdot\mathbf{d}\boldsymbol{x} = 0$$

and most importantly, it means that there is a conserved energy

$$\text{Energy } = \frac{1}{2}m\dot{\mathbf{x}}^2 + m\Phi(\mathbf{x}) + q\phi(\mathbf{x})$$

Gauss' law then becomes

$$\nabla^2\Phi = 4\pi G\rho(x)$$

and

$$\nabla^2\phi = -\frac{\rho_e(\mathbf{x})}{\varepsilon_0}$$

This is the *Poisson equation.* The goal is to solve for $\Phi(\mathbf{x})$ for some fixed "source" $\rho(\mathbf{x})$. If $\rho = 0$, then we solve

$$\nabla^2\Phi = 0.$$

This is the *Laplace equation.*

5.2 The Laplace and Poisson Equations

We write

$$\nabla^2\psi(\mathbf{x}) = -\rho(\mathbf{x})$$

Solve for $\psi(\mathbf{x})$ given a *source* $\rho(\mathbf{x})$ *and* suitable boundary conditions.

Note. $\nabla^2\psi = 0$ is linear, so if ψ_1 and ψ_2 are solutions, then so is $\psi_1 + \psi_2$.

Solutions to the Laplace equation act as complementary solutions to Poisson.

Isotropic Solutions

$\nabla^2\psi = 0$ is a PDE. But with suitable symmetry, it becomes an ODE. For example, spherical symmetry $\implies \psi = \psi(r)$ $(r^2 = x^2 + y^2 + z^2)$.

$$\begin{aligned}\nabla^2\psi &= \frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} + \frac{2}{r} \\ &= \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\psi}{\mathrm{d}r}\right) \\ &= 0 \\ \implies \psi &= \frac{A}{r} + B\end{aligned}$$

(A, B constants).

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We can also look in cylindrical polar coordinates and seek solutions of the form

$$\psi = \psi(r)$$

$(r^2 = x^2 + y^2)$. Now we have

$$\nabla^2\psi = \frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\psi}{\mathrm{d}r} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\psi}{\mathrm{d}r}\right) = 0$$

$$\implies \psi = A\log r + B$$

Note. In $\mathbb{R}^n$, with $n \ge 3$, $\nabla^2\psi = 0$ has the symmetric solution

$$\psi = \frac{A}{r^{n-2}} + B$$

We can get more solutions by differentiating, for example if $\psi = \frac{1}{r}$ is a solution in $\mathbb{R}^3$, then so too

$$\tilde{\psi}(\mathbf{x}) = \mathbf{d}\cdot\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{d}\cdot\mathbf{x}}{r^3}$$

with $\mathbf{d}$ a constant vector. (This is a potential for a *dipole* in electromagnetism).

Boundary Conditions

Often boundary conditions are important. For example: Solve

$$\nabla^2 \psi = \begin{cases} -\rho_0 & r \leq R \\ 0 & r > R \end{cases}$$

with ρ_0 constant. (for example gravitational potential for a planet with constant density). To get a unique solution, we require

- $\psi(r = 0)$ non-singular
- $\psi(r) \to 0)$ as $r \to \infty$
- ψ and ψ' continuous.

Can check that if $\psi(r) = r^p$ then

$$\nabla^2 = p(p+1)r^{p-2}$$

(in $\mathbb{R}^3$). With spherical symmetry, we have

$$\psi(r) = \frac{A}{r} + B - \frac{1}{6}\rho_0 r^2 \qquad r \leq R$$

$\psi(r = 0)$ non-singular $\implies A = 0$. For $r > R$,

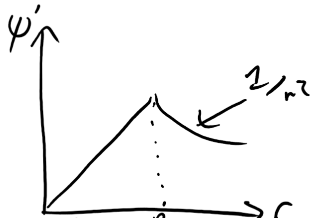
$$\psi(r) = \frac{c}{r} + D \qquad r > R$$

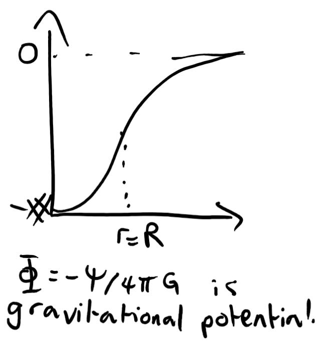
$\psi(r \to \infty) \to 0 \implies D = 0$. Then we patch these together at $r = R$:

$$\psi(r = R) = B - \frac{1}{6}\rho_0 R^2 = \frac{C}{R}$$

$$\psi'(r = R) = -\frac{1}{3}\rho_0 R = -\frac{c}{R^2}$$

$$\implies \psi(r) = \begin{cases} \frac{1}{6}\rho_0(3R^2 - r^2) & r \leq R \\ \frac{1}{3}\rho_0 \frac{R^3}{r} & r > R \end{cases}$$





General Results

If we solve $\nabla^2\psi = -\rho$ in some region $V \subset \mathbb{R}^3$, then there are two common boundary conditions on ∂V.

