Dynamics and Relativity

April 28, 2022

Contents

Start of lecture 1

1 Introduction

Books:

- Classical Mechanics - Douglas Gregory (more examples)
- Classical Mechanics - Tom Kibble & Frank Berkshire (more "chatty")

Lecture Notes - David Tong PDF online.
Lecture on Saturday 5th March will be pre-recorded & online only.

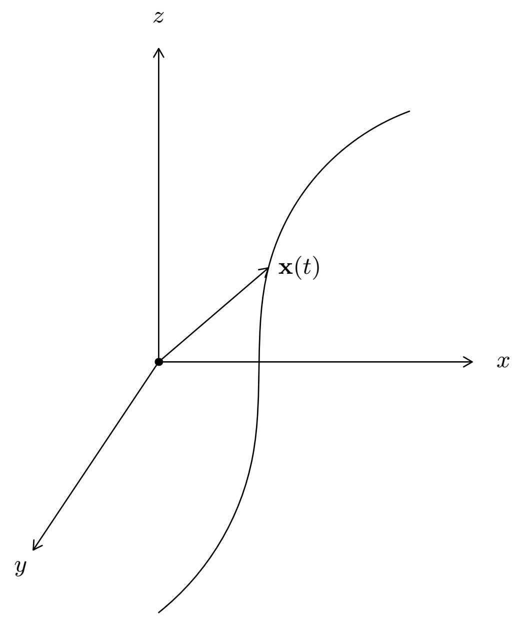
Newtonian Mechanics - Definitions

Definition (Particle)**.** A *particle* is an object of insignificant size. For now, its only attribute is its position.

For large objects, we take the centre of mass to define the position and treat them like a particle.
To describe the position, we pick a *reference frame*:

Definition (Reference frame)**.** A *reference frame* is a choice of origin and 3 coordinate axes.

With respect to this frame, a particle sweeps out a trajectory $\mathbf{x}(t)$. (sometimes, we may write $\mathbf{r}(t)$).



The *velocity* of the particle is $\mathbf{v} = \dot{\mathbf{x}} := \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$. The *acceleration* of the particle is $\mathbf{a} = \ddot{\mathbf{x}} := \frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}$.

Note. The derivative of a vector is defined using components:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \\ \frac{\mathrm{d}z}{\mathrm{d}t} \end{pmatrix}.$$

Given two vector functions $\mathbf{f}(t)$ and $\mathbf{g}(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{f}, \mathbf{g}) = \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} \cdot \mathbf{g} + \mathbf{f} \cdot \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{f} \times \mathbf{g}) = \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} \times \mathbf{g} + \mathbf{f} \times \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}.$$

(proof by components)

Newton's Laws of Motions

The framework of Newtonian mechanics rests on three axioms, known as Newton's Laws:

N1 Left alone, a particle moves with constant velocity.

N2 The rate of change of momentum is proportional to the force.

N3 Every action has an equal and opposite reaction.

Inertial Frames and The First Law

For many reference frames, N1 isn't true! It only holds for frames that are not themselves accelerating. Such frames are called *intertial frames*. In an intertial frame, $\ddot{\mathbf{x}} = \mathbf{0}$ when left alone ($\mathbf{F} = \mathbf{0}$).

A better framing of the first law is:

N1' Inertial frames exist.

Inertial frames provide the setting for much of what follows in this course. For most purposes, this room approximates an inertial frame.

Galilean Relativity

Inertial frames are not unique. Given an inertial frame S, in which a particle has coordinate $\mathbf{x}$, we can construct another inertial frame S' in which the coordinates of the particle are given by $\mathbf{x}'$.

- Translations: $\mathbf{x}' = \mathbf{x} + \mathbf{a}$, where $\mathbf{a}'$ is a constant.
- Rotations and reflections: $\mathbf{x}' = R\mathbf{x}$ where R is a 3×3 matrix with $R^\top R = I$ (orthogonal matrix).
- Boosts: $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$ where $\mathbf{v}$ is a constant.

For each of these, if there is no force on a particle, we have that $\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = 0$ since S is an inertial frame, which implies that $\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = 0$ so S' is an inertial frame. The *Galilean principle of relativity* tells us that the laws of physics stay the same

- At every point in space.
- No matter which direction you face.
- No matter what constant velocity you move at.
- At all moments in time.

(these are experimentally tested facts).

There is no such thing as "absolutely stationary", but notice that acceleration is absolute - you don't have to accelerate relative to something.

Start of lecture 2

1.1 Newton's Second Law

The *equation of motion* for a particle subjected to a force $\mathbf{F}$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{\mathbf{x}}) = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$$

$\mathbf{p} = m\dot{\mathbf{x}}$ is *momentum*, and the force $\mathbf{F}$ can depend on position and velocity. m is *inertial mass*. It is a measure of the reluctance of a particle to move.
When $\frac{m}{t} = 0$ (true in most situations), we have

$$\boxed{m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})}$$

The equation of motion is a second order differential equation $\implies$ we need to specify two initial conditions for each degree of freedom.
For example, $\mathbf{x} \in \mathbb{R}^3 \implies$ 3 degrees of freedom i.e. 6 initial conditions are needed.

There are two steps in any Newtonian mechanics problem:

- Write down the equation(s).
- Solve it.

2 Forces

2.1 Potentials in One Dimension

Consider a particle moving in a line with position $x(t)$. Suppose that $F = F(x)$, i.e. it depends on position, not on velocity. We define a *potential energy* $V(x)$ by

$$F = -\frac{\mathrm{d}V}{\mathrm{d}x} \qquad \text{or} \qquad V(x) = -\int_{x_0}^{x} \mathrm{d}x' F(x')$$

Note. The x' does note derivative here; x' is a dummy variable.

The equation of motion is

$$m\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x} \tag{$*$}$$

Claim. The energy $E = \frac{1}{2}m\dot{x}^2 + V(x)$ is conserved (i.e. $\dot{E} = 0$) for any trajectory which obeys ($*$).

Proof.

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= m\dot{x}\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\dot{x} \\ &= \dot{x}\left(m\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\right) \\ &= 0 && \text{by } (*) \end{aligned}$$

□

Note. If $F = F(x, \dot{x})$, there is no conserved quantity.

Example (Harmonic Oscillator)**.**

$$V = \frac{1}{2}kx^2$$

Then $(*)$ becomes $m\ddot{x} = -kx$. The general solution is

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

where $\omega := \sqrt{\frac{k}{m}}$ is the *angular frequency.* A and B are integration constants. It's simple to show that

$$E = \frac{1}{2}m\dot{x} + \frac{1}{2}kx^2$$

is constant. The time take to complete a cycle is the *period* $T = \frac{2\pi}{\omega}$.

For a general potential $V(x)$, the conserved quantity allows use to 'solve' any one-dimensional problem.

$$\begin{aligned} E &= \frac{1}{2}m\dot{x}^2 + V(x) \\ \implies \frac{\mathrm{d}x}{\mathrm{d}t} &= \pm\sqrt{\frac{2}{m}(E - V(x))} \\ \implies t - t_0 &= \pm\int_{x_0}^{x} \frac{\mathrm{d}x'}{\sqrt{\frac{2}{m}(E - V(x'))}} \end{aligned}$$

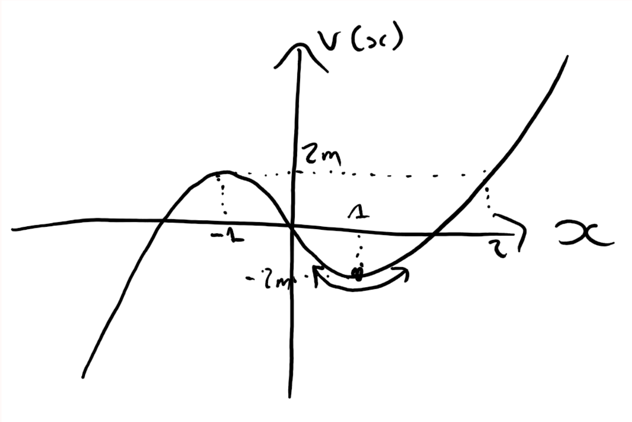
which is 'the solution' - we 'just' need to do the integral.

Motion in a Potential

Sometimes even if you can't do the integral, it's simple to get a qualitative picture of the solution.

Example.

$$V(x) = m(x^3 - 3x).$$



Drop particle at $x = x_0 \implies E = V(x_0)$.

$x_0 = \pm 1 \implies$ particle stays there $\forall t$ ($*$ says there's no force).

- $x_0 \in (-1, 2) \implies$ particle oscillates back and forth in dip.
- $x_0 > 2$. Particles keeps on going to $-\infty$.
- $x_0 = 2$ is a special case. It reaches $x = -1$ but how long does it take? Write $x = -1 + \varepsilon$ and as $\varepsilon \to 0 \implies V(x)\mathbf{u}2m - 3m\varepsilon^2$

$$t - t_0 = \int_{x_0}^{x} \frac{\mathrm{d}x'}{\sqrt{\frac{2}{m}(E - V(x'))}} = -\int_{\varepsilon_0}^{\varepsilon} \frac{\mathrm{d}\varepsilon'}{\sqrt{6}\varepsilon'} = -\frac{1}{\sqrt{6}} \ln\left(\frac{\varepsilon}{\varepsilon_0}\right)$$

so $t \to \infty$ as $\varepsilon \to 0$, i.e. it takes infinite time.

The initial energy is all potential. It turns this into kinetic energy and falls down the dip.

Start of lecture 3

Equilibrium Points

A particle placed at an *equilibrium point* x_0 will stay there for all time.

$$m\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}$$

implies equilibrium points obey

$$\left.\frac{\mathrm{d}V}{\mathrm{d}x}\right|_{x_0}$$

i.e. critical points of V. We can look at motion near equilibrium point. Taylor expanding,

$$V(x) \approx V(x_0) + \frac{1}{2}(x - x_0)^2 V''(x) + \cdots$$

- $V''(x) > 0 \implies$ minimum of V, potential of harmonic oscillator

  $$m\ddot{x} \approx V''(x_0)(x - x_0)$$

  This point is *stable*. Particle oscillates with frequency $\omega = \sqrt{\frac{V''(x_0)}{m}}$.

- $V''(x_))$ $\implies$ maximum of V $\implies$ point is *unstable*.

  $$x - x_0 \approx Ae^{\alpha t} + Be^{-\alpha t}$$

  with

  $$\alpha = \sqrt{-\frac{V''(x_0)}{m}}.$$

- $v''(x_0) = 0 \implies$ more work needed.

Example (the pendulum)**.** The equation of motion is

$$\ddot{\theta} = -\frac{g}{x}\sin\theta$$

The energy is

$$E = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta.$$

- $E > mgl \implies \dot{\theta} \neq 0$ for all t.
- $E < mgl \implies \dot{\theta} = 0$ at some point θ_0. $\implies$ oscillates back and forth and

  $$E = -mgl\cos\theta_0.$$

Using the general solution for 1-dimensional system,

$$T = \int_0^{\theta_0} \frac{\mathrm{d}\theta}{\sqrt{\frac{2E}{ml^2} + \left(\frac{2g}{l}\right)\cos\theta}} = 4\sqrt{\frac{l}{g}}\int_0^{\theta_0} \frac{\mathrm{d}\theta}{\sqrt{2\cos\theta - 2\cos\theta_0}}$$

but, for small θ, $\cos\theta \approx 1 - \frac{\theta^2}{2}$:

$$T \approx 4\sqrt{\frac{l}{g}}\int_0^{\theta_0} \frac{\mathrm{d}\theta}{\sqrt{\theta_0^2 - \theta^2}} = 2\pi\sqrt{\frac{l}{g}}$$

(Note: independent of θ_0).
This is the result for the harmonic oscillator, of course.

Start of lecture 4

2.2 Potential in 3D

Suppose $\mathbf{F} = \mathbf{F}(\mathbf{x})$.

Claim. There exists a conserved energy if and only if the force is of the form $\mathbf{F} = -\nabla V$, i.e. $F_i = -\frac{\partial V(\mathbf{x})}{\partial x^i}$. Then,

$$E = \frac{1}{2}m\dot{\mathbf{x}} \cdot \dot{\mathbf{x}} + V(\mathbf{x})$$

is conserved.

Proof. (To prove E is conserved if $\mathbf{F} = -\nabla V$):

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= m\ddot{\mathbf{x}} \cdot \dot{\mathbf{x}} + \frac{\partial V}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} \\ &= \dot{\mathbf{x}} \cdot (m\ddot{\mathbf{x}} + \nabla V) \\ &= 0 \end{aligned}$$

so E is conserved.

To prove $\mathbf{F} = -\nabla V$ if E is conserved, we introduce the *work done* W on a particle, which moves from $\mathbf{x}(t_1)$ to $\mathbf{x}(t_2)$ along a trajectory C.

$$\begin{aligned} W := \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} \\ &= \int_{t_1}^{t_2} \mathbf{F} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\mathrm{d}t \\ &= m\int_{t_1}^{t_2} \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}\mathrm{d}t \\ &= m\int_{t_1}^{t_2} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{x}} \cdot \dot{\mathbf{x}})\mathrm{d}t \\ &= T(t_2) - T(t_1) \end{aligned}$$

where $T := \frac{1}{2}m\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}$ is the kinetic energy. If we want a conserved energy $E = T + V$

$$\implies W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = V(\mathbf{x}(t_1)) - V(\mathbf{x}(t_2)).$$

depends only on the end-points, not on the trajectory C, i.e. for a closed path C,

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = 0$$

A result from vector calculus states that this can only hold if $\mathbf{F} = -\nabla V$. □

Forces of this form are called *conservative*.

Note. $\mathbf{F} = -\nabla V \iff \nabla \times \mathbf{F} = 0$.

Central Forces

A special class of force has potential $V(r)$, with $r = |\mathbf{x}|$

$$\implies \mathbf{F}(r) = -\nabla V = -\frac{\mathrm{d}v}{\mathrm{d}r}\hat{\mathbf{r}}.$$

i.e. points towards/away from the origin. These are *central forces*. Central forces have an additional conserved quantity called *angular momentum*.

$$\mathbf{L} = m\mathbf{x} \times \dot{\mathbf{x}}.$$

Note that it depends on choice of origin.

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = m\dot{\mathbf{x}} \times \dot{\mathbf{x}} + m\mathbf{x} \times \ddot{\mathbf{x}} = \mathbf{x} \times \mathbf{F} = 0$$

because $\mathbf{F} = \mathbf{x}$. For non-central forces,

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \boldsymbol{\tau}$$

where $\boldsymbol{\tau} = \mathbf{x} \times \mathbf{F}$ is the *torque*.

2.3 Gravity

Gravity is a conservative force. Fix a particle of mass M at the origin. A particle of mass m in its presence experiences a potential energy.

$$V(r) = -\frac{GMm}{r}$$

where $G = 6.67 \times 10^{-11}\mathsf{m}^3\mathsf{kg}^{-1}\mathsf{s}^{-2}$ is Newton's constant.

$$\implies \mathbf{F} = -\nabla V = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

It's common to define the *gravitational field* $\phi(r) = -\frac{GM}{r}$, a property of the mass M alone.
The gravitational field due to many fixed masses M_i at position $\mathbf{r}_i$ is

$$\phi(\mathbf{r}) = -G\sum_i \left(\frac{M_i}{|\mathbf{r} - \mathbf{r}_i|}\right).$$

The potential energy of mass m is then $V = m\phi$.

Claim. The external gravitational field of a spherically symmetric object of mass M is

$$\phi(r) = -\frac{GM}{r}$$

i.e. as if all the mass were at $r = 0$.

Proof. Volume integral (see vector calculus course). □

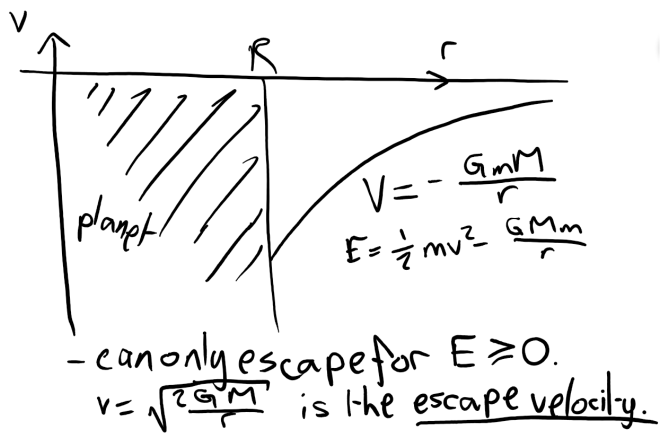
Some 1d Problems

Sit on a spherical planet of radius R. If one only moves a distance $x \ll R$ above the surface, then

$$V(R+z) = -\frac{GMm}{R+z} = -\frac{GMm}{R}\left(1 - \frac{z}{R} + \cdots\right) \approx \text{constant} + \frac{GM}{R^2} m \cdot z$$

$$V \approx \text{constant} + mgz$$

How fast must we jump to escape the gravitational pull of the earth?



Start of lecture 5

Inertial versus Gravitational Mass

Mass has appeared in a couple of formulae:

$$\mathbf{F} = m_I \ddot{\mathbf{x}}$$

$$\mathbf{F} = -\frac{GM_G m_G}{r^2}\hat{\mathbf{r}}$$

here m_I denotes *inertial mass* and m_G denotes *gravitational mass.*

Conceptually, these are very different quantities. Experimentally, we know that $m_I = m_G$ (to 1 part in 10^{13}) - only explained by Einstein's general theory of relativity.

2.4 Electromagnetism

The force experienced by a particle with electric charge q is

$$\mathbf{F} = q(\mathbf{E}(\mathbf{x}) + \dot{\mathbf{x}} \times \mathbf{B}(\mathbf{x})).$$

(E is *electric field* and B is *magnetic field*.) *This is the Lorentz Force Law.*

The electron has a charge $-1.6 \times 10^{-19}\mathsf{C}$ (C = Coulomb). $\mathbf{E}$ and $\mathbf{B}$ are functions of space and (in principle) time. We'll only consider time independent fields, where we can write $\mathbf{E}$ as

$$\mathbf{E} = -\nabla\phi$$

for some *electric potential* $\phi \implies$ the electric force is conservative.

Claim. For t-independent $\mathbf{E}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ the energy

$$E = \frac{1}{2}m\dot{\mathbf{x}} \cdot \dot{\mathbf{x}} + q\phi(\mathbf{x})$$

is conserved.

Proof.

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= m\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} + q\nabla\phi \cdot \dot{\mathbf{x}} \\
&= \dot{\mathbf{x}}(m\ddot{\mathbf{x}} + q\nabla\phi) \\
&= q\dot{\mathbf{x}} \cdot (\dot{\mathbf{x}} \times \mathbf{B}(\mathbf{x})) \\
&= 0 && \text{since perpendicular}
\end{aligned}$$

□

Point Charges

A fixed particle of charge Q produces an electric field

$$\mathbf{E} = -\nabla\left(\frac{Q}{4\pi\varepsilon_0 r}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}$$

ε_0 is the *permitivity of free space*, and is equal to approximately $8.85\times10^{-12}\mathsf{m}^{-3}\mathsf{kg}^{-1}\mathsf{s}^2\mathsf{C}^2$. The resulting force on a particle of charge q, $\mathbf{F} = q\mathbf{E}$, is the *Coulomb force*.

$$\mathbf{F} = q\mathbf{E} = \frac{qQ}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}$$

(Note that the signs of q and Q tell you whether the force is attractive or repulsive.)

Motion in Constant $\mathbf{B}$

Set $\mathbf{E} = \mathbf{0}$ and consider $\mathbf{B}(0,0,B)$, $\mathbf{F} = q\dot{\mathbf{x}} \times \mathbf{B}$. In components,

$$m\ddot{x} = qB\dot{y} \tag{1}$$
$$m\ddot{y} = -qB\dot{x} \tag{2}$$
$$m\ddot{z} = 0 \tag{3}$$

so constant velocity in z direction.
Trick: let $\xi = x + iy$, then

$$(1) + i(2) \implies m\ddot{\xi} = -iqB\dot{\xi} \implies \xi = \alpha e^{i\omega t} + \beta$$

with $\omega : +\frac{qB}{m}$, the *cyclotron frequency*. For initial conditions, pick at $t = 0$, $\mathbf{x} = 0$, $\dot{\mathbf{x}} = (0, -v, 0)$, then at $t = 0$, $\xi = 0$ and $\dot{\xi} = -iv$.

$$\implies \xi = \frac{v}{w}(e^{i\omega t} - 1).$$

$$\implies x = \frac{v}{w}(\cos\omega t - 1),$$
$$y = -\frac{v}{w}\sin\omega t$$
$$\implies \left(x + \frac{v}{w}\right)^2 + y^2 = \left(\frac{v}{w}\right)^2.$$

The particle moves around the circle with period $T = \frac{2\pi}{\omega} = \frac{2\pi m}{qB}$.

Note. T is independent of v. But the faster the particle, the bigger the circle.

A Comment on Solving Using Vectors

Sometimes better off keeping everything in vector form.

$$m\ddot{\mathbf{x}} = q\dot{\mathbf{x}} \times \mathbf{B}$$

$$\implies m\ddot{\mathbf{x}} \cdot \mathbf{B} = q(\dot{\mathbf{x}} \times \mathbf{B}) \cdot \mathbf{B} = 0.$$

i.e. constant velocity in the $\mathbf{B}$ direction. This is only one equation. We get two others by taking the cross product of the top equation with $\mathbf{B}$. In this case, it's best to integrate first

$$m\dot{\mathbf{x}} = q\mathbf{x} \times \mathbf{B} + \mathbf{C}$$

now take $\times\mathbf{B}$ of this:

$$m\dot{\mathbf{x}} \times \mathbf{B} = q(\mathbf{x} \times \mathbf{B}) \times \mathbf{B} + \mathbf{C} \times \mathbf{B}$$

and plug into the original equation.

Start of lecture 6

2.5 Friction

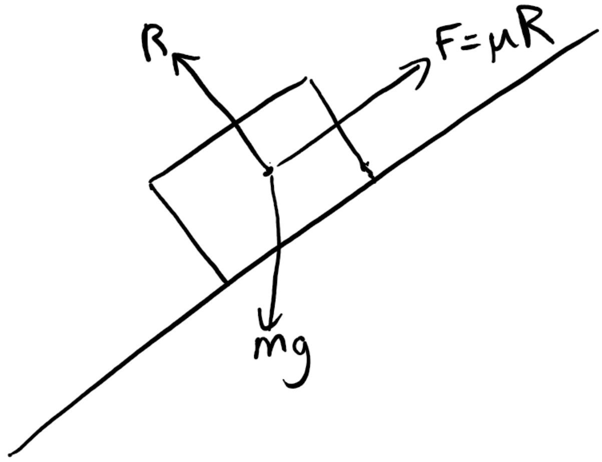
Energy is conserved at the atomic level. But mechanical energy appears not to be conserved in many everyday processes. This is summarised by friction.

Dry Friction

When solid objects are in contact, the friction force is of the form

$$F = \mu R.$$

Here μ is the *coefficient of friction*, and it is a dimensionless constant (no units). For example, the value for aluminium on aluminium is about 0.3. It is an empirical property which depends on the two materials touching. R is the *reaction force*.



Fluid Drag

When an object moves through a fluid (liquid or gas), it experiences *drag*. This is typically of two different kinds:

- Linear drag

  $$\mathbf{F} = -\gamma \mathbf{v}$$

  for some constant γ. Applies for objects moving slowly in a viscous fluid (for a spherical object of radius L, Stokes' formula gives

  $$\gamma = 6\pi\zeta L$$

  where L is the viscosity of fluid).

- Quadratic drag

  $$\mathbf{F} = -\gamma |\mathbf{v}| \mathbf{v}$$

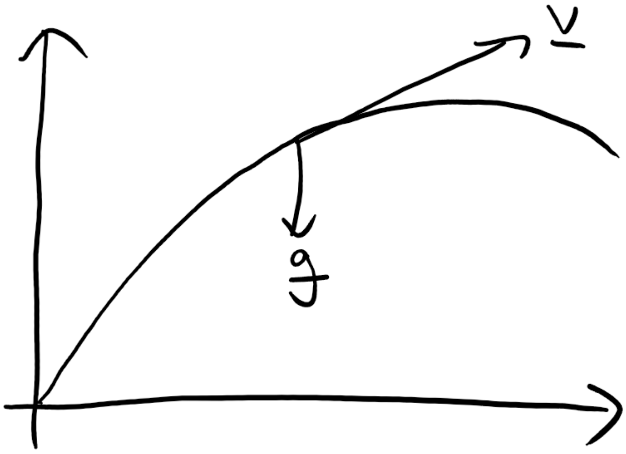
  for some constant γ. Applies for objects moving in less viscous fluids, for example objects falling in air (usually have $\gamma \propto L^2$).

Note. Mechanical energy isn't conserved for either of these forces. (Quadratic drag is easier to understand than linear).

An example

Consider a projectile moving under a constant gravitational force and experiencing linear drag (ball in treacle). At time $t = 0$, throw ball with velocity $\mathbf{u}$. The equation of motion is

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\mathbf{g} - \gamma\mathbf{v}$$



Trick: solve for $\mathbf{v}$ first, then $\mathbf{x}$.
Introduce an integrating factor:

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{\gamma t/m}\mathbf{v}) = e^{\gamma t/m}\mathbf{g}$$

$$\implies e^{\gamma t/m}\mathbf{v} = \frac{m}{\gamma}e^{\gamma t/m}\mathbf{g} + \mathbf{c}$$

where $\mathbf{c}$ is an integration constant. Now using $\mathbf{v} = \mathbf{u}$ at $t = 0$ implies that $\mathbf{c} = \mathbf{u} - \frac{m}{\gamma}\mathbf{g}$. So

$$\dot{\mathbf{x}} = \mathbf{v} = \frac{m\mathbf{g}}{\gamma} + \left(\mathbf{u} - \frac{m}{\gamma}\mathbf{g}\right)e^{-\gamma t/m}$$

$$\implies \mathbf{x} = \frac{m\mathbf{g}t}{\gamma} - \frac{m}{\gamma}\left(\mathbf{u} - \frac{m}{\gamma}\mathbf{g}\right)e^{-\gamma t/m} + \mathbf{b}.$$

(for some other integration constant $\mathbf{b}$). Now using $\mathbf{x} = 0$ at $t = 0$,

$$\implies \mathbf{x} = \frac{m\mathbf{g}t}{\gamma} + \frac{m}{\gamma}\left(\mathbf{u} - \frac{m}{\gamma}gbf\right)(1 - e^{-\gamma t/m}).$$

In components,

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} u\cos\theta \\ 0 \\ u\sin\theta \end{pmatrix} \qquad \mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}$$

$$\implies x = \frac{m}{\gamma} u\cos\theta(1 - e^{-\gamma t/m})$$

Note that it never gets further than $\frac{m}{\gamma} u\cos\theta$.

$$z = -\frac{mgt}{\gamma} + \frac{m}{\gamma}\left(u\sin\theta + \frac{mg}{\gamma}\right)(1 - e^{-\gamma t/m})$$

(units of γ are mass per unit time).

3 Dimensional Analysis

Physical quantities have units, or *dimensions*. In any equation, the units have to be constant / equal in each term. For most problems, it's convenient to introduce three fundamental dimensions

- Length L
- Mass M
- Time T

The dimension $[Y]$ of quantity Y should be expressible in terms of L, M and T. For example

- $[\text{area}] = L^2$
- $[\text{speed}] = LT^{-1}$
- $[\text{acceleration}] = LT^{-2}$
- $[\text{force}] = MLT^{-2}$ (from $F = ma$)
- $[\text{energy}] = ML^2T^{-2}$ (from $E = \frac{1}{2}mv^2$)

Similarly constants can have dimensions. For example

- $[G] = M^{-1}L^2T^{-2}$ (from $F = -\frac{GMm}{r^2}$)

For some problems, one needs even less than three dimensions, and for other problems more (for example charge).

Start of lecture 7

Scaling (Bridgeman's Theorem)

Dimensionful quantities can only appear in equation as powers, for example L^α for some α - can never have more complicated functions. For example

$$e^x = 1 + x + \frac{x^2}{2} + \cdots$$

x must be dimensionless.

Suppose we want to compute a quantity Y with dimensions

$$[Y] = M^\alpha L^\beta T^\gamma.$$

Let's suppose we wish to express Y in terms of other quantities X_i, $i = 1, \ldots, n$. Let's pick for example X_1, X_2, X_3. We assume that they are *dimensionally independent* (i.e. we can build dimensions of length, mass and time from them). Then, we can write

$$Y = cX_1^{a_1}X_2^{a_2}X_3^{a_3}$$

where c is a dimensionless constant. In general it can be a dimensionless combination of X_i. In the case where there are no dimensionless combinations, c is just a number.

Example (Pendulum)**.** We want to know period T. Obviously, $[T] = T$. It can depend upon:

- mass m, $[m] = M$
- length l, $[l] = L$
- gravity g, $[g] = LT^{-2}$
- starting angle θ_0, $[\theta_0] = 0$, since $\theta_0 \equiv \theta_0 + 2\pi \implies [\theta_0] = [2\pi] = 0$.

The only dimensionless quantity is θ_0,

$$\implies T = c(\theta_0) g^{a_1} m^{a_2} l^{a_3}$$

Take dimensions:

$$\begin{aligned} T &= [g^{a_1}][m^{a_2}][l^{a_3}] \\ &= L^{a_1+a_3} T^{-2a_1} M^{a_2} \end{aligned}$$

$$\implies a_1 = -\frac{1}{2}, \qquad a_2 = 0, \qquad a_3 = +\frac{1}{2}.$$

$$\implies T = c(\theta_0)\sqrt{\frac{l}{g}}$$

(often we write $T \sim \sqrt{\frac{l}{g}}$). So if we take two pendulums of different lengths l_1 and l_2, released at angle θ_0, then

$$\frac{T_1}{T_2} = \sqrt{\frac{l_1}{l_2}}.$$

4 Central Forces

Study the three-dimensional motion of a particle obeying

$$m\ddot{\mathbf{x}} = -\nabla V(r); \qquad r := |\mathbf{x}|.$$

This could describe:

- the position of a particle $\mathbf{x}$ in a fixed potential
- the separation $\mathbf{x}$ of two interacting particles where

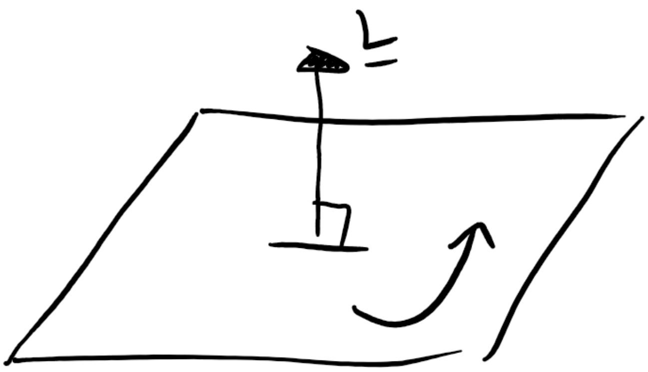
$$m = \frac{m_1 m_2}{m_1 + m_2}.$$

Note. When one particle is very heavy, this second case reduces to the first.

The angular momentum is conserved

$$\mathbf{L} = m\mathbf{x} \times \dot{\mathbf{x}} \implies \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = m\mathbf{x} \times \ddot{\mathbf{x}} = -\mathbf{x} \times \nabla V,$$

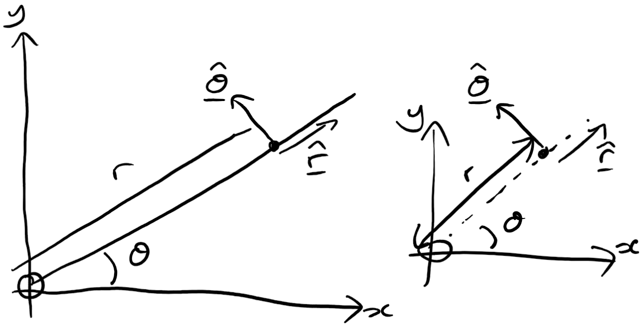
which is 0 since $\mathbf{x} \parallel \nabla V$, so $\mathbf{L}$ is a fixed vector. But, by construction, $\mathbf{L} \cdot \mathbf{x} = 0$. This is the equation for a plane so all motion takes place in a two-dimensional plane perpendicular to $\mathbf{L}$ (note $\mathbf{L} \cdot \mathbf{x} = 0$ too).



4.1 Polar Coordinates in the Plane

A particle sits at coordinates

$$x = r\cos\theta \qquad y = r\sin\theta$$



and we define unit vectors

$$\hat{\mathbf{r}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad \hat{\boldsymbol{\theta}} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

These form an orthonormal basis at each point, but they depend on θ.

$$\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \hat{\boldsymbol{\theta}} \qquad \frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}\theta} = \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} = -\hat{\mathbf{r}}.$$

The position of a particle is $\mathbf{x} = r\hat{\mathbf{r}}$, so the velocity is

$$\dot{\mathbf{x}} = \dot{r}\hat{\mathbf{r}} + r\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}\theta}\dot{\theta} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \tag{4.1}$$

$\dot{\theta}$ is *angular velocity*. The acceleration is

$$\ddot{\mathbf{x}} = \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}\theta}\dot{\theta} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}\theta}\dot{\theta} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \tag{4.2}$$

Example (Circular Motion)**.** A particle moving with constant angular speed in a circle has $\dot{r} = 0$, $\dot{\theta} = \omega$

$$(4.1) \implies \dot{\mathbf{x}} = r\omega\hat{\boldsymbol{\theta}}$$

and so the speed is $|\dot{\mathbf{x}}| = r\omega$. The acceleration is

$$(4.2) \implies \ddot{\mathbf{x}} = -r\omega^2\hat{\mathbf{r}} \implies a = |\ddot{\mathbf{x}}| = r\omega^2.$$

If we want a particle to move like this, Newton's second law states that we must supply a *centripetal force* toward to the origin.