- Dirichlet (D): Fix $\psi(\mathbf{x}) = f(\mathbf{x})$ on ∂V.
- Neumann (N): Fix $\mathbf{n} \cdot \nabla\psi(\mathbf{x}) = g(\mathbf{x})$ on ∂V, with $\mathbf{n}$ the outward pointing normal.

Notation. Sometimes this is written as

$$\frac{\mathrm{d}\psi}{\mathrm{d}\mathbf{n}} \equiv \mathbf{n} \cdot \nabla\psi$$

or even

$$\frac{\mathrm{d}\psi}{\mathrm{d}n} \equiv \mathbf{n} \cdot \nabla\psi$$

Claim. There is a unique solution to the Poisson equation on V with either D or N boundary conditions specified on each ∂V.

Note. Unique up to constant for N.

Proof. Let ψ_1 and ψ_2 satisfy Poisson and

$$\psi = \psi_1 - \psi_2$$

Then $\nabla^2\phi = 0$ with $\psi = 0$ or $\mathbf{n} \cdot \nabla\psi = 0$ on each ∂V.

$$\begin{aligned}
\int_v \nabla \cdot (\psi\nabla\psi)\mathrm{d}V &= \int_V (\nabla\psi \cdot \nabla\psi + \psi\nabla^2\psi)\mathrm{d}V \\
&= \int_V |\nabla\psi|^2 \mathrm{d}V \\
&= \int_{\partial V} \psi\nabla\psi \cdot \mathbf{ds} && \text{by Divergence Theorem} \\
&= \int_{\partial V} \psi(\mathbf{n} \cdot \nabla\psi)\mathrm{d}s \\
&= 0 && \text{by one of the boundary conditions} \\
\implies |\nabla\psi|^2 &= 0 && \text{in } V \\
\implies \nabla\psi &= 0 && \text{in } V \\
\implies \psi &\text{is constant}
\end{aligned}$$

If Dirichlet $\implies \psi = 0$ on $\partial V \implies \psi = 0$ everywhere. □

Note. Strictly for bounded V, but we can work harder and extend to, for example $\mathbb{R}^3$.

Note.
- If we can find, for example an isotropic solution, then this is *the* solution.
- Sometimes there may be *no* solution.

  Example. Solve $\nabla^2\psi = \rho(x)$ with $\mathbf{n} \cdot \nabla\psi = g(\mathbf{x})$ on ∂V. Then

  $$\int_V \nabla^2\psi \mathrm{d}V = \int_{\partial V} \nabla\psi \cdot \mathbf{d}\boldsymbol{s}$$

  so a solution can exist only if

  $$\int_V \rho \mathrm{d}v = \int_{\partial V} g(\mathbf{x}) \mathrm{d}s$$

- The proof uses Green's first identity:

  $$\int_V \psi\nabla^2\psi \mathrm{d}V = -\int \nabla\phi \cdot \nabla\psi \mathrm{d}V + \int_S \phi\nabla\psi \cdot \mathbf{d}\boldsymbol{s}$$

  (with $\phi = \psi$) This follows from the divergence theorem. Or by anti-symmetry

  $$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\mathrm{d}V = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot \mathbf{d}\boldsymbol{s}$$

  This is Green's second identity.

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Harmonic Functions

Solutions to the Laplace equation

$$\nabla^2\psi = 0$$

are called *harmonic functions*.

Claim (The mean value property)**.** If ψ is harmonic in a region V that includes the ball with boundary

$$S_r : |\mathbf{x} - \mathbf{a}| = R$$

then

$$\psi(\mathbf{a}) = \overline{\psi}(R) := \frac{1}{4\pi R^2} \int S_R \psi(\mathbf{x}) \mathrm{d}S$$

i.e. the value in the middle of the sphere is equal to the average over the boundary of the ball.

Proof. In spherical polar coordinates

$$\mathrm{d}S = r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

$$\overline{\psi}(r) = \frac{1}{4\pi} \int \mathrm{d}\phi \int \mathrm{d}\theta \sin\theta \psi(x, \theta, \phi)$$

$$\begin{aligned}
\frac{\mathrm{d}\overline{\psi}}{\mathrm{d}r}(R) &= \frac{1}{4\pi} \int \mathrm{d}\phi \int \mathrm{d}\theta \sin\theta \frac{\partial \psi}{\partial r}(R, \theta, \phi) \\
&= \frac{1}{4\pi R^2} \int_{S_R} \frac{\partial \psi}{\partial r} \mathrm{d}S \implies \frac{\mathrm{d}\overline{\psi}}{\mathrm{d}r}(R) = \frac{1}{4\pi R^2} \int_{S_R} \nabla \psi \cdot \mathbf{d}\boldsymbol{S} \\
&= \frac{1}{4\pi R^2} \int_{\text{Ball}} \nabla^2 \psi \mathrm{d}V \\
&= 0
\end{aligned}$$

by the divergence theorem. But $\overline{\psi}(R) \to \psi(\mathbf{a})$ as $R \to 0$ hence $\overline{\psi}(R) = \psi a$ for all R. □

Claim. A harmonic function can have neither a maximum nor a minimum in the interior of V. The max / min lie on ∂V.

Proof. If $\exists$ a local maximum at $\mathbf{a}$ then $\exists\ \varepsilon$ such that $\psi(\mathbf{x}) < \psi(\mathbf{a})$ for all $|\mathbf{x} - \mathbf{a}| < \varepsilon$. But this contradicts that $\overline{\psi}(R) = \psi(\mathbf{a})$ for $0 < R < \varepsilon$. □

Note. Saddle points are allowed. The Hessian is

$$\frac{\partial^2 \psi}{\partial x^i \partial x^j}$$

has eigenvalues λ_i, but $\nabla^2 \psi = 0 \implies \sum_i \lambda_i = 0$ so λ_i must be both positive and negative. (This has a loophole when all $\lambda_i = 0$ which is closed by our previous proof).

Integral Solutions

We want to solve the Poisson equation

$$\nabla^2\psi = -\rho(\mathbf{x})$$

for a fixed $\rho(\mathbf{x})$. Consider

$$\psi(\mathbf{x}) = \frac{\lambda}{4\pi r}$$

for λ fixed. Previously we showed that this solves $\nabla^2\psi = 0$, at least if $r \neq 0$. By what happens at $r = 0$? Something must be going on because

$$\begin{aligned}\int_V \nabla^2\psi \mathrm{d}V &= \int_S \nabla\psi \cdot \mathbf{d}\boldsymbol{S} \\ &= -\lambda\end{aligned}$$

We can't have $\nabla^2\psi = 0$ everywhere! Instead, $\psi = \frac{\lambda}{4\pi r}$ must actually solve the Poisson equation for some source $\rho(\mathbf{x})$. But we know $\rho(\mathbf{x}) = 0$ for all $\mathbf{x} \neq 0$. And we must have

$$\int \rho(\mathbf{x})\mathrm{d}V = \lambda$$

The source is the 3D Dirac delta function:

$$\rho(\mathbf{x}) = \lambda\delta^3(\mathbf{x})$$

Here $\delta^3(\mathbf{x})$ is an infinite spike at the origin, such that

$$\int_V f(\mathbf{x})\delta^3(\mathbf{x})\mathrm{d}V = f(\mathbf{x} = 0)$$

In particular

$$\int_V \delta^3(\mathbf{x})\mathrm{d}V = 1$$

So, we've learned that $\psi = \frac{\lambda}{4\pi r}$ does *not* solve the Laplace equation, but

$$\nabla^2\psi = -\lambda\delta^3(\mathbf{x}) \implies \psi(\mathbf{x}) = \frac{\lambda}{4\pi r}$$

Claim. $\nabla^2\psi = -\rho$ has the integral solution

$$\psi(\mathbf{x}) = \frac{1}{4\pi}\int_{V'} \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\mathrm{d}V'$$

(V' should include any region with $\rho(\mathbf{x}') = 0$)

Proof. Intuition is that you sum over "$\frac{1}{r}$" solutions, weighted by $\rho(\mathbf{x}')$ for each $\mathbf{x}'$.