Start of lecture 8

4.2 Back to Central Forces

Since $V = V(r) \implies \nabla V = \frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}}$. From Newton's second law and (4.2) we have

$$m(\ddot{r} - r\dot{\theta})\hat{\mathbf{r}} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}}$$

equate $\hat{\boldsymbol{\theta}}$ components

$$\implies r\ddot\theta + 2\dot r\dot\theta = 0$$

$$\iff \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot\theta) = 0$$

$$\iff l = r^2\theta$$

is conserved. This is actually the magnitude (perhaps with a sign) of the angular momentum per unit mass. To see this, look at

$$\mathbf{L} = m\mathbf{x}\times\dot{\mathbf{x}} \stackrel{(4.1)}{=} mr\hat{\mathbf{r}}\times(\cancel{\dot r\hat{\mathbf{r}}} + r\dot\theta\hat{\boldsymbol{\theta}}) = mr^2\dot\theta(\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}).$$

Equate $\hat{\mathbf{r}}$ components: $m(\ddot r - r\dot\theta^2) = -\frac{\mathrm{d}V}{\mathrm{d}r}$.
Replace $\dot\theta$ with l:

$$\implies m\ddot r = -\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{ml^2}{r^3} : + -\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r}$$

where V_{eff} is the "*effective potential*":

$$V_{\text{eff}}(R) = V(r) + \frac{ml^2}{2r^2}$$

($\frac{ml^2}{2r^2}$ is called the *angular momentum barrier.*)

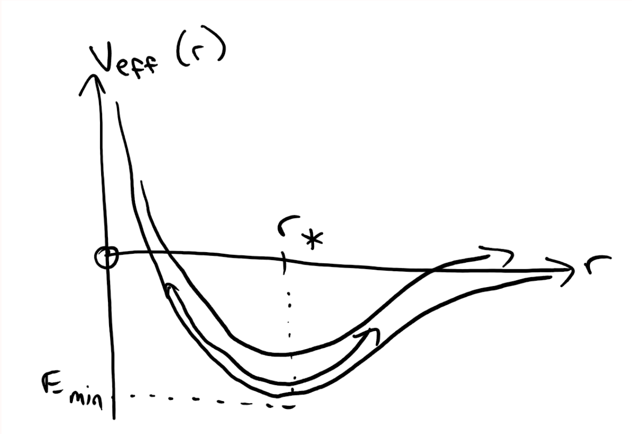
This problem has been reduced to the kind of one-dimensional problem we looked at before. The energy of the particle is

$$\begin{aligned}
E &= \frac{1}{2}m\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} + V(r)\\
&= \frac{1}{2}m\dot r^2 + \frac{1}{2}mr^2\dot\theta^2 + V(r)\\
&= \frac{1}{2}m\dot r^2 + \frac{ml^2}{2r^2} + V(r)\\
&= \frac{1}{2}m\dot r^2 + V_{\text{eff}}(r)
\end{aligned}$$

Example (Inverse Square Law)**.**

$$V = -\frac{k}{r} \implies V_{\text{eff}} = -\frac{k}{r} + \frac{ml^2}{2r^2}.$$

$(k > 0)$



The minimum is at $r_* = \frac{ml^2}{k}$

$$\implies E_{\text{min}} = V_{\text{eff}}(r_*) = -\frac{k^2}{2ml^2}.$$

Kinds of motion:

$E = E_{\text{min}} \implies$ particle sits at $r = r_*$. But

$$\dot{\theta} = \frac{l}{r^2} \implies \text{particle moves in a circle at constant angular speed.}$$

- $E_{\text{min}} < E < 0 \implies$ particle oscillates back and forth. Meanwhile, $\dot{\theta} = \frac{l}{r^2}$ also changes. This is a non-circular orbit. (Definitions: The smallest value of r reached by the particle is the *periapsis* (*perehelion* for orbiting the sun). The furthest distance is the *apoapsis* (*aphelion* for the sun).)
- $E > 0 \implies$ this motion is not an orbit. The particle escapes to ∞.

Stability of Circular Orbits

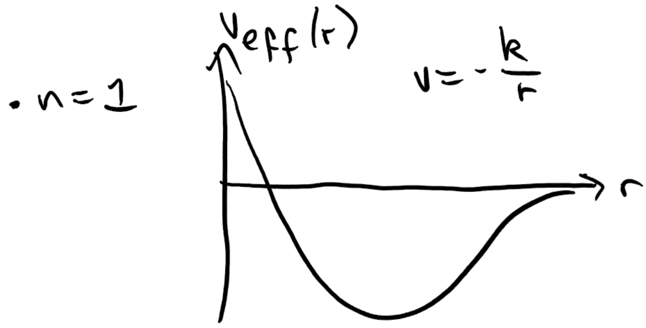
Circular orbits are equilibrium points, r_*, of V_{eff}:

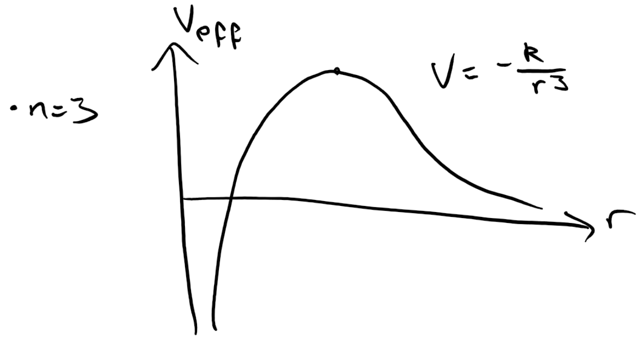
$$V'_{\text{eff}}(r_*) = 0.$$

They are stable if it's a minimum, i.e. $V''_{\text{eff}}(r_*) > 0$.

For $V(r) = -\frac{k}{r^n}$ for $n \geq 1$

$$\implies V_{\text{eff}} = -\frac{k}{r^n} + \frac{ml^2}{2r^2}.$$





Can check that circular orbits are stable for $(n+1) - 3 < 0 \implies n < 2$. (In a univserse with d space dimensions, potential energy of gravity is $V \sim \frac{1}{r^{d-2}} \implies$ circular orbits not stable in $d > 3$).

The Orbit Equations

If we want to compute $r(t)$, we use

$$E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r).$$

and integrate

$$\implies t = \pm\sqrt{\frac{m}{2}} \int \frac{\mathrm{d}r}{\sqrt{E - V_{\text{eff}}(r)}}$$

Here, we'll instead try to understand the shape of the trajectory by computing $r(\theta)$ (can then use $l = r^2\dot{\theta}$ to get $\theta(t)$ and $r(t)$).
A trick: define $u = \frac{1}{r}$. If $r = r(\theta)$ then

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\dot{\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{l}{r^2} = -l\frac{\mathrm{d}u}{\mathrm{d}\theta}$$

and

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-l\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) = -l\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2}\dot{\theta} = -l^2\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2}\frac{1}{r^2} = -l^2u^2\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2}$$

The equation of motion is

$$m\ddot{r} - \frac{ml^2}{r^3} = -\frac{V}{r} = F(r)$$

$$-ml^2u^2\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} - ml^2u^3 = F\left(\frac{1}{u}\right)$$

$$\implies \boxed{\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = -\frac{1}{ml^2u^2}F\left(\frac{1}{u}\right)}$$

(this is known as the *orbit equation*).

Start of lecture 9

The Kepler Problem

$$V(r) = -\frac{km}{r} \implies F(r) = -\frac{km}{r^2}$$

$k = GM$ for gravity, $k = -\frac{qQ}{4\pi\varepsilon_0}$ for Coulomb force.
The orbit equation is

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{k}{l^2}$$

This is a harmonic oscillator with a displaced centre:

$$u = A\cos(\theta - \theta_0) + \frac{k}{l^2}$$

Pick $A > 0$.
We choose our polar coordinates such that the closest point (periapsis) occurs at $\theta = 0 \implies \theta_0 = 0$.

$$\implies r = \frac{r_0}{e\cos\theta + 1}$$

where

$$r_0 := \frac{l^2}{k} \qquad \text{and} \qquad e :+ \frac{Al^2}{k}$$

This is the equation of a *conic section*. e is the *eccentricity*. First look at attractive forces ($k > 0$):

<u>$e < 1$ ellipses:</u>

$$\frac{r}{r_0} \in \left[\frac{1}{1+e}, \frac{1}{1-e}\right]$$

$\implies r$ is positive and bounded.

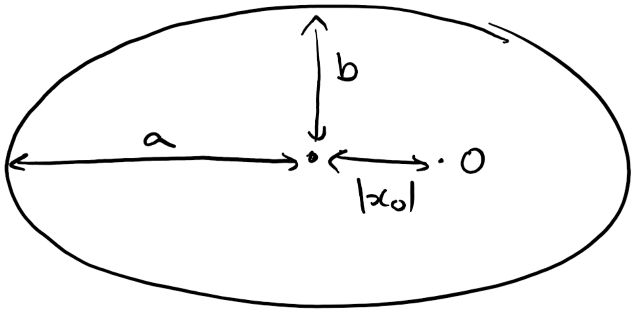
$$r = r_0 - re\cos\theta$$

$$\implies x^2 + y^2 = (r_0 - ex)^2$$

$$\implies \cdots$$
$$\implies \frac{(x - x_0)^2}{a^2} + \frac{y^2}{b^2} = 1$$

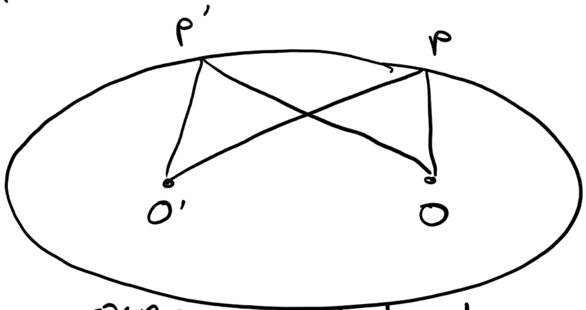
which is the formula of an ellipse where

$$x_0 := -\frac{er_0}{1 - e^2}$$
$$a^2 := \frac{r_0^2}{(1 - e^2)^2}$$
$$b^2 := \frac{r_0^2}{1 - e^2} < a^2$$



O is the origin and sits at a focus of the ellipse, a distance $|x_0| = ea$ from the centre. When $e = 0$, $a = b$ and ellipse $\to$ circle.

Aside:



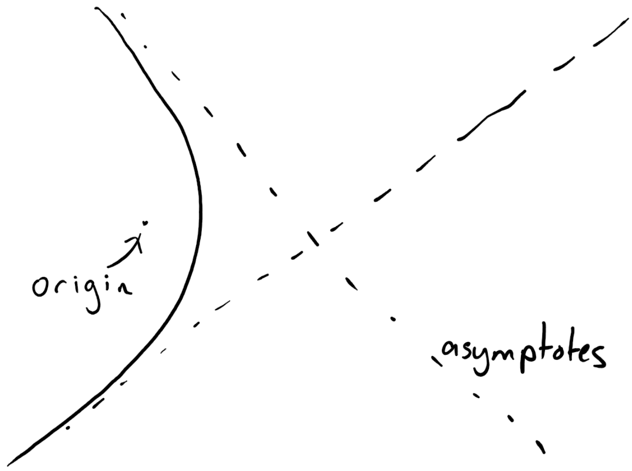
O'PO= equidistant.
- proof by messy algebra

<u>$e > 1$ hyperbolae:</u> $r \to \infty$ as $\cos\theta \to -\frac{1}{e}$. Repeat steps above:

$$\implies \frac{1}{a^2}\left(x - \frac{r_0 e}{e^2 - 1}\right)^2 - \frac{y^2}{b^2} = 1$$

with

$$a^2 = \frac{r_0^2}{(e^2-1)^2}, \qquad b^2 = \frac{r_0^2}{e^2-1}.$$



The Energy of the Orbit

$$\begin{aligned} E &= \frac{1}{2}m\dot{r}^2 + \frac{ml^2}{2r^2} - \frac{km}{r} \\ &= \frac{1}{2}m\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \frac{l^2}{r^4} + \frac{ml^2}{2r^2} - \frac{km}{r} \\ &= \cdots \\ &= \frac{1}{2l^2}mk^2(e^2-1) \end{aligned}$$

$$\left(\dot{r} = -\frac{l}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)$$

- $e < 1 \implies E < 0$, bound orbits.
- $e > 1 \implies E > 0$, unbound orbit.

A repulsive force:

$$V = -\frac{km}{r}$$

$k > 0$ is attractive, $k < 0$ is repulsive. We get

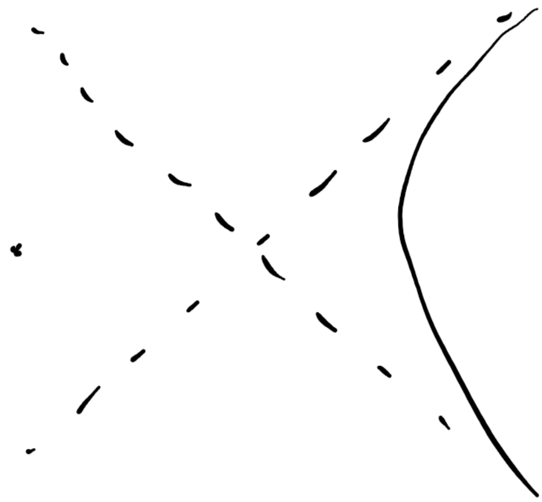
$$r = \frac{|r_0|}{e\cos\theta - 1}$$

with

$$|r_0| = \frac{l^2}{|k|}$$

$$e = \frac{Al^2}{|k|}$$

but $r > 0 \implies e > 1$ for solutions to make sense. Solutions now look like



$r \to \infty$ at $\cos\theta = +\frac{1}{e}$ i.e. $\theta < \frac{\pi}{2}$.

Kepler's Laws of Planetary Motion

In 1605, Kepler stated three laws obeyed by all planets (in the solar system).

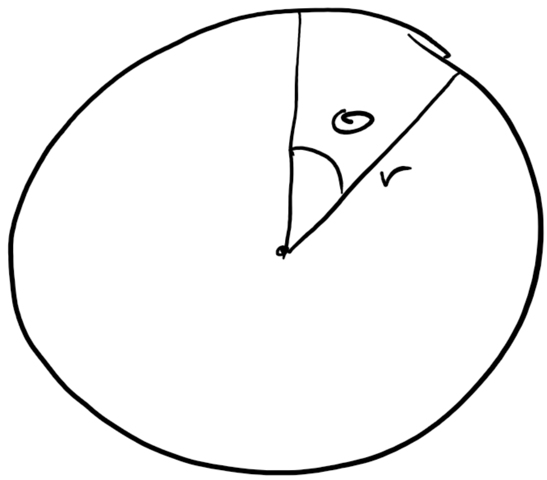
K1 Each planet moves on an ellipse with the sun at one focus.

K2 The line between the planet and the sun sweeps out equal areas in equal times.

K3 The period of the orbit is proportional to $(\text{radius})^{3/2}$.

Let's see how they follow from Newton's laws:

K2 follows from conservation of angular momentum.



$$\delta A = \frac{1}{2} r^2 \delta\theta$$

$$\implies \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2} r^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{2} l$$

is constant. This then is true for any central force.

K3 holds on dimensional grounds if we assume the inverse square law:

$$F = -\frac{GMm}{r^2} \implies [GM] = L^3 T^{-2}$$

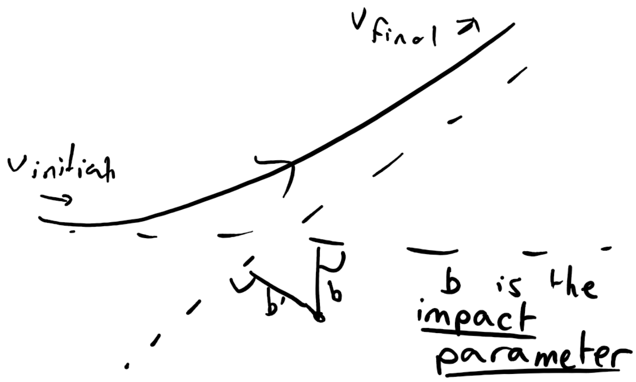
so if R is some definition of radius, the period T must obey $T^2 \propto \frac{R^3}{GM}$.

K1 requires the full solution to inverse square law.

Start of lecture 10

Scattering

Consider a potential $V(r)$ such that $V(r) \to 0$ as $r \to \infty$. For a repulsive potential, the trajectory will look something like



Conservation of energy implies

$$V_{\text{initial}} = v_{\text{final}} := v$$

In a central potential, we have conservation of angular momentum, i.e.

$$l = |\mathbf{x} \times \dot{\mathbf{x}}| = \frac{|\mathbf{L}|}{m}$$

is conserved.