$$\nabla^2\psi(\mathbf{x}) = \frac{1}{4\pi}\int_{V'}\rho(\mathbf{x}')\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\mathrm{d}V'$$

(here ∇ differentiates $\mathbf{x}$ and treats $\mathbf{x}'$ as constant) but

$$\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -4\pi\delta^3(\mathbf{x}-\mathbf{x}')$$

(this is our previous result that $\nabla^2\frac{1}{r} = -4\pi\delta^3(\mathbf{x})$ but with the origin shifted to $\mathbf{x}'$)

$$\begin{aligned}\implies \nabla^2\psi &= -\int_{V'}\rho(\mathbf{x}')\delta^3(\mathbf{x}-\mathbf{x}')\mathrm{d}V' \\ &= -\rho(\mathbf{x})\end{aligned}$$

□

This powerful technique is known as the *Green's function* approach.

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6 Tensors

6.1 What is a Tensor?

Not any list of n numbers constitutes a vector in $\mathbb{R}^n$. They come with certain responsibilities.

We start with a point $\mathbf{x} \in \mathbb{R}^n$. To attach some coordinates to this, we first introduce a basis $\{\mathbf{e}_i, i = 1, \dots, n\}$ such that

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

And we write $\mathbf{x} = x_i \mathbf{e}_i$. We call $x_i = (x_1, \dots, x_n)$ a "vector". It's a set of labels to specify $\mathbf{x}$.

Alternatively, we could use

$$\mathbf{e}'_i = R_{ij} \mathbf{e}_j \tag{$*$}$$

We insist that $\mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij}$.

$$\implies R_{ik} R_{jk} \mathbf{e}_k \cdot \mathbf{e}_l = R_{ik} R_{jk} = \delta_{ij}$$

$$\implies RR^\top = \mathbb{1}$$

Such matrices are called *orthogonal*. We write $R \in O(n)$. We have

$$\det RR^\top = (\det R)^2 = 1 \implies \det R = \pm 1$$

If $\det R = +1$, then R corresponds to a rotation and we write $R \in SO(n)$ (special orthogonal).
If $\det R = -1$, it is a reflection + rotation. Under a change of basis, $\mathbf{x}$ doesn't change. We have

$$\mathbf{x} = x_i \mathbf{e}_i = x'_i \mathbf{e}'_i = x'_i R_{ij} \mathbf{e}_j$$

$$\mathbf{x} \cdot \mathbf{e}_k = x_k = x'_i R_{ik}$$

inverting:

$$\implies x'_i = R_{ij} x_j$$

A *tensor* T is a generalisation of these ideas to an object with more indices. When measured with respect to the basis $\{\mathbf{e}_i\}$, a *tensor of rank* p (or p-*tensor*) has indices

$$T_{i_1 \cdots i_p}$$

Under a change of basis $(*)$ we have the *tensor transformation rule*

$$T'_{i_1 \cdots i_p} = R_{i_1 j_1} \cdots R_{i_p j_p} T_{j_1 \cdots j_p}$$

Note. 0-tensor is a number
1-tensor is a vector
2-tensor is a matrix such that $T'_{ij} = R_{ik}R_{jl}T_{kl}$.

Example. There is one special rank 2 tensor in $\mathbb{R}^n$:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

This is the same in all bases since

$$\delta'_{ij} = R_{ik}R_{jl}\delta_{kl} = \delta_{ij}$$

It's an example of an *invariant* tensor.

Tensors as Maps

There is an equivalent, coordinate independent view. A p-tensor is a multi-linear map

$$T : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{p} \to \mathbb{R}$$

such that

$$T(\mathbf{a}, \mathbf{b}, \cdots, \mathbf{c}) = T_{i_1 \cdots i_p} a_{i_1} b_{i_2} c_{i_p}$$

(multi-linear = linear in each entry seperately).