Claim.

$$l_{\text{initial}} = bv_{\text{initial}}$$

Proof. Suppose $V = 0$. The particle follows a straight horizontal line. At the closest point to the origin, its angular momentum per unit mass is bv_{initial}, but by conservation of l, this must be the same as $l_{\text{initial}} = bv_{\text{initial}}$. □

In the presence of V i.e. the potential, we have

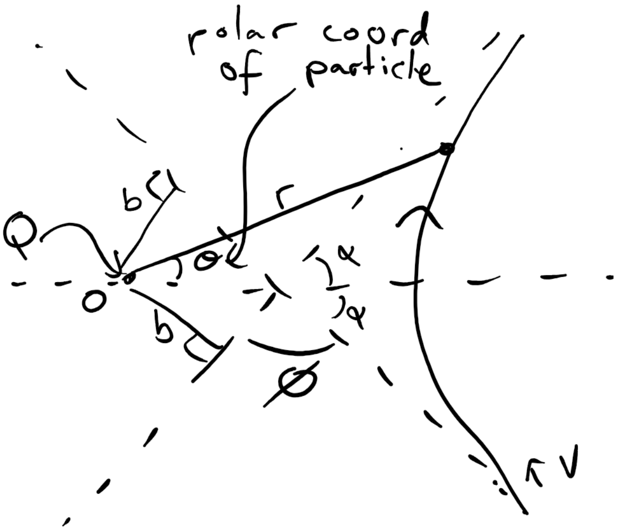
$$l_{\text{initial}} = l_{\text{final}} = b' v_{\text{final}} \implies b = b'$$

Rutherford Scattering

A repulsive Coulomb potential

$$V = \frac{qQ}{4\pi\varepsilon_0 r}$$

(replace $k = -\frac{qQ}{4\pi\varepsilon_0 m}$)



ϕ is the *scattering angle*. In the short term though, it's more useful to consider α. ($\phi = \pi - 2\alpha$)

Question: How does ϕ depend on v and b?
Recall from last lecture for a repulsive potential

$$r = \frac{|r_0|}{e\cos\theta - 1} \to \infty$$

when $\theta = \alpha \implies \cos\alpha = \frac{1}{e}$. ($e > 1 \implies \alpha < \frac{\pi}{2}$)
Energy of particle (last lecture)

$$\begin{aligned} E &= \frac{mk^2}{2l^2}(e^2 - 1) \\ &= \frac{1}{2}mv^2 \\ &= \frac{mk^2}{2l^2}\tan^2\alpha \end{aligned}$$

sub in $l = bv$:

$$\implies \frac{1}{\tan^2\alpha} = \frac{k^2}{b^2v^4} \qquad \text{or} \qquad \frac{1}{\tan\alpha} = \frac{|k|}{bv^2}$$

Note.

$$\tan\frac{\phi}{2} = \tan\left(\frac{\pi}{2} - \alpha\right) = \frac{1}{\tan\alpha}$$

$$\implies \boxed{\phi = 2\tan^{-1}\left(\frac{|k|}{bv^2}\right)}$$

This is the scattering formula.

5 Systems of Particles

So far we've only covered the motion of a single particle. We now describe N interacting particles. Put a label $i = 1, \ldots, N$ on everything. The i-th particle has mass m_i, position $\mathbf{x}_i$ and momentum $\mathbf{p}_i$.
For each particle,

N2 is $\dot{\mathbf{p}}_i = \mathbf{F}_i$, force on i-th particle. The force can be split into two parts:

$$\mathbf{F}_i = \mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{ij}$$

where $\mathbf{F}_i^{\text{ext}}$ is an external force on the i-th particle, and $\mathbf{F}_{ij}$ is the force on the i-th particle due to the j-th particle.

N3 is $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$. This holds for gravitational and Coulomb forces.

5.1 Centre of Mass Motion

The total mass of the system $M = \sum_i m_i$. The *centre of mass* is defined as

$$\mathbf{R} := \frac{1}{M} \sum_{i=1}^{N} m_i \mathbf{x}_i$$

The total momentum of the system is captured by the centre of mass

$$\underline{\mathbf{P}} := \sum_{i=1}^{N} \mathbf{p}_i = M\dot{\mathbf{R}}$$

$$\begin{aligned}
\dot{\underline{\mathbf{P}}} &= \sum_i \dot{\mathbf{p}}_i \\
&= \sum_i \left(\mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{ij} \right) \\
&= \sum_i \mathbf{F}_i^{\text{ext}} + \sum_{i<j} (\mathbf{F}_{ij} + \mathbf{F}_{ji}) \\
&= \sum_i \mathbf{F}_i^{\text{ext}}
\end{aligned}$$

This is important: the motion of the centre of mass is only due to the external forces.

Conservation of Momentum

If $\sum_i \mathbf{F}_i^{\text{ext}} = 0$, $\underline{\mathbf{P}}$ is conserved (i.e. $\dot{\underline{\mathbf{P}}} = 0$).

Start of lecture 11

Conservation of Angular Momentum

Total angular momentum

$$\mathbf{L} = \sum_i \mathbf{x}_i \times \mathbf{p}_i$$

$$\begin{aligned}
\implies \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} &= \sum_i \mathbf{x}_i \times \dot{\mathbf{p}}_i \\
&= \sum_i \mathbf{x}_i \times \left(\mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{ij} \right) \\
&= \boldsymbol{\tau} + \sum_i \sum_{j \neq i} \mathbf{x}_i \times \mathbf{F}_{ij}
\end{aligned}$$

where

$$\boldsymbol{\tau} := \sum_i \mathbf{x}_i \times \mathbf{F}_i^{\text{ext}}$$

is the *total external torque.*
We write the second term as

$$\begin{aligned}
\sum_i \sum_{j \neq i} \mathbf{x}_i \times \mathbf{F}_{ij} &= \sum_{i<j} \mathbf{x}_i \times \mathbf{F}_{ij} + \sum_{j<i} \mathbf{x}_i \times \mathbf{F}_{ij} \\
&= \sum_{i<j} \mathbf{x}_i \times \mathbf{F}_{ij} + \sum_{j<i} \mathbf{x}_i \times (-\mathbf{F}_{ji}) \\
&= \sum_{i<j} \mathbf{x}_i \times \mathbf{F}_{ij} + \sum_{i<j} \mathbf{x}_j \times (-\mathbf{F}_{ij}) \\
&= \sum_{i<j} (\mathbf{x}_i - \mathbf{x}_j) \times \mathbf{F}_{ij}
\end{aligned}$$

which is 0 if $\mathbf{F}_{ij}$ is parallel to $(\mathbf{x} - \mathbf{x}_j)$, for example in gravity and electromagnetism.

This is a stronger version of Newton's third law:

N3' $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ and is parallel to $(\mathbf{x}_i - \mathbf{x}_j)$.

If N3' holds, then $\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \boldsymbol{\tau}$.

Conservation of Energy

The kinetic energy

$$T = \frac{1}{2} \sum_i m_i \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_i$$

We write

$$\mathbf{x}_i = \mathbf{R} + \mathbf{y}_i$$

where $\mathbf{R}$ is the position of the centre of mass. Since

$$\sum_i m_i \mathbf{x}_i = M\mathbf{R}$$

$$\implies \sum_i m_i \mathbf{y}_i = 0$$

Then

$$\begin{aligned}
T &= \frac{1}{2}\sum_i m_i(\dot{\mathbf{R}} + \dot{\mathbf{y}}_i)^2 \\
&= \frac{1}{2}\sum_i m_i \dot{\mathbf{R}}^2 + \dot{\mathbf{R}} \cdot \cancel{\sum_i m_i \dot{\mathbf{y}}_i} + \frac{1}{2}\sum_i m_i \dot{\mathbf{y}}_i \cdot \dot{\mathbf{y}}_i \\
&= \frac{1}{2}\sum_i m_i \dot{\mathbf{R}}^2 + \frac{1}{2}\sum_i m_i \dot{\mathbf{y}}_i \cdot \dot{\mathbf{y}} \\
&= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\sum_i m_i \dot{\mathbf{y}}^2
\end{aligned}$$

where $\frac{1}{2}M\dot{\mathbf{R}}^2$ is the energy of the centre of mass, and $\frac{1}{2}\sum_i m_i \dot{\mathbf{y}}_i^2$ is the energy due to motion around the centre of mass.
We repeat the analysis of work done in 3D (section 2.2). The i-th particle moves on a trajectory C_i. Then

$$T(t_2) - T(t_1) = \sum_i \int_{C_i} \mathbf{F}_i^{\text{ext}} \cdot \mathrm{d}\mathbf{x}_i + \sum_i \sum_{i \neq j} \int_{C_i} \mathbf{F}_{ij} \cdot \mathrm{d}\mathbf{x}_i$$

We can define a potential energy if

- $\mathbf{F}_i^{\text{ext}} = -\nabla V_i(\mathbf{x}_i)$ (summation convention is *not* used here)
- $\mathbf{F}_{ij} = -\nabla V_{ij}(|\mathbf{x}_i - \mathbf{x}_j|)$ (summation convention is *not* used here)

Remember $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. The energy

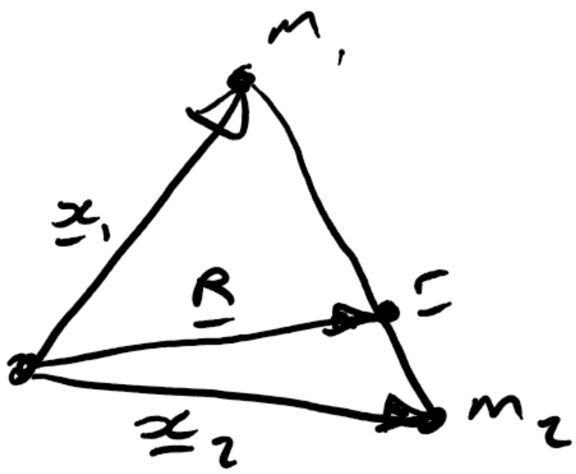
$$E = T + \sum_i V_i(\mathbf{x}_i) + \sum_i \sum_{j \neq i} V_{ij}(|x_i - x_j|)$$

is conserved.

The 2 Body Problem as a 1 Body Problem

Consider 2 particles with $\mathbf{F}_i^{\text{ext}} = 0$. The centre of mass $M\mathbf{R} = m_1\mathbf{x}_1 + m_2\mathbf{x}_2$ and the relative separation is

$$\begin{aligned}
\mathbf{r} &= \mathbf{x}_1 - \mathbf{x}_2 \\
\mathbf{x}_1 &= \mathbf{R} + \frac{m_2}{M}\mathbf{r} \\
\mathbf{x}_2 &= \mathbf{R} - \frac{m_1}{M}\mathbf{r}.
\end{aligned}$$



Since $\mathbf{F}_i^{\text{ext}} = 0$, $\ddot{\mathbf{R}} = 0$ (see last lecture). Meanwhile

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot{\mathbf{x}}_1 - \ddot{\mathbf{x}}_2 \\ &= \frac{1}{m_1}\mathbf{F}_{12} - \frac{1}{m_2}\mathbf{F}_{21} \\ &= \frac{m_1 + m_2}{m_1 m_2}\mathbf{F}_{12} \\ \implies \mu\ddot{\mathbf{r}} &= \mathbf{F}_{12}(r)\end{aligned}$$

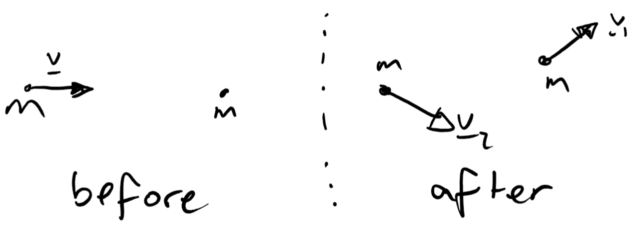
where μ is the *reduced mass*

$$\mu := \frac{m_1 m_2}{m_1 + m_2}$$

This is the same as the equation for one particle of mass μ and position $\mathbf{r}$.

Collisions

elastic collisions are ones in which both kinetic energy and momentum are conserved - these come from conservative inter-particle forces between the two colliding particles. Take



conservation of energy

$$\frac{1}{2}m\mathbf{v}^2 = \frac{1}{2}m\mathbf{v}_1^2 + \frac{1}{2}m\mathbf{v}_2^2$$

conservation of momentum

$$m\mathbf{v} = m\mathbf{v}_1 + m\mathbf{v}_2$$

hence

$$\mathbf{v}^2 = \mathbf{v}_1^2 + \mathbf{v}_2^2 + 2\mathbf{v}_1 \cdot \mathbf{v}_2$$

Then combining with conservation of energy gives

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0.$$

i.e. one of the final state particles is stationary or they travel at right angles.

Note. We have some information about the final state from conservation laws but not a unique outcome.
Not surprising: $\mathbf{v}_1, \mathbf{v}_2$ have 6 parameters, but 4 constraint equations: 1 energy, 3 momentum.

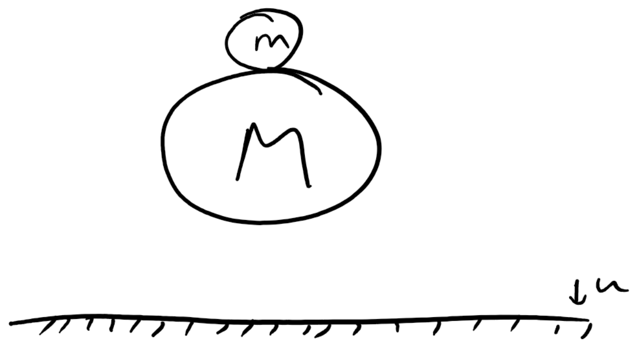
Impulse

When particles are subject to short, sharp shocks (for example in collisions) one talks of impulse rather than force. If a force acts for a short time Δt, the *impulse* experienced by the particle is

$$\mathbf{I} = \int_t^{t+\Delta t} \mathbf{F} \mathrm{d}t \overset{(N2)}{=} \Delta \mathbf{p}$$

Bouncing Balls

2 equations, 2 unknowns: after collision, small ball has speed v and big ball has speed V.



Best to think of balls as slightly separated. Conservation of E:

$$mu^2 + Mu^2 = mv^2 + MV^2$$

Conservation of p:

$$Mu - mu = mv + MV$$

One can check David Tong's note for the full solution, but one can solve to get

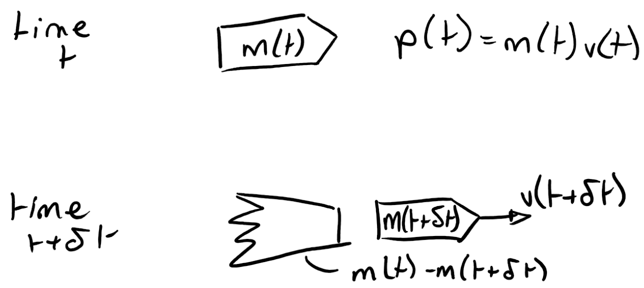
$$\boxed{v = \frac{3M - m}{M + m} u}$$

Start of lecture 12

Variable Mass Problem

N2 really is $\mathbf{F} = \dot{\mathbf{p}}$ where $p = m\dot{\mathbf{x}}$ is momentum. This coincides with the more familiar $\mathbf{F} = m\ddot{\mathbf{x}}$ only when m doesn't change with t, for example where it *does*: *Rockets.*

A rocket moves in a straight line with velocity $\mathbf{v}(t)$. The mass of the rocket $m(t)$ changes with time because it propels itself forward by spitting fuel out behind.
Suppose the fuel is emitted at speed u relative to the rocket.
Question: what is $v(t)$?
analysis



momentum conserved (later)

$$p(t + \delta t) = p_{\text{rocket}}(t + \delta t) + p_{\text{fuel}}(t + \delta t)$$

$$\begin{aligned}
p_{\text{rocket}}(t + \delta t) &= m(t + \delta t)v(t + \delta t) \\
&= \left(m(T) + \delta t\frac{\mathrm{d}m}{\mathrm{d}t}\right)\left(v(t) + \delta t\frac{\mathrm{d}v}{\mathrm{d}t}\right) + O(\delta t^2) \\
&= m(t)v(t) + \delta t\left(m\frac{\mathrm{d}v}{\mathrm{d}t} + v\frac{\mathrm{d}m}{\mathrm{d}t}\right) + O(\delta t^2) \\
p_{\text{fuel}} &= [m(t) - m(t + \delta t)][v(t + \delta t) - u] \\
&= -\delta\frac{\mathrm{d}m}{\mathrm{d}t}[v(t) - u] + O(\delta t^2)
\end{aligned}$$

substitute in

$$p(t+\delta t) = p(t) + \left(m\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}m}{\mathrm{d}t}\right)\delta t + O(\delta t^2)$$

N2:

$$\lim_{\delta t \to 0} \frac{p(t+\delta t) - p(t)}{\delta t} = \boxed{F = m(t)\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}m}{\mathrm{d}t}}$$

(the boxed part is Tsiolkovsky rocket equation (1903)). For example rocket in deep space (i.e. $F = 0$) Here, the Tsiolkovsky equation becomes

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{u}{m(t)}\frac{\mathrm{d}m}{\mathrm{d}t}$$

integrate

$$\implies v(t) = v_0 + u\ln\left(\frac{m_0}{m(t)}\right)$$

where the rocket (and fuel inside) has mass m_0 when its speed is v_0.

Note. burning fuel only increases speed logarithmically!

We shall assume that the fuel is burnt at a constant rate:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\alpha \implies m(t) = m_0 - \alpha t$$

Note. $\alpha > 0 \implies \frac{\mathrm{d}m}{\mathrm{d}t} < 0$.

sub in:

$$v(t) = v_0 - u\ln\left(1 - \frac{\alpha t}{m_0}\right)$$

Note. Only makes sense for $t < \frac{m_0}{\alpha}$, since at $t = \frac{m_0}{\alpha}$, all of the mass in the rocket is burnt, including the rocket ($m = 0$). Assuming this, integrate once more to yield the distance travelled:

$$x = v_0 t + \frac{u m_0}{\alpha}\left[\left(1 - \frac{\alpha t}{m_0}\right)\ln\left(1 - \frac{\alpha t}{m_0}\right) + \frac{\alpha t}{m_0}\right].$$

6 Rigid Bodies

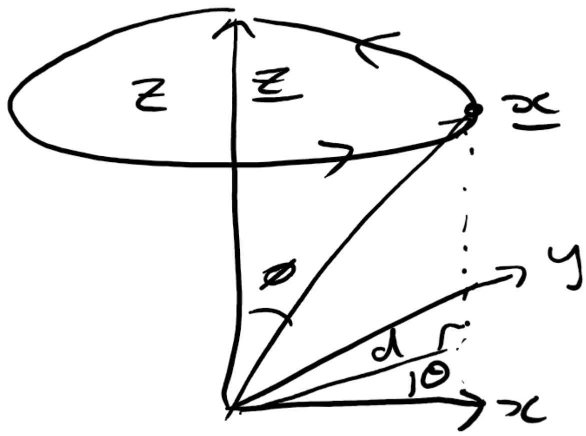
A *rigid body* is an extended object comprised of N particles that are constrained so that the relative distance between any two, $|\mathbf{x}_i - \mathbf{x}_j|$, is fixed.
A rigid body can undergo two types of motion:

- its centre of mass can move
- it can rotate

We'll begin with rotations of rigid bodies.