The tensor transformation rule ensures that the map is independent of the choice of basis.

$$\begin{aligned} T(\mathbf{a}, \mathbf{b}, \cdots \mathbf{c}) &= T'_{i_1 \cdots i_p} a'_{i_1} b'_{i_2} \cdots c'_{i_p} \\ &= (R_{i_1 j_1} \cdots R_{i_p j_p}) T_{j_1 \cdots j_p} \times (R_{i_1 k_1} a_{k_1}) \cdots (R_{i_p k_p} c_{k_p}) \\ &= T_{j_1 \cdots j_p} a_{j_1} b_{j_2} \cdots c_{j_p} \end{aligned}$$

Alternatively, we can think of a tensor as a map between lower rank tensors. For example, a p-tensor can be viewed as a map

$$T : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{p-1} \to \mathbb{R}^n$$

The map is

$$a_i = T_{i j_1 \cdots j_{p-1}} b_{j_1} \cdots c_{j_{p-1}}$$

This is the way that tensors originally appear in maths and physics, typically as a map from vectors to vectors.

$$\mathbf{u} = T\mathbf{v} \implies u_i T_{ij} v_j$$

T is a matrix but, importantly, transforms as a tensor so the equation holds in all bases

$$T'_{ij} = R_{ik} R_{jl} T_{kl}$$

or

$$T' = RTR^\top$$

Tensor Operations

- If S, T are tensors of the same rank, then so is $S + T$ and λT for $\lambda \in \mathbb{R}$.
- If S is a p-tensor and T is a q-tensor then we can form a $(p+q)$-tensor known as the tensor product

  $$(S \otimes T)_{i_1 \cdots i_p j_1 \cdots j_q} = S_{i_1 \cdots i_p} T_{j_1 \cdots j_q}$$

  for example, given two vectors $\mathbf{a}$ and $\mathbf{b}$ we can form the matrix

  $$(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$$

- If T is a p-tensor then we can construct a $(p-2)$-tensor by *contraction*:

  $$\delta_{ij} T_{ijk_1 \cdots k_{p-2}} = T_{iik_1 \cdots k_{p-2}}$$

  for example

  $$T_r T = T_{ii}$$

  for a 2-tensor.

We can combine the tensor product and contraction. If P is a p-tensor and Q is a q-tensor, we can form a $(p+q-2)$-tensor. For example, contraction on the first index gives

$$P_{ik_1 \cdots k_{p-1}} Q_{il_1 \cdots l_{q-1}}$$

for example given vectors $\mathbf{a}, \mathbf{b}$,

$$\delta_{ij} a_i b_j = \mathbf{a} \cdots \mathbf{b}$$

is a zero-tensor. This is just the usual inner-product. Another example is matrix multiplication.

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How do we know if a bunch of numbers $T\ldots$ form a tensor?

If T is a $(p+q)$-tensor then for every q-tensor u,

$$v_{i_1 \cdots i_p} = T_{i_1 \cdots i_p j_1 \cdots j_q} u_{j_1 \cdots j_q}$$

is a p-tensor.

Conversely, if v is a p-tensor for every q-tensor u, then T is a $(p+q)$-tensor. This is the *quotient rule.*

Proof. Consider

$$u_{j_1\cdots j_q} = c_{j_1}\cdots d_{j_q}$$

By assumption

$$v_{i_1\cdots i_p} = T_{i_1\cdots i_p j_1\cdots j_q} c_{j_1}\cdots d_{j_q}$$

is a tensor, so

$$a_{i_1}\cdots b_{i_p} v_{i_1\cdots i_p} = T_{i_1\cdots i_p j_1\cdots j_q} a_{i_1}\cdots b_{i_p} c_{j_1}\cdots d_{j_q}$$

is a scalar, $\forall\ \mathbf{a},\cdots\mathbf{b},\mathbf{c},\cdots\mathbf{d}$ hence T must be a $(p+q)$-tensor. □

(Anti)-Symmetry

A tensor that obeys

$$T_{ijp\cdots q} = \pm T_{jip\cdots q}$$

is said to be (anti)-symmetric in i, j (anti for $-$). This is a basis independent statement:

$$\begin{aligned} T'_{ijp\cdots q} &= R_{ik}R_{jl}R_{pr}\cdots R_{qs}T_{klr\cdots s} \\ &= \pm R_{ik}R_{jk}R_{pr}\cdots R_{qs}T_{lkr\cdots s} \\ &= \pm T'_{jip\cdots q} \end{aligned}$$

If T is (anti)-symmetric in all indices it is said to be *totally (anti)-symmetric*. A totally anti-symmetric p-tensor in $\mathbb{R}^n$ has $\binom{n}{p}$ independent components, and vanishes in $p > n$.