6.1 Angular Velocity

Consider a single particle rotating in a circle around the z-axis.



The position is

$$\mathbf{x} = (d\cos\theta, d\sin\theta, z)$$

$$\implies \dot{\mathbf{x}} = (-\dot{\theta}d\sin\theta, \dot{\theta}d\cos\theta, 0)$$

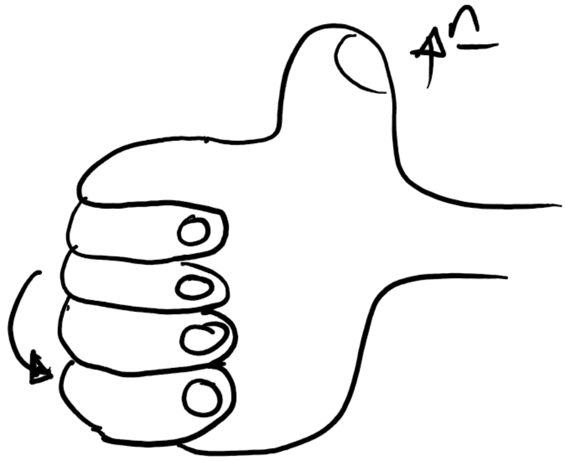
We can write $\dot{\mathbf{x}}$ by introducing a new vector

$$\boldsymbol{\omega} = \dot{\theta}\hat{\mathbf{z}} \qquad \text{and} \qquad \dot{\mathbf{x}} = \boldsymbol{\omega} \times \mathbf{x}$$

we call $\boldsymbol{\omega}$ the *angular velocity.* In general, we write

$$\boldsymbol{\omega} = \omega\hat{\mathbf{n}}$$

where ω is the angular speed, $|\dot{\theta}|$ and $\hat{\mathbf{n}}$ is the unit vector in direction of motion in the right-handed sense.



(curl fingers of right hand in direction of motion and the thumb points at $\hat{\mathbf{n}}$)
The speed of the particle is $v = |\dot{\mathbf{x}}| = \omega d$ where $d = |\hat{\mathbf{n}}\times\mathbf{x}| = |\mathbf{x}|\sin\phi$ is the perpendicular distance to the axis of rotation. The kinetic energy of the particle is

$$\begin{aligned} T &= \frac{1}{2}m\dot{\mathbf{x}}^2 \\ &= \frac{1}{2}m(\boldsymbol{\omega}\times\mathbf{x})\cdot(\boldsymbol{\omega}\times\mathbf{x}) \\ &= \frac{1}{2}md^2\omega^2 \end{aligned}$$

Moment of Inertia

For a rigid body, all N particles rotate with the same angular velocity, so

$$\dot{\mathbf{x}}_i = \boldsymbol{\omega}\times\mathbf{x}_i \qquad i = 1,\dots,N$$

This ensures that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x}_i - \mathbf{x}_j)^2 &= 2(\dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j)\cdot(\mathbf{x}_i - \mathbf{x}_j) \\ &= 2[\boldsymbol{\omega}\times(\mathbf{x}_i - \mathbf{x}_j)]\cdot(\mathbf{x}_i - \mathbf{x}_j) \\ &= 0 \end{aligned}$$

The kinetic energy is

$$T = \frac{1}{2}\sum_i m_i\dot{\mathbf{x}}_i\cdot\dot{\mathbf{x}}_i := \frac{1}{2}\mathbf{T}\omega^2$$

where

$$\mathbf{T} := \sum_{i=1}^{N} m_i d_i^2$$

is the *moment of inertia* of the body.

Start of lecture 13

$$I = \sum_{i=1}^{N} m_i d_i^2$$

moment of inertia.

The angular momentum of the body is

$$L = \sum_i m_i \mathbf{x}_i \times \dot{\mathbf{x}}_i = \sum_i m_i \mathbf{x}_i \times (\boldsymbol{\omega} \times \mathbf{x}_i)$$

For $\boldsymbol{\omega} = \omega \hat{\mathbf{n}}$ ($\hat{\mathbf{n}}$ is unit vector),

$$\begin{aligned} \mathbf{L} \cdot \hat{\mathbf{n}} &= \omega \sum_i m_I (\mathbf{x}_i \times (\hat{\mathbf{n}} \times \mathbf{x}_i)) \cdot \hat{\mathbf{n}} \\ &= \omega \sum_i m_i |\mathbf{x}_i \times \hat{\mathbf{n}}|^2 \end{aligned}$$

the identity can be proven using components:

$$\begin{aligned} \mathbf{x} \times (\hat{\mathbf{n}} \times \mathbf{x}) \cdot \hat{\mathbf{n}} &= \varepsilon_{lkm} x_l \varepsilon ijk \hat{n}_i x_j n_k \\ &= (\varepsilon_{mlk} \hat{n}_m x_l)(\varepsilon_{ijk} \hat{n}_i x_j) \\ &= (\mathbf{x} \times \hat{\mathbf{n}}) \cdot (\mathbf{x} \times \hat{\mathbf{n}}) \end{aligned}$$

but $d_i = |\mathbf{x}_i \times \hat{\mathbf{n}}|$, so

$$\mathbf{L} \cdot \hat{\mathbf{n}} = I\omega$$

Recall that $\dot{\mathbf{L}} = \boldsymbol{\tau}$, the torque. If $\boldsymbol{\tau} = \tau \hat{\mathbf{n}}$, then $\tau = I\dot{\omega}$.

Calculating the Moment of Inertia

We often treat rigid bodies as continuous , with density distribution $\rho(\mathbf{x})$ instead of discrete masses m_i. Then, the mass is

$$M = \int \rho(\mathbf{x}) \mathrm{d}V$$

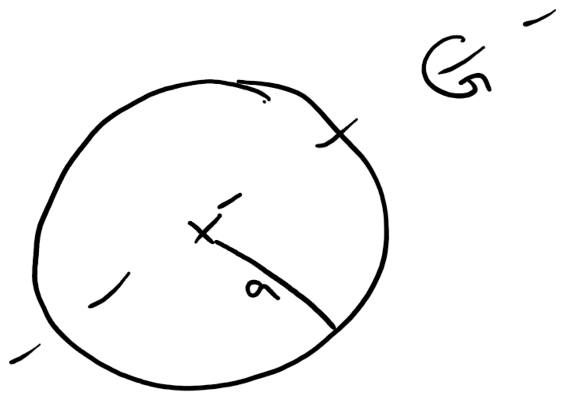
and the moment of inertia is

$$I = \int \rho(\mathbf{x}) x_\perp^2 \mathrm{d}V$$

where $x_\perp = x \sin\phi$ is the perpendicular distance to the axis of rotation.

Examples

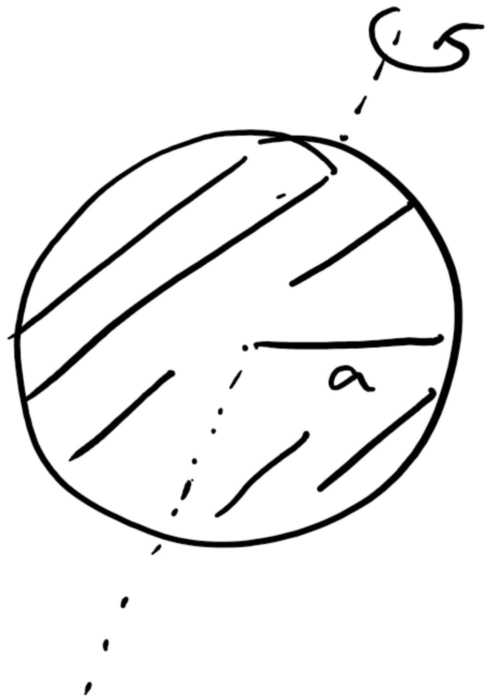
- Circular hoop, axis of rotation through centre and perpendicular to the hoop. Mass M, radius a.



$$I = \int \rho a^2 \mathrm{d}V = a^2 \int \rho(\mathbf{x}) \mathrm{d}V = a^2 M$$

- Disc radius a.

  (a) axis through centre, perpendicular to plane of disc (spinning plate).

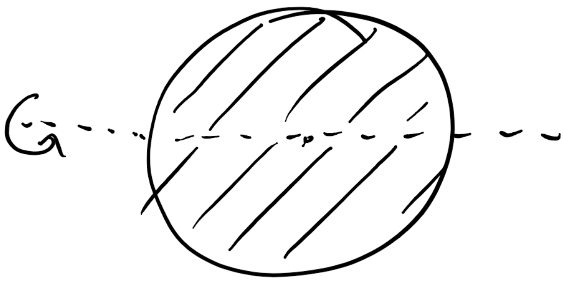


$$M = \pi a^2 \rho$$

(from now on, put $\rho(\mathbf{x}) = \rho$, a constant within the body).

$$I = \int_0^a r\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \rho r^2 = \frac{1}{2} M a^2$$

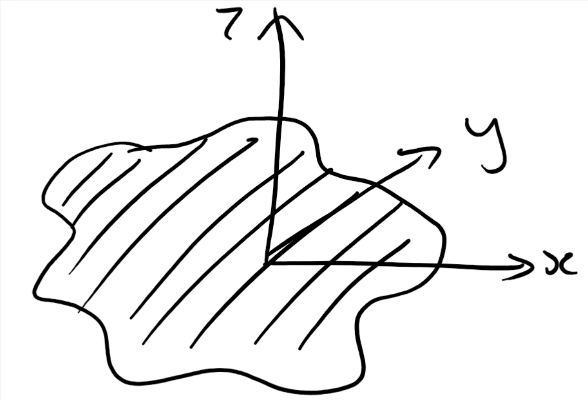
(b) axis of rotation through centre, in the plane of the disc (coin flip).



$$I = \int_0^a r\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \rho (r \sin\theta)^2 = \frac{1}{4} M a^2$$

Definition. 2 dimensional objects such as the disc are called *laminas.* (a) and (b) illustrate a general fact about laminas. If we take the z-axis perpendicular to the lamina, we obtain the...

Theorem (Perpendicular Axis Theorem)**.** Consider a 2 dimensional object lying in the xy-plane.



$$I_x = \int \rho y^2 \mathrm{d}A$$

$$I_y = \int \rho x^2 \mathrm{d}A$$

while

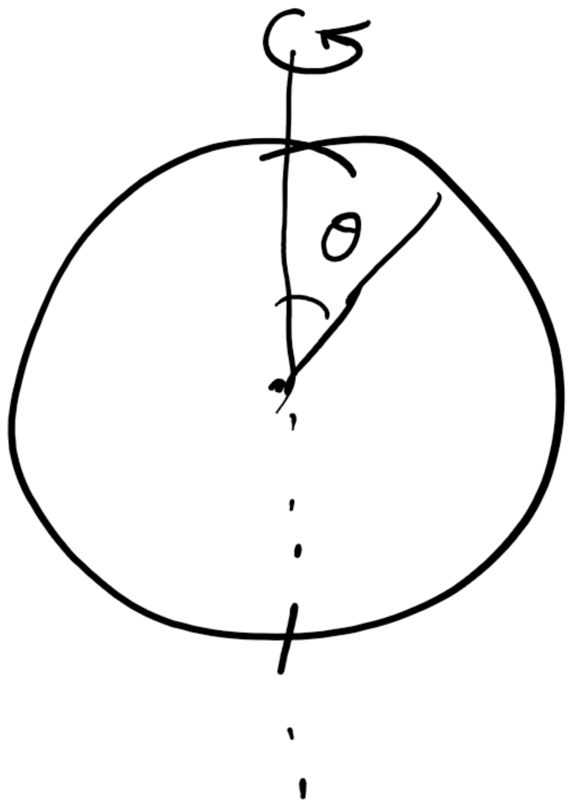
$$I_z = \int \rho r^2 \mathrm{d}A = \int \rho(x^2 + y^2) \mathrm{d}A$$

hence

$$\boxed{I_z = I_x + I_y}$$

so for example for a disc, by symmetry $I_x = I_y$, so $I_z = 2I_x$, which is consistent with the previous computations.

- A sphere, radius a, mass $M = \frac{4}{3}\pi\rho a^3$. Axis of rotation through centre

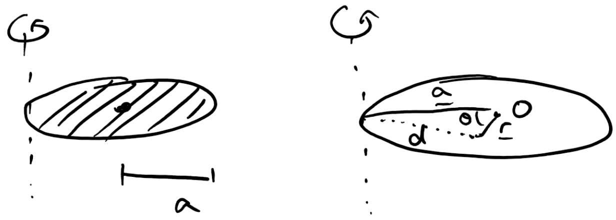


spherical polar coordinates with $\theta = 0$ pointing along axis. $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$.

$$
\begin{aligned}
I &= \int_0^a r^2 \mathrm{d}r \int_0^\pi \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \rho \frac{(r\sin\theta)^2}{x_\perp^2} \\
&= \frac{8}{15}\pi\rho a^5 \\
&= \frac{2}{5} M a^2
\end{aligned}
$$

(note that all the examples have been consistent with dimensional analysis, $[I] = [M][L]^2$)

- disc again about an axis perpendicular to disc, passing through a point on circumference.



$$d^2 = (\mathbf{r} - \mathbf{a})^2 = r^2 + a^2 - 2ar\cos\theta$$

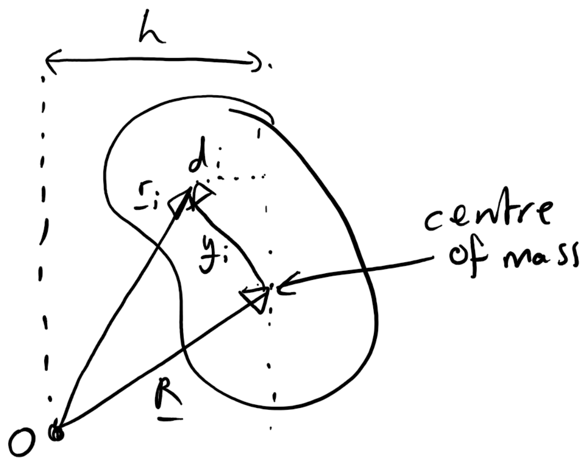
$$\begin{aligned} I &= \rho \int_0^a r \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta (r^2 + a^2 - 2ar\cos\theta) \\ &= \frac{3}{2}\pi\rho a^2 \\ &= \frac{3}{2} M a^2 \end{aligned}$$

We could have done this quickly using the...

Theorem (Parallel Axis Theorem)**.** A rigid body has mass M and moment of inertia I_{CoM} about an axis which passes through the centre of mass. Let I be the moment of inertia about a parallel axis that lies a distance h away. Then

$$\hat{\mathbf{n}} I = I_{\text{CoM}} + Mh^2.$$

Proof. Let $\mathbf{r}_i = \mathbf{R} + \mathbf{y}_i$ where $\sum_i m_i \mathbf{y}_i = 0$.



Then

$$
\begin{aligned}
I &= \sum_i m_I(\hat{\mathbf{n}} \times \mathbf{r}_i) \cdot (\hat{\mathbf{n}} \times \mathbf{r}_i) \\
&= \sum_i m_i(\hat{\mathbf{n}} \times (\mathbf{R} + \mathbf{y}_i)) \cdot (\hat{\mathbf{n}} \times (\mathbf{R} + \mathbf{y}_i)) \\
&= \sum_i m_i[\underbrace{(\hat{\mathbf{n}} \times \mathbf{y}_i) \cdot (\hat{\mathbf{n}} \times \mathbf{y}_i)}_{d_i^2} + \underbrace{2(\hat{\mathbf{n}} \times \mathbf{y}_i) \cdot (\hat{\mathbf{n}} \times \mathbf{R})}_{\propto \sum_i m_i y_i = 0} + \underbrace{(\hat{\mathbf{n}} \times \mathbf{R}) \cdot (\hat{\mathbf{n}} \times \mathbf{R})}_{h^2}] \\
&= I_{\text{CoM}} + Mh^2
\end{aligned}
$$

□

Note.

$$I = I_{\text{CoM}} + Ma^2 = \frac{3}{2}Ma^2$$

Start of lecture 14

The Inertia Tensor (non-examinable)

I is not inherent to the body itself - it also depends on the axis of rotation. A more refined quantity, which is a property only of the rigid body and contains the necessary information to compute the moment of inertia about any axis is the *inertia tensor* $\mathcal{I}$, a 3×3 matrix.
It was already implicit in our expression of kinetic energy of a rotating object:

$$
\begin{aligned}
T &+ \sum_i \frac{1}{2} m_i(\boldsymbol{\omega} \times \mathbf{x}_i) \cdot (\boldsymbol{\omega} \times \mathbf{x}_i) \\
&= \sum_i \frac{1}{2} m_I\{|\boldsymbol{\omega}||\mathbf{x}_i|^2 - |\boldsymbol{\omega} \cdot \mathbf{x}_i|^2\} \\
&= \frac{1}{2}\boldsymbol{\omega}^\top \mathcal{I} \boldsymbol{\omega}
\end{aligned}
$$

where the elements of the matrix are expressed in terms of m_i and the components of $\mathbf{x}_i$.

$$\mathbf{x}_i = ((x_i)_1, (x_i)_2, (x_i)_3) := ((x_i)_a)_{a=1,2,3}$$

i.e.

$$\mathcal{I}_{ab} = \sum_i m_i\{|x_i|^2 \delta_{ab} - (\mathbf{x}_i)a(\mathbf{x}_i)_b\}$$

Moment of inertia $\mathcal{I}_{\hat{\mathbf{n}}}$ about an axis $\hat{\mathbf{n}}$ is

$$\mathcal{I}_{\hat{\mathbf{n}}} = \hat{\mathbf{n}}^\top \mathcal{I} \hat{\mathbf{n}}$$

and one can show that $\mathbf{L} = \mathcal{I}\boldsymbol{\omega}$. Hence angular momentum not necessarily in the same direction as the angular velocity.

6.2 Motion of Rigid Bodies

The mos general motion of a rigid body can be described by its centre of mass following a trajectory $R(t)$, together with a rotation around the centre of mass.

$$\mathbf{r}_i = \mathbf{R} + \mathbf{y}_i \implies \dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \dot{\mathbf{y}}_i$$

If the body rotates around the centre of mass with a velocity $\boldsymbol{\omega}$,

$$\dot{\mathbf{y}}_i = \boldsymbol{\omega} \times \mathbf{y}_i \implies \dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{R}) \qquad (*)$$

The kinetic energy is

$$\begin{aligned} T &= \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \sum_i m_i \dot{\mathbf{y}}_i \cdot \dot{\mathbf{y}}_i \\ &= \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \mathbf{T} \omega^2 \end{aligned}$$

Motion with Rotation About a Different Point

The centre of mass is the most natural point to choose. But we could describe the motion as the trajectory of some other point $\mathbf{Q}$ and rotation about $\mathbf{Q}$.
Put $\mathbf{r}_i = \mathbf{Q}$ in $(*)$:

$$\dot{\mathbf{Q}} = \dot{\mathbf{R}} + \boldsymbol{\omega} \times (\mathbf{Q} - \mathbf{R})$$

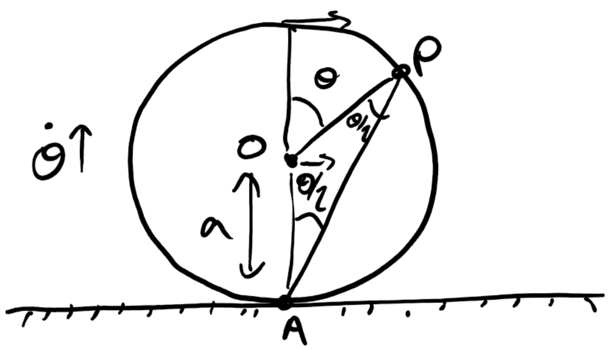
Now use this to eliminate $\dot{\mathbf{R}}$ in $(*)$ to get the motion of any particle with respect to $\mathbf{Q}$:

$$\implies \dot{\mathbf{r}}_i = \dot{\mathbf{Q}} + \boldsymbol{\omega} + \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{Q})$$

Comparing to $(*)$, we see that angular velocity $\boldsymbol{\omega}$ is the same.

Example: Rolling

A hoop, radius a.



If the hoop moves without slipping, there is a relationship between rotational and translational motion. The *no-slip condition* is the requirement that point A (defined as the point of contact with the ground) is stationary, i.e. has no speed with respect to the ground.

The angular speed about centre is $\dot{\theta}$. But can also think of the hoop as rotating about A; also with angular speed $\dot{\theta}$ hence horizontal velocity of origin is $v = a\dot{\theta}$. Horizontal velocity of top of hoop is $2a\dot{\theta}$.

Question: What is speed of general point P?
Think of it rotating about A:

$$AP = 2a\cos\left(\frac{\theta}{2}\right) \implies \text{speed of } P \text{ relative to } A \text{ is}$$

$$v' = 2a\cos\left(\frac{\theta}{2}\right)\dot{\theta}$$

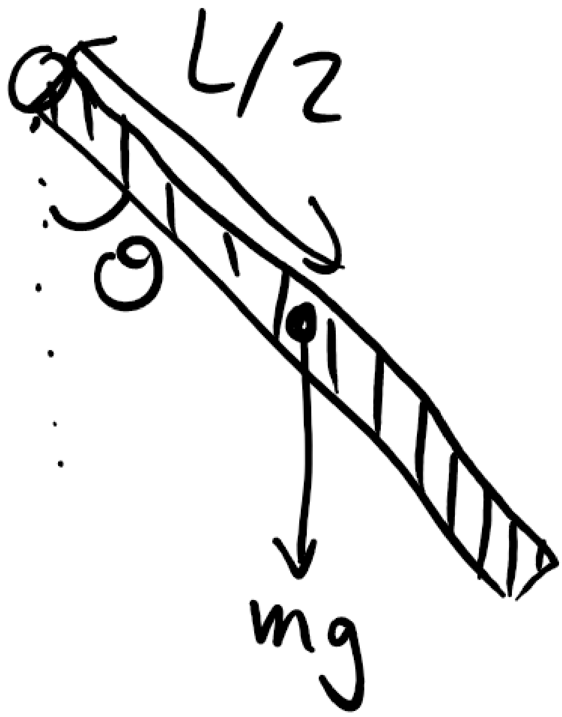
(check: $\theta = 0$, $v' = 2a\dot{\theta}$, and $\theta = \pi$, $v = 0$ as expected.)

Note. Velocity of P is not tangent to the hoop! This would only be the case if the hoop were rotating about a fixed point. It's actually perpendicular to AP, reflecting the fact that P is rotating *and* moving forward as the hoop moves.

Comment: despite the presence of friction, this is one case where mechanical energy is conserved - because point of contact with the ground doesn't move with respect to the ground, friction does no work.

Start of lecture 15

Example (Swinging Rod).
This is a pendulum made from a heavy rod of mass m, length L.



Viewing the rod as rotating about the pivot, the kinetic energy is

$$T = \frac{1}{2} I \dot{\theta}^2 \qquad \text{with} \qquad I = \frac{1}{3} m L^2$$

(from before).

Note. Could also look at this from the point of view of the centre of mass. As we saw before, the angular speed is still $\dot{\theta} =$ angular speed about the pivot. Speed of the centre of mass is

$$\frac{L}{2}\dot{\theta} \qquad \text{and} \qquad T = \frac{1}{2} m \left(\frac{L\dot{\theta}}{2} \right)^2 + \frac{1}{2} I_{\text{CoM}} \dot{\theta}^2$$

parallel axis theorem implies that

$$I = I_{\text{CoM}} + m \left(\frac{L}{2} \right)^2$$

so T calculated both ways is identical.

Compared to the pivot, the rod is at a height

$$
\begin{aligned}
-\frac{L}{2}\cos\theta \implies E &= \frac{1}{2}I\dot{\theta}^2 - \frac{mgL}{2}\cos\theta \\
\implies \dot{E} &= \dot{\theta}\left(I\ddot{\theta} + \frac{mgL}{2}\sin\theta\right) = 0 \\
\implies I\ddot{\theta} &= -\frac{mgL}{2}\sin\theta
\end{aligned}
$$

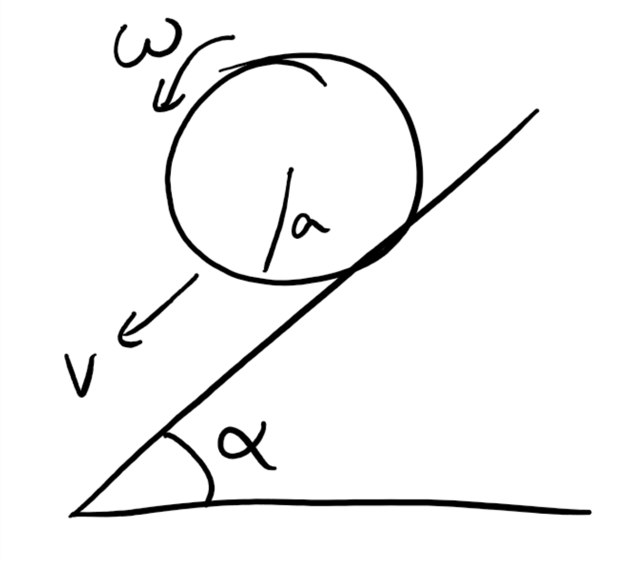
is equation of motion. Pendulum:

$$\ddot{\theta} = -\frac{g}{L}\sin\theta$$

Rod:

$$\ddot{\theta} = -\frac{3g}{2L}\sin\theta$$

Example (Rolling Disc)**.** A disc of mass M rolls down a slope with the plane of the disc being vertical.



The moment of inertia perpendicular to the plane of the disc is $I = \frac{1}{2}Ma^2$ (from before). Let speed be v and angular speed be ω. No slip condition implies that $v = a\omega$. The kinetic energy is

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{I}{a^2} + M\right)v^2$$

($\frac{1}{2}Mv^2$ is translational energy and $\frac{1}{2}I\omega^2$ is rotational energy). Total energy is ($x =$ distance down slope):

$$E = \frac{1}{2}\left(\frac{I}{a^2} + M\right)\dot{x}^2 - Mgx\sin\alpha$$

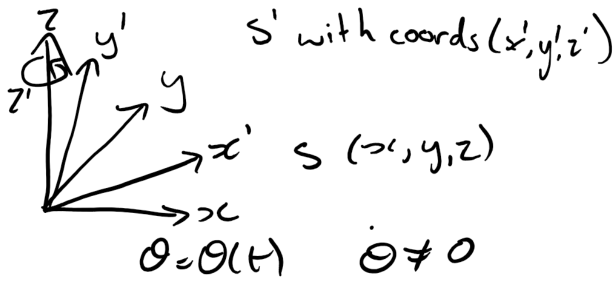
($\dot{E} = 0$ to get equation of motion):

$$\left(\frac{I}{a^2} + M\right)\ddot{x} = Mg\sin\alpha.$$

7 Non-Inertial Frames

7.1 Rotating Frames

Let S be an inertial frame. S' is a non-inertial frame, rotating with respect to S.



Question: If you sit in S', what do Newton's Laws look like?
Consider a particle stationary in S'. Then from the perspective of S, it moves with velocity

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$$

with $\boldsymbol{\omega} = \dot{\theta}\hat{\mathbf{z}}$ in the diagram. In general, $\boldsymbol{\omega}$ points along the direction of rotation. We can apply this formula to the axes of S'. Let $\mathbf{e}'_i$ $(i = 1, 2, 3)$ be unit vectors pointing in x', y' and z' directions S'. Then these rotate, with

$$\boxed{\dot{\mathbf{e}}'_i = \boldsymbol{\omega} \times \mathbf{e}'_i}$$

(this is the main formula that allows us to understand motion in rotating forces).

Velocity and Acceleration in Rotating Frames

Consider a particle now moving on a trajectory in S'. The position of the particle as measured in S is

$$\mathbf{r} = r_i \mathbf{e}_i$$

(note summation convention). Measuring the same point in S', we write

$$\mathbf{r} = r'_i \mathbf{e}'_i$$

The velocity in frame S is

$$\dot{\mathbf{r}} = \dot{r}_i \mathbf{e}_i$$

(because unit vectors in S don't change with time) while in frame S' the velocity is

$$\begin{aligned}\dot{\mathbf{r}} &= \dot{r}'_i \mathbf{e}'_i + r'_i \dot{\mathbf{e}}'_i \\ &= \dot{r}'_i \mathbf{e}'_i + r'_i(\boldsymbol{\omega} \times \mathbf{e}'_i) \\ &= \dot{r}'_i \mathbf{e}'_i + (\boldsymbol{\omega} \times \mathbf{r})\end{aligned}$$

Now we introduce some new notation to highlight the physical difference between the two frames.
The velocity of the particle as seen by an observer in S is

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_S := \dot{r}_i\mathbf{e}_i$$

The velocity of the particle as seen by an observer in S' is:

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} = \dot{r}'_i\mathbf{e}'_i.$$

From above, we have

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_S = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{r}$$

For acceleration, we just take another time derivative. In S, we get $\ddot{\mathbf{r}} = \ddot{r}_i\mathbf{e}_i$ whereas in S' we get

$$\ddot{\mathbf{r}} = \ddot{r}_i\mathbf{e}'_i + \dot{r}'_i\mathbf{e}_i + \dot{r}'_i(\boldsymbol{\omega} \times \mathbf{e}_i)$$

—ONLINE LECTURE CUT OFF HERE—

Start of lecture 16

$$\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_S = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} + 2\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad (*)$$

7.2 Newton's Laws in a Rotating Frame

In the inertial frame S, we have

$$\mathbf{F} = m\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_S$$

Then, from $(*)$ in S' we have

$$m\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{S'} = Fbf - 2m\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

To explain the motion of the particle viewed from S', we must include the extra terms on the RHS. They are called *fictitious forces.* They are necessary to explain the seeming departure from uniform motion of a free particle due to the rotating frame.

- $2m\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'}$ *Coriolis force*
- $m\dot{\boldsymbol{\omega}} \times \mathbf{r}$ *Euler force*
- $m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ *Centrifugal force*

The most familiar non-inertial frame is this room. Earth rotates once a day around N-S axis and once a year around the sun, which rotates around the galaxy.
$R_{\text{earth}} \approx 6000\text{km}$, earth spins with an angular frequency of

$$\omega_{\text{rot}} = \frac{2\pi}{1 \text{ day}} \approx 7 \times 10^{-5}\text{s}^{-1}$$

angular frequency of orbit:

$$\omega_{\text{orb}} = \frac{2\pi}{1 \text{ year}} \approx 2 \times 10^{-7}\text{s}^{-1}$$

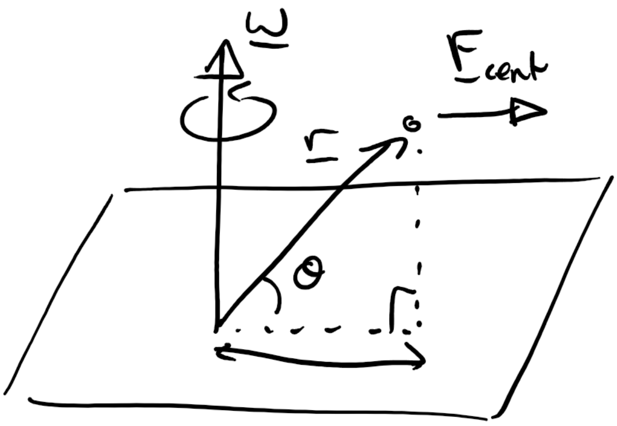
Note.

$$\frac{\omega_{\text{rot}}}{\omega_{\text{orb}}} \approx 365 = \frac{T_{\text{orb}}}{T_{\text{rot}}}$$

Here, we won't consider the Euler force.

7.3 Centrifugal Force

$$\mathbf{F}_{\text{cent}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = -m(\boldsymbol{\omega} \cdot \mathbf{r})\boldsymbol{\omega} + m\omega^2\mathbf{r}$$



$\boldsymbol{\omega} \times \mathbf{r}$ points into the paper so $-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ points away from the axis of rotation, as shown.

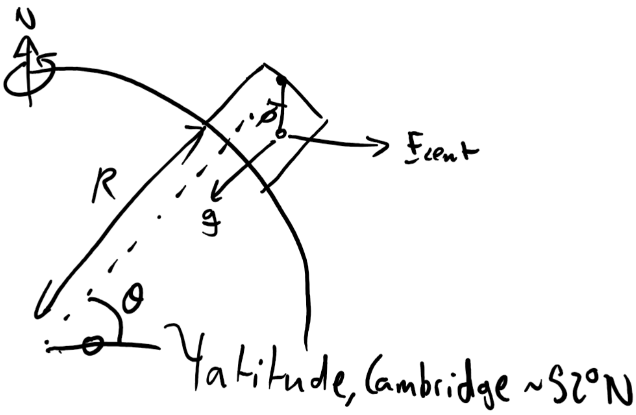
$$|\mathbf{F}_{\text{cent}}| = m\omega^2 r \cos\theta = m\omega^2 d$$

$\mathbf{F}_{\text{cent}}$ doesn't depend on the velocity of the particle. In fact, it's a conservative force:

$$\mathbf{F}_{\text{cent}} = -\nabla V \qquad \text{with} \qquad V = -\frac{m}{2}|\boldsymbol{\omega} \times \mathbf{r}|^2$$

Example: Apparent Gravity

Suspend a piece of string from the ceiling. Centrifugal deflects the string (a bit) from pointing straight down to the centre of the earth. But by how much?