In $\mathbb{R}^3$, a 2-tensor T_{ij} decomposes as

$$S_{ij} = \frac{1}{2}(T_{ij} + T_{ij})$$

$$A_{ij} = \frac{1}{2}(T_{ij} - T_{ji})$$

and S_{ij} further decomposes as

$$S_{ij} = P_{ij} + \frac{1}{3}Q\delta_{ij}$$

where P_{ij} is traceless (i.e. $P_{ii} = 0$) and the trace of S_{ij} is Q.

In $\mathbb{R}^3$ we have another invariant tensor ε_{ijk} (see below) and we can write

$$A_{ij} = \varepsilon_{ijk}B_k \iff B_k\frac{1}{2}\varepsilon_{klm}A_{lm}$$

So a 3×3 matrix can be written as

$$T_{ij} = P_{ij} + \varepsilon_{ijk}B_k\frac{1}{3}Q\delta_{ij}$$

with P, B and Q are themselves tensors.

Invariant Tensors

A tensor that obeys

$$T'_{i_1\cdots i_p} = R_{i_1 j_1} \cdots R_{i_p j_p} T_{j_1 \cdots j_p} = T_{i_1 \cdots i_p}$$

for all R is called an *invariant tensor* or is said to be *isotropic*.

Any rank 0 tensor is isotropic. There are no rank 1 isotropic tensors. There is a rank 2, and in $\mathbb{R}^3$, a rank 3 invariant tensor:

- δ_{ij} with $\delta'_{ij} = R_{ik} R_{jl} \delta_{kl} = \delta_{il}$
- ε_{ijk} with

$$\begin{aligned} \varepsilon'_{ijk} &= R_{il} R_{jm} R_{kn} \varepsilon_{lmn} \\ &= (\det R) \varepsilon_{ijk} \\ &= \varepsilon_{ijk} \end{aligned}$$

Claim. The only isotropic tensors in $\mathbb{R}^3$ of rank $1 \leq p \leq 3$ are

$$T_{ij} = \alpha \delta_{ij}$$

and

$$T_{ijk} = \beta \varepsilon_{ijk}$$

with α, β constant.

Proof. Look for a rank 1 tensor. Must have

$$T'_i = R_{ij} T_j = T_i$$

for

$$R_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

hence $T'_1 = -T_1$ and $T'_2 = -T_2$ so $T_1 = T_2 = 0$. A similar argument gives $T_3 = 0$.

Look for a rank 2 tensor:

$$T'_{ij} = \tilde{R}_{ik} \tilde{R}_{jl} T_{kl} = T_{ij}$$

with

$$\tilde{R}_{ij} = \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

($\frac{\pi}{2}$ rotation about z-axis) This gives $T'_{13} = T_{23}$ and $T'_{23} = -T_{13}$ hence $T_{13} = T_{23} = 0$. Also $T'_{11} = T_{22}$. Similar arguments show that $T_{ij} = 0$ for $i \neq j$ and $T_{11} = T_{22} = T_{33} = \alpha$

$$\implies T_{ij} = \alpha\delta_{ij}$$

For rank 3,

$$T'_{ijk} = R_{il}R_{jp}R_{kq}T_{lpq} = T_{ijk}$$

Use

$$R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix} \implies T'_{133} = -T_{133}, T'_{111} = -T_{111}$$

$\implies T_{ijk} = 0$ unless i, j, k distinct. Use $R = \tilde{R}$ to show that $T'_{123} = -T_{213}$

$$\implies T_{ijk} = \beta\varepsilon_{ijk}$$

□

All higher rank invariant tensors in $\mathbb{R}^3$ are built from ε_{ijk} and δ_{ij}, for example isotropic rank 4 tensor has the most general form

$$T_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

for α, β, γ some constants.

Invariant Integrals

We can sometimes use this to do integrals, for example

$$T_{ij\cdots k} = \int_V f(r)x_i x_j \cdots x_k \mathrm{d}V$$

(V is a spherically symmetric region and $r = |\mathbf{x}|$). Under a rotation

$$\begin{aligned} T'_{ij\cdots k} &= R_{ip}R_{jq}\cdots R_{kr}T_{pq\cdots r} \\ &= \int_V f(r)x'_i x'_j \cdots x'_k \mathrm{d}V \end{aligned}$$

($x'_i = R_{ip}x_p$). Change variables to x'. Both $r = |\mathbf{x}|$ and V are invariant

$$\implies T'_{ij\cdots k} = T_{ij\cdots k}$$

so must be proportional to an invariant tensor.