Question: How bis is ϕ?
The effective acceleration is due to the combination of gravity and the centrifugal force

$$\mathbf{g}_{\text{eff}} = \mathbf{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

We resolve the central force

$$\begin{aligned}\mathbf{F}_{\text{cent}} &= |\mathbf{F}| \cos\theta \hat{\mathbf{r}} - |\mathbf{F}| \sin\theta \hat{\boldsymbol{\theta}} \\ &= m\omega^2 r \cos^2\theta \hat{\mathbf{r}} - m\omega^2 r \cos\theta \sin\theta \hat{\boldsymbol{\theta}}\end{aligned}$$

(Note at the north pole, $\theta = \frac{\pi}{2}$ and $F_{\text{cent}} = 0$, as expected). Giving the effective acceleration:

$$\begin{aligned}\mathbf{g}_{\text{eff}} &= -g\hat{\mathbf{r}} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \\ &= (-g + \omega^2 R \cos^2\theta)\hat{\mathbf{r}} - \omega^2 R \cos\theta \sin\theta\end{aligned}$$

We can also resolve tension in the string

$$\mathbf{T} = T\cos\phi \hat{\mathbf{r}} + T\sin\phi \hat{\boldsymbol{\theta}}$$

which balances the $m\mathbf{g}_{\text{eff}}$ force.

$$\mathbf{T} + m\mathbf{g}_{\text{eff}} = 0 \implies \tan\phi = \frac{\omega^2 R \cos\theta \sin\theta}{g - \omega^2 R \cos^2\theta}$$

(Note $\omega^2 R = 3 \times 10^{-2}\text{ms}^{-2}$). ϕ is (approximately) maximised at

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta \sin\theta) = 0$$

i.e. $\theta \approx 45°$.

$$\tan\phi \approx \frac{\omega^2 R}{q} \approx o(10^{-3})$$

so very small. At the equator, $\theta = 0$, $\phi = 0$. But gravity is weaker there:

$$\mathbf{g}_{\text{eff}} \,|_{\text{equator}} = g - \omega^2 R \implies g - g_{\text{eff}} = 10^{-2}\mathsf{ms}^{-2}$$

Example (Rotating Water Bucket)**.** Gravity acts down on water and centrifugal force pushes it to the side.

Question: What's the shape of the surface?
Assume water spins with bucket. Consider a water molecule, mass m on the surface. It's potential energy is

$$V = mgz - \frac{1}{2}m\omega^2 r^2$$

If the water molecule could change its energy by moving along the surface, then it would. But we're looking for the equilibrium shape of surface i.e. each point on it has equal potential energy. Put $V =$ constant:

$$z = \frac{\omega^2}{2g}r^2 + \text{constant}$$

i.e. a parabola!

Start of lecture 17

7.4 Coriolis Force

The Coriolis force is

$$\mathbf{F}_{\text{cor}} = -2m\boldsymbol{\omega} \times \mathbf{v}$$

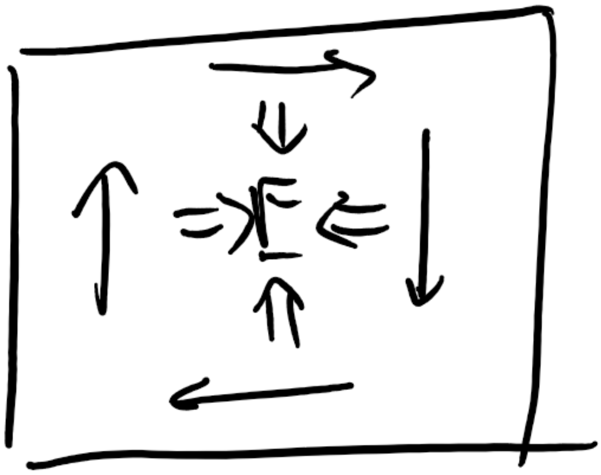
where

$$\mathbf{v} = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'}$$

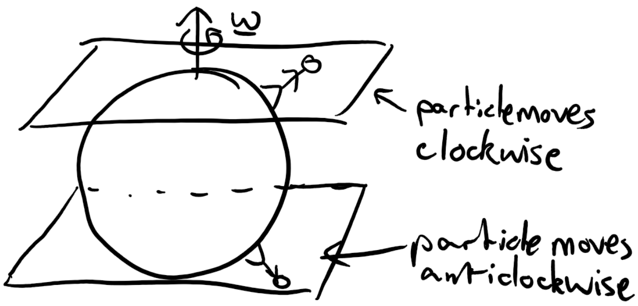
($\mathbf{v} = 0$ for the water in the bucket). Let's ignore the centrifugal force for now.

- It's velocity dependent and independent of position.
- It's mathematically related to the Lorentz force for a constant magnetic field $\implies$ particles move in circles.

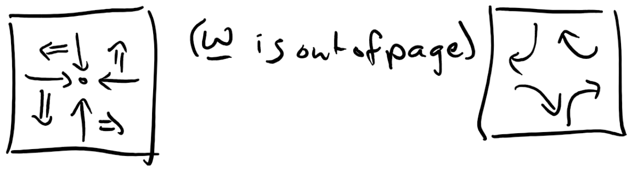
$\boldsymbol{\omega}$ is out of the page.



$\implies$ a free particle moves in a clockwise direction.



This is a similar effect that leads to hurricanes. Suppose there's a low pressure region. Particles in the fluid move radially towards the low pressure and are deflected by the Coriolis force:

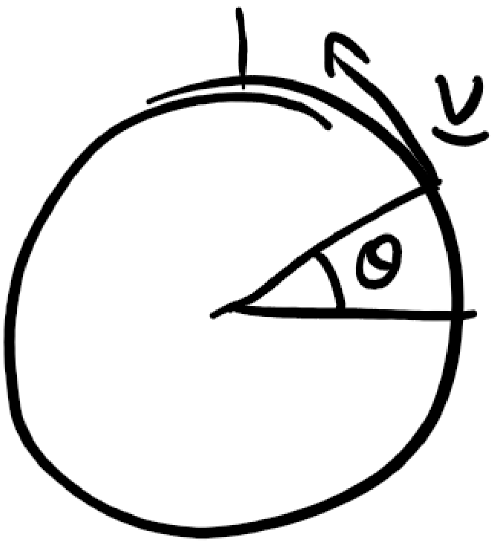


end result is that swirls in an anti-clockwise direction (opposite to a free particle). Hurricanes in the Northern hemisphere rotate in an anticlockwise direction. (Southern hemisphere - clockwise).

Note. Our discussion assumed that particles move in a plane perpendicular to $\boldsymbol{\omega}$. This isn't true on the surface of the earth.

For a particle travelling due North, the Coriolis force points East with size

$$|\mathbf{F}_{\text{cor}}| = 2mv\omega \sin\theta$$



For a particle travelling East, it's a little trickier: $\mathbf{F}_{\text{cir}}$ has a component pointing upward. Looking at the component tangent to Earth,

$$|\mathbf{F}_{\text{cor}} \cdot \hat{\boldsymbol{\theta}}| = 2mv\omega \sin\theta$$

again.

Note. $\theta = 0 \implies \mathbf{F}_{\text{cor}}$ and indeed, there are no hurricanes within about 500 miles of the equator.

Example (Balls and Tower)**.** Climb up a tower of height h and drop a ball.
Question: Does it land in front or behind the tower?
Answer: In front - a bit counter intuitive.

$$\ddot{\mathbf{r}} = \mathbf{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} \approx \mathbf{g} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} \qquad (*)$$

(we neglect $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ because it is $O(\omega^2) +$ very small, so it's fine after $\dot{\mathbf{r}}$ is above a certain value). Integrating once:

$$\implies \dot{\mathbf{r}} = \mathbf{g}t - 2\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0).$$

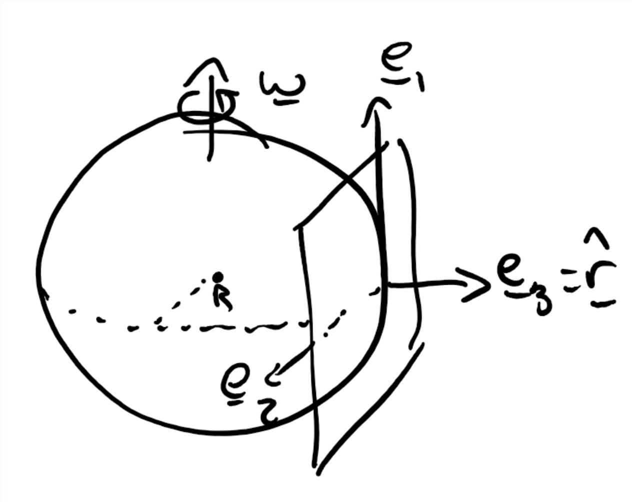
($\mathbf{r}_0$ is an integration constant - the initial position). Substitute into $(*)$ to get:

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times \mathbf{g}t$$

Now integrate twice:

$$\implies \mathbf{r} = \mathbf{r}_0 + \frac{1}{2}gt^2 - \frac{1}{3}\boldsymbol{\omega} \times \mathbf{g}t^3$$

We'll look at this effect on the equator



$$\begin{aligned}
\mathbf{g} &= -g\mathbf{e}_3 \\
\boldsymbol{\omega} &= \omega\mathbf{e}_1 \\
\mathbf{r}_0 &= (R+h)\mathbf{e}_3 \\
\implies \mathbf{r} &= \left(R + h - \frac{1}{2}gt^2\right)\mathbf{e}_3 - \frac{1}{3}\omega g t^3 \mathbf{e}_2
\end{aligned}$$

$\mathbf{e}_2$ is Westerley but displacement is in the Easterley direction. Particle hits ground when $t^2 = \frac{2h}{g}$, as usual.

$$\implies \text{displacement } = \frac{1}{3}\omega g \left(\frac{2h}{g}\right)^{3/2} \text{ East}$$

(ball drops in *front* of tower).

Note. This is really conservation of angular momentum. When dropped, $l = (R+h)^2\omega$ so the speed close to the ground is v:

$$Rv = (R+h)^2\omega \implies v = \frac{(R+h)^2}{R}\omega > R\omega = v_{\text{earth}}$$

Brief Discussion of Foucault's Pendulum

A pendulum at the North pole stays aligned with its inertial force while the Earth rotates underneath. An observer on the earth sees the swing process throughout the day (see David Tong's notes for algebra)

$$\zeta := x + iy \qquad \zeta = e^{-i\omega t \sin\theta}\left(a\cos\sqrt{\frac{g}{l}}t + B\sin\sqrt{\frac{g}{l}}t\right)$$

Pendulum rotates with period $\frac{1\text{ day}}{\sin\theta}$.

Larmor Precession

Consider a charged particle orbiting around a second, fixed particle, under the influence of the Coulomb force. Add this to a constant magnetic field $\mathbf{B}$. Equation of motion:

$$m\ddot{\mathbf{r}} = -\frac{k}{r^2} + q\dot{\mathbf{r}} \times \mathbf{B}; \qquad k = \frac{qQ}{4\pi\varepsilon_0}$$

In a rotating frame ($\dot{\boldsymbol{\omega}} = 0$):

$$\begin{aligned} m\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{S'} &= -\frac{k}{r^2}\hat{\mathbf{r}} + q\dot{\mathbf{r}} \times \mathbf{B} - 2m\boldsymbol{\omega}\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} - m(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})) \\ &= -2m\boldsymbol{\omega}\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} - m(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})) - \frac{k}{r^2}\hat{\mathbf{r}} + q\left[\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{r}\right] \times \mathbf{B} \end{aligned}$$

Pick $\boldsymbol{\omega}$ to cancel $\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'}$ terms, i.e. $\boldsymbol{\omega} = -\frac{q\mathbf{B}}{2m}$

$$\implies m\ddot{\mathbf{r}} = -\frac{k}{r^2}\hat{\mathbf{r}} + \frac{q^2}{4m}\mathbf{B} \times (\mathbf{B} \times \mathbf{r})$$

(when $B^2 \ll \frac{4mk}{q^2r^2}$ we can neglect the last term). We go back to solutions with elliptic motion. Transform back out of rotating frame to show that the ellipses precess with angular speed $\omega = \frac{qB}{2m}$, the Larmor frequency ($\frac{1}{2}$ the cyclotron frequency).

Start of lecture 18

8 Special Relativity

When particles travel very fast, Newtonian mechanics breaks down and is replaced by Einstein's theory of special relativity. The effects of special relativity only become apparent when the speed of the particles approaches the speed of light (in a vacuum)

$$c = 3.00 \times 10^8 \text{ms}^{-1}$$

$$\text{speed of sound in air} \approx 300\text{ms}^{-1}$$

$$\text{escape velocity} \approx 10^4\text{ms}^{-1}$$

The theory of special relativity rests on 2 postulates:

(1) The principle of relativity - the laws of physics are the same in all inertial frames (Galilean principle of relativity).

(2) The speed of light in vacuous (= in a vacuum) is the same in all inertial frames.

Postulate 2 is weird - 'common sense' says that a car moving at speed v should have its light travelling at speed $c + v$ with respect to the ground. (This is not compatible with Galilean relativity).

Consider 2 inertial frames, S and S', moving relative to each other with $\mathbf{v} = (v, 0, 0)$. Galilean transformations would relate the coordinates as:

$$x' = x - vt \quad y' = y, \quad z' = z \quad t' = t.$$

A ray of light travels in the x-direction in frame S with speed $c \implies$ it traces out trajectory $\frac{x}{t} = c$.

$$\text{Galilean transformation} \implies \frac{x'}{t'} = \frac{x - vt}{t} = c - v$$

Only way to incorporate postulate 2 (and 1) is to change the definition of time.

8.1 Lorentz Transformations

S, S' moving relative speed v in x-direction. Ignore y and z for now. Want to know relationship between x', t', x and t. The most general form is

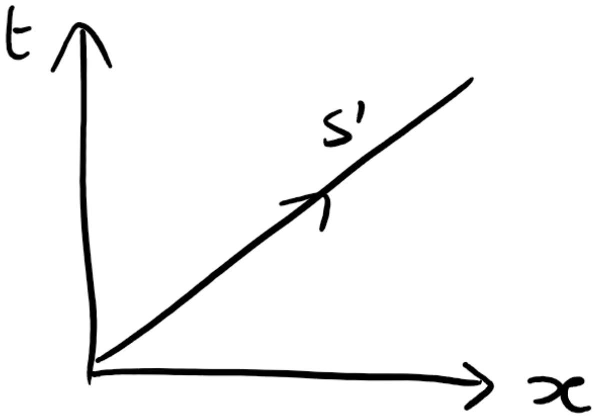
$$x' = f(x, t) \qquad t' = g(x, t)$$

for some functions f and g.

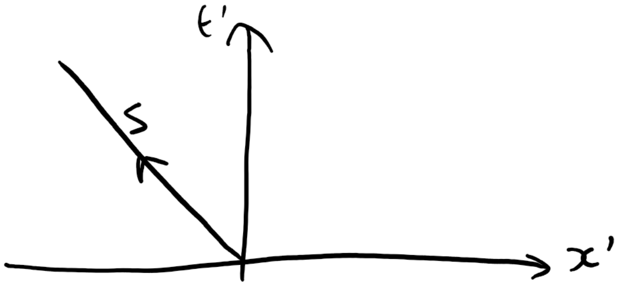
Left alone, a particle should travel in a straight line viewed in S and S'. (both inertial frames). So, the map $(x, t) \mapsto (x', t')$ takes straight lines to straight lines (i.e. it is linear, for example $x' = \alpha_1 x + \alpha_2 t$, $t' = \alpha_3 x + \alpha_4 t$; α_i are functions of v). S, S' move at relative speed v. Assume that origins coincide $\implies$ lines $x = vt$ must map to

$$x' = 0 \implies x' = \gamma(x - vt) \tag{8.1}$$

for some $\gamma = \gamma(v)$ (it shouldn't depend on 1D direction of $\mathbf{v}$, i.e. $\gamma(v) = \gamma(-v)$; from rotational invariance). (position of observer sitting origin of S').



But from the perspective of S', S moves with velocity $-v$.



Same argument as (8.1)

$$\implies x = \gamma(x' + vt') \tag{8.2}$$

so

$$ct' = \gamma(c - v)t \qquad \text{and} \qquad ct = \gamma(c + v)t'$$

so

$$\boxed{\gamma = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}}}$$

- important!

Note.
- $v \ll c$: $\gamma \approx 1 \implies$ Special Relativity $\to$ Galilean transformations
- $v \to c$: $\gamma \to \infty$
- $v > c$: γ is imaginary - unphysical

(8.1) reads $\boxed{x' = \gamma(x - vt)}$ while

$$x = \gamma(x' + vt') = \gamma[\gamma(x - vt) + vt']$$

$$\implies \boxed{t' = \gamma\left(t - \frac{v}{c^2}x\right)}$$

(this is the *Lorentz transformations* (= *Lorentz boosts*)). They tend to Galilean transformations in $\lim \frac{v}{c} \to 0$. ($t' = t$). Inverting,

$$x = \gamma(x' + vt')$$

$$t = \gamma\left(t' + \frac{v}{c^2}x\right)$$

(note same, with $v \mapsto -v$). The directions perpendicular to direction of motion have the trivial transformation law $y' = y$, $z' = z$.