Example. Consider the integral over 3D ball of radius R:

$$T_{ij} = \int_V \rho(r) x_i x_j \mathrm{d}V$$

Necessarily $= \alpha\delta_{ij}$ for some α. Take the trace:

$$\implies \int_V \rho(r) r^2 \mathrm{d}V = 3\alpha$$

($\delta_{ii} = 3$)

$$\implies T_{ij} = \frac{1}{3}\delta_{ij} \int_V \rho(r) r^2 \mathrm{d}V$$

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Tensor Fields

A tensor field over $\mathbb{R}^3$ assigns a tensor $T_{i\cdots k}(\mathbf{x})$ to each point $\mathbf{x} \in \mathbb{R}^3$. This generalises the vector field

$$\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$$

to

$$T : \mathbb{R}^3 \to \mathbb{R}^m$$

with $m =$ # components of the tensor.

Tensor fields have one further operation: we can differentiate to build higher rank tensors.

Example. If ϕ is a scalar field then

$$\nabla\phi = \frac{\partial\phi}{\partial x^i}\mathbf{e}_i$$

is a vector field, and so

$$\frac{\partial\phi}{\partial x^i}$$

transforms as a 1-tensor.

More generally, if T is a p-tensor field then we can construct a $(p+q)$-tensor field

$$X_{i_1\cdots i_q j_1 \cdots j_p}(\mathbf{x}) = \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_q}} T_{j_1\cdots j_p}(\mathbf{x})$$

To check that this is indeed a tensor, we use

$$x_i' = R_{ij} x_j \implies x_j = R_{ij} x_i'$$

$$\implies \frac{\partial}{\partial x_i'} = \frac{\partial x_j}{\partial x_i'} \frac{\partial}{\partial x_j} = R_{ij} \frac{\partial}{\partial x^j}$$

($\frac{\partial}{\partial x}$ transforms as a tensor.)

6.2 Physical Examples

The simplest examples of tensors are just matrices.

In a material, an applied electric field $\mathbf{E}$ will give a current $\mathbf{J}$ is given by

$$J_i = \sigma_{ij} E_j$$

where σ_{ij} is the *conductivity tensor*. This is the grown-up version of *Ohm's law*.

Note. In 3D, isotropic materials necessarily have

$$\sigma_{ij} = \sigma \delta_{ij}$$

with σ the conductivity.
In 2D (i.e. thin materials) then isotropy means

$$\begin{aligned} \sigma_{ij} &= \delta_{xx}\delta_{ij} + \delta_{xy}\varepsilon_{ij} \\ &= \begin{pmatrix} \delta_{xx} & \delta_{xy} \\ -\delta_{xy} & \delta_{xx} \end{pmatrix} \end{aligned}$$

(σ_{xy} is the *Hall conductivity*).

In Newtonian mechanics, a rigid body has

$$\mathbf{L} = I\boldsymbol{\omega}$$

($\mathbf{L}$ is angular momentum, $\boldsymbol{\omega}$ is angular velocity), where I is the inertia tensor. If the body is made of particles of mass m_a, rotating as

$$\dot{\mathbf{x}}_a = \boldsymbol{\omega} \times \mathbf{x}_a$$

then

$$\begin{aligned} \mathbf{L} &= \sum_a m_a \mathbf{x}_a \times \dot{\mathbf{x}}_a \\ &= \sum_a m_a \mathbf{x}_a \times (\boldsymbol{\omega} \times \mathbf{x}_a) \\ &= \sum_a m_a (|\mathbf{x}_a|^2 \boldsymbol{\omega} - (\mathbf{x}_a \cdot \boldsymbol{\omega})\mathbf{x}_a) \\ \implies \mathbf{L} &= I_{ij}\omega_j \end{aligned}$$

with

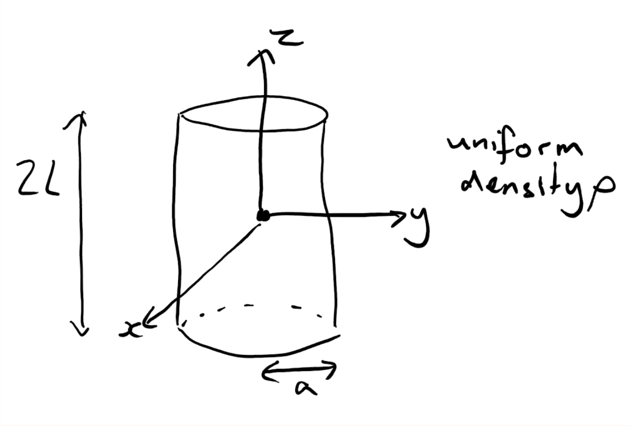
$$I_{ij} = \sum_a m_a (|\mathbf{x}_a|^2 \delta_{ij} - (\mathbf{x}_a)_i (\mathbf{x}_a)_j)$$

For a continuous object,

$$I_{ij} = \int_V \rho(\mathbf{x})(|\mathbf{x}|^2 \delta_{ij} - x_i x_j) \mathrm{d}V$$

Example (A Sphere)**.** A ball of radius R and density $\rho(r)$ has

$$\begin{aligned} I_{ij} &= \int_V \rho(r)(r^2\delta_{ij} - x_i x_j)\mathrm{d}V \\ &= \frac{8\pi}{3}\delta_{ij}\int_0^R \mathrm{d}r\rho(r)r^4 \end{aligned}$$

Example (A Cylinder). 

$$M = 2\pi a^2 L\rho$$

In cylindrical polar,

$$x = r\cos\phi \qquad x = r\sin\phi$$

$$\begin{aligned}
I_{33} &= \int_V \rho(x^2+y^2)\mathrm{d}V \\
&= \rho\int_0^{2\pi}\mathrm{d}\phi\int_0^a \mathrm{d}r\int_{-L}^{+L}\mathrm{d}z\cdot r\cdot r^2 \\
&= \rho\pi L a^4 \\
I_{33} &= \int_V \rho(y^2+z^2)\mathrm{d}V \\
&= \rho\int_0^{2\pi}\mathrm{d}\phi\int_0^a \mathrm{d}r\int_{-L} +Lr(r^2\sin^2\phi+z^2) \\
&= \rho\pi a^2 L\left(\frac{1}{2}a^2+\frac{2}{3}L^2\right) \\
&= I_{22} && \text{by symmetry} \\
I_{13} &= -\rho\int_V xz\mathrm{d}V \\
&= -\rho\int_0^{2\pi}\mathrm{d}\phi\int_0^a \mathrm{d}r\int_{-L}^{+L}\mathrm{d}z r^2 z\cos\phi \\
&= -\rho\int_0^{2\pi}\mathrm{d}\phi\cos\phi C \\
&= 0
\end{aligned}$$

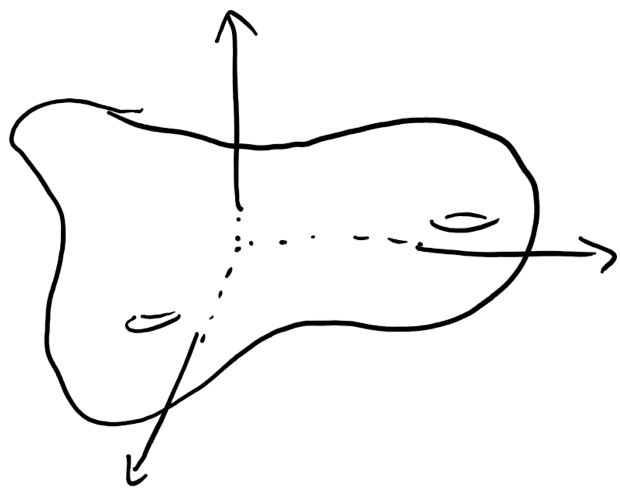
All other off diagonal entries vanish similarly, so for a cylinder

$$I = \mathrm{diag}\left(M\left(\frac{a^2}{4}+\frac{L^2}{3}\right), M\left(\frac{a^2}{4}+\frac{L^2}{3}\right), \frac{1}{2}Ma^2\right)$$

For a general body, and a general choice of basis, I_{ij} will not be diagonal. However, $I_{ij} = I_{ji}$ so there exist an $R \in \mathrm{SO}(3)$ such that

$$I' = RIR^\top = \mathrm{diag}(I_1, I_2, I_3)$$

i.e. every body has a preferred set of axes such that I is diagonal.



From $\mathbf{L} = I\boldsymbol{\omega}$, if the angular velocity, $\boldsymbol{\omega}$ is aligned with one of these axes then $\mathbf{L} \parallel \boldsymbol{\omega}$. Otherwise $\mathbf{L}$ is not parallel to $\boldsymbol{\omega}$ and this is the reason things wobble! (see classical dynamics).