Checking the Constancy of the Speed of Light

A light ray travelling in the y-direction in S has trajectory $x = 0$, $y = ct$. In S', trajectory is

$$x' = -vt' \qquad \text{and} \qquad y' = \frac{ct'}{\gamma}$$

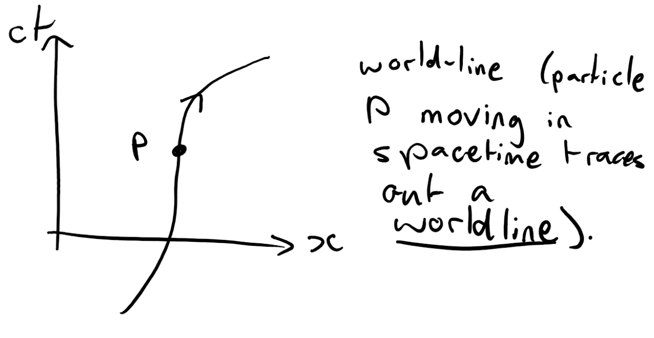
and the speed of light is

$$v'^2 = \left(\frac{x'}{t'}\right)^2 = v^2 + \frac{c^2}{\gamma^2} = c^2$$

i.e. the same, so postulate 2 holds.

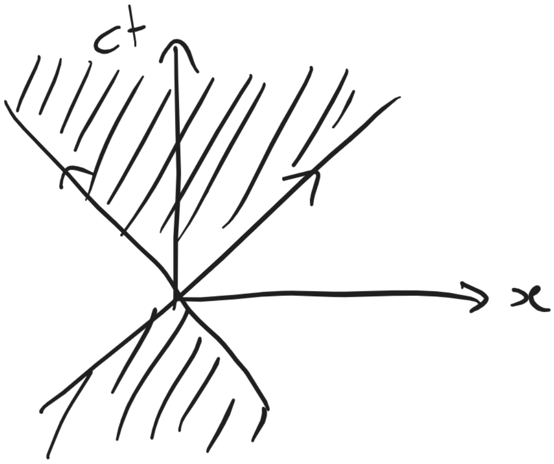
Space-time Diagrams

In a fixed inertial frame S, draw a direction of space - say x coordinate along horizontal coordinate and ct along vertical coordinate.

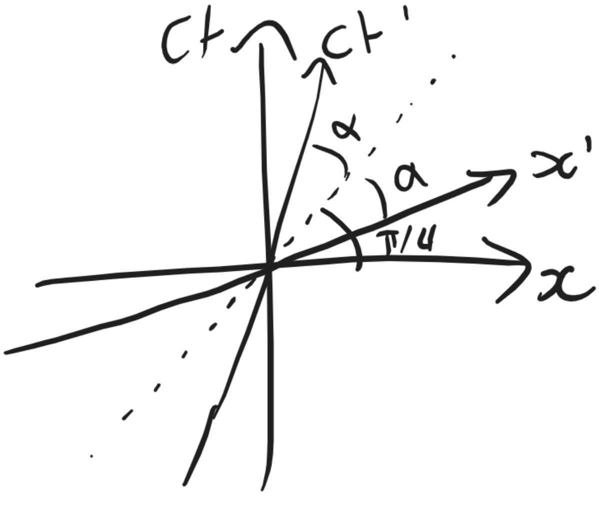


The union of space and time is called *Minkowski spacetime*. Each point P represents an *event*. We label the coordinates of P as (ct, x) (note backwards convention)!

Start of lecture 19



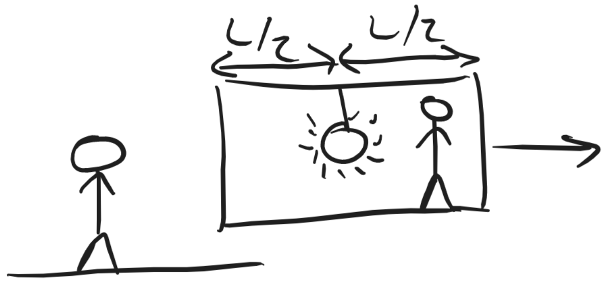
Light rays travel at $\frac{\pi}{4}$ angle with respect to the axes. We'll see later that particles can't travel with $v > c$. (i.e. particle worldlines are steeper than $\frac{\pi}{4}$). We could also draw the axes of S', moving with velocity $\mathbf{v} = (v, 0, 0)$ with respect to S. The t' axis sits at $x' = 0$ or $x = vt$. The x' axis sits at $t' = 0$ so, from the Lorentz transformation, $ct = \frac{vx}{c}$.



Axes are symmetrical about the light ray diagonal, reflecting that the speed of light is also c in S'.

8.2 Relativistic Physics

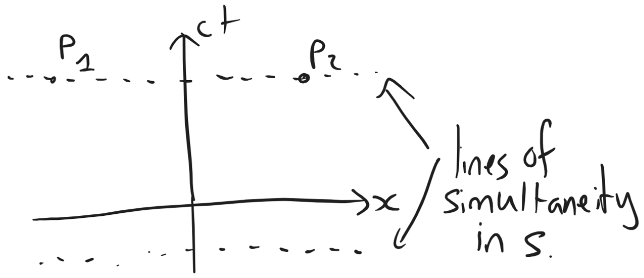
Speeding train



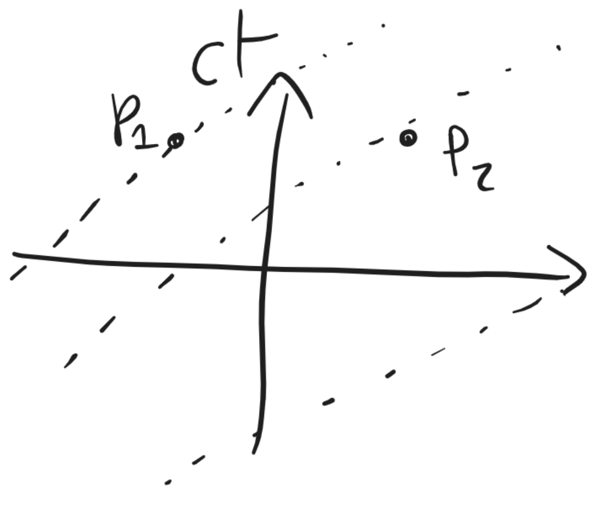
Observer on train sees light hit front and back simultaneously. Station master on ground: sees light rays going forward and back at same speed c, but while the light is travelling, the train moves, so the station master sees the light hit the back of the train first.

Simultaneity

An observer in S decides that events P_1 and P_2 occur simultaneously if $t_1 = t_2$



But in S', simultaneous events occur with equal t', or $t - \frac{vx}{c^2} = \text{constant}$

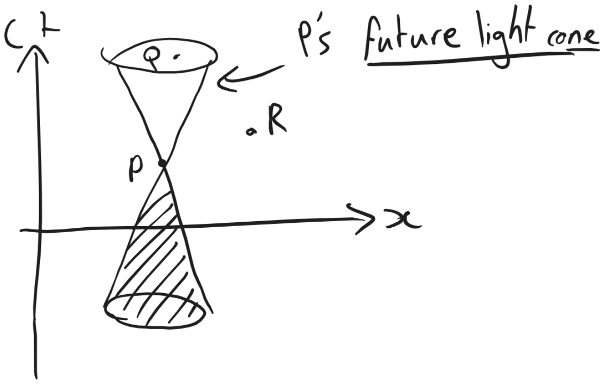


$\cdots$ = lines of simultaneity in S'. P_1 and P_2 occur at *different* t' (as measured in S').

Causality

Question: If observers disagree on temporal ordering of events, where does that leave the idea of cause and effect? Thankfully, this still holds: there are only some events which observers disagree about.

Because Lorentz boosts are only possible for $v < c$, the lines of simultaneity are at an angle $< \frac{\pi}{4}$.



All observers agree that Q occurred after P. But they can disagree on the temporal ordering of P and R. If nothing travels faster than c, then P can only be influenced by events in its past light cone (the reflection of the future light cone about P).

Time Dilation

A clock sitting in S' ticks at intervals T', for example tick 1 occurs at coordinates $(ct', x') = (ct'_1, 0)$, tick 2 occurs at coordinates $(ct', x') = (c(t'_1 + T'), 0)$ etc.

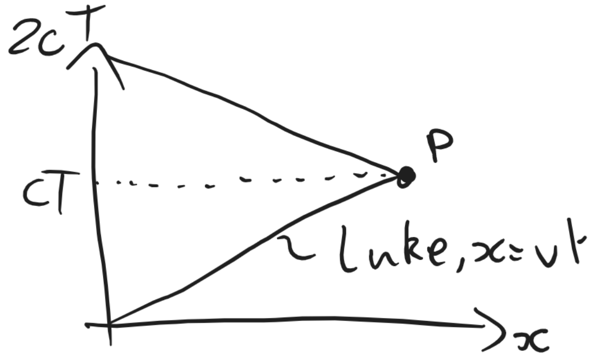
Question: What are the intervals between the ticks in S?
Clock sits at $x' = 0$. Lorentz transformations implies that $\gamma = \left(t' + \frac{vx'}{c^2}\right) \implies T = \gamma T'$.
Since $\gamma > 1$, time runs slower in S.

Twin Paradox

Two twins: Luke & Leia.

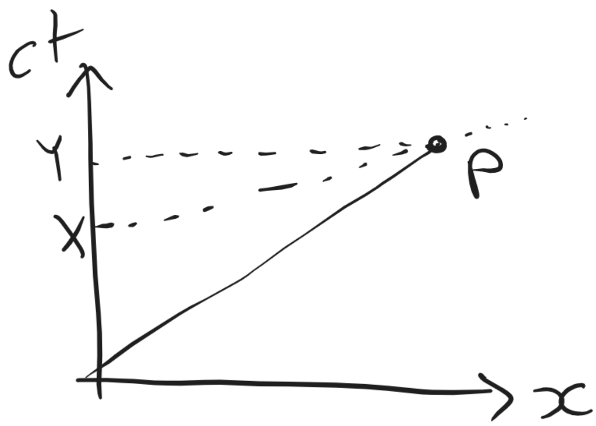
- Leia stays at home
- Luke heads to a planet P
- When he arrives at P, he turns around and comes back at the same speed.

Leia's perspective:



When Luke returns, Leia has aged $2T$. Luke has aged $\frac{2T}{\gamma}$, i.e. he's younger than Leia.

Question: can't we use the same argument from Luke's perspective to say that Leia is younger than him? Why aren't things symmetric? Resolution is that Luke's not in an inertial frame at P - spoils symmetry.
When Luke arrives at P, he thinks Leia is at simultaneous event X.



Point P is $(ct, x) = (cT, vT)$. Time Luke thinks it takes to make journey is

$$T' = \gamma\left(T - \frac{v^2}{c^2}T\right) = \frac{T}{\gamma}.$$

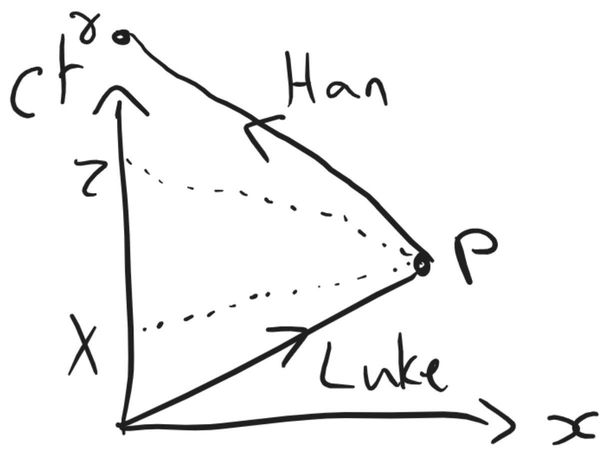
Point X occurs at $x = 0$ and $t' = T' = \frac{1}{\gamma}$, since it's simultaneous from Luke's perspective, with his arrival at P. So

$$t' = \gamma\left(t - \frac{v^2 x}{c^2}\right) \implies t = \frac{T'}{\gamma} = \frac{T}{\gamma^2}.$$

Leia's age from Luke's perspective when he arrives at P is t, i.e. Luke thinks Leia is younger than him by a factor $\frac{1}{\gamma}$ $\to$ so far, it's symmetric.

Start of lecture 20

Things change when Luke turns around. Suppose instead that Luke meets his friend Han who's just leaving P towards Leia. On this journey, Han thinks Leia ages by $\frac{T}{\gamma^2}$.



But he thinks Leia is at point Z when he leaves

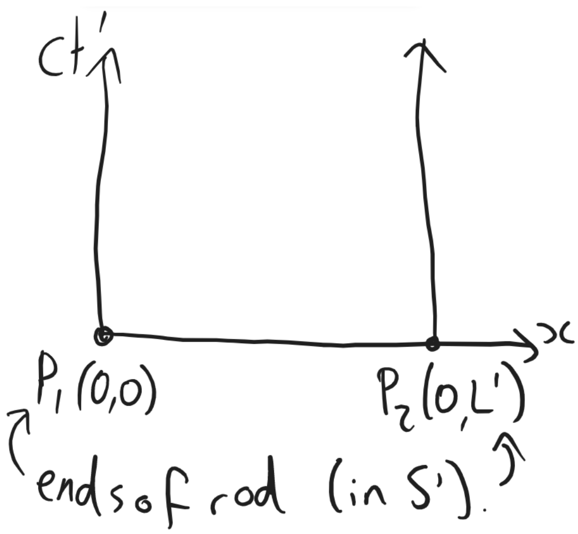
$$\left(t = 2T - \frac{T}{\gamma^2}\right)$$

The asymmetry arises (between Luke and Leia) when Luke turns around. At this point, he sees Leia age rapidly from $X \to Z$.

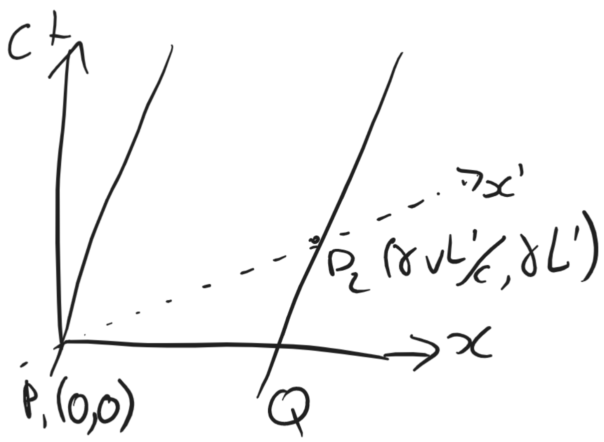
Length Contraction

A rod of length L' is stationary in frame S'.

Question: What is its length in frame S?
In S'



The length of the rod is the distance between the two end points *at equal times*. In S



(Lorentz transforms $x' = f(x,t)$, $t' = g(x,t)$, Inverse transformations $x = f'(x',t')$, $t = g'(x',t')$). P_2 is at $t = \frac{\gamma v L'}{c^2}$ and $x = \gamma L'$. Need to measure L at equal time in S. Follow P_2 back to Q, Q sits at

$$x = \gamma L' - vt = \gamma L' - \frac{\gamma v^2 L'}{c^2} = \frac{L'}{\gamma}$$

The length of the rod in S is $L = \frac{L'}{\gamma}$, i.e. moving rods are shorter.

Example. Consider a barn of length L, ladder of length $2L$. Question: does it fit?

- From perspective of barn, ladder length is $\frac{2L}{\gamma}$, so if $\gamma \geq 2$, ladder firs.
- From perspective of ladder holder, barn length $= \frac{L}{\gamma}$, so doesn't fit.

So whether ladder fits or not depends on the frame. (So this is a matter of simultaneity: events about the front and back being in the barn).

Addition of Velocities

A particle moves with velocity u' in S', which moves with respect to S. What is the velocity $\mathbf{u}$ in S?

The worldline of the particle in S'

$$x' = u't' \tag{$*$}$$

In S,

$$u = \frac{x}{t} = \frac{\gamma\left(x' + vt'\right)}{\gamma\left(t' + \frac{vx'}{c^2}\right)}$$

(inverse Lorentz transformations)

$$(*) \implies \boxed{u = \frac{u' + v}{1 + \frac{u'v}{c^2}}}$$

- $u'v \ll c^2 \implies$ Galilean addition of velocities
- $u' = c \implies u = c$, consistent with speed of light being constant in any frame.

8.3 Geometry of Spacetime

There is a quantity that all observer agree on. Consider two events P_1 with coordinates (ct_1, x_1), P_2 with coordinates (ct_2, x_2).

$$\Delta t := t_1 - t_2 \qquad \Delta x := x_1 - x_2$$

The *invariant interval* is defined as

$$\Delta s^2 = c^2(\Delta t)^2 - (\Delta x)^2$$

Claim: All observers agree on value of Δs^2: have

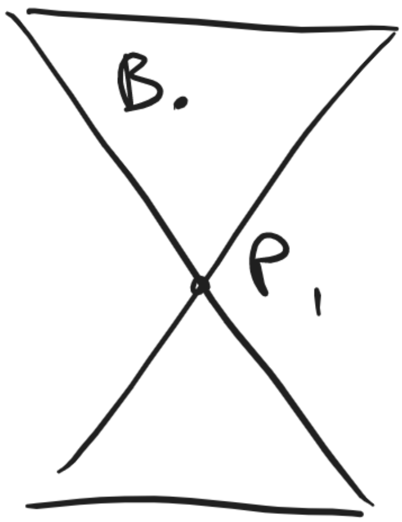
$$\begin{aligned}
\Delta s^2 &= \gamma^2 \left(c\Delta t' + \frac{v\Delta x'}{c} \right)^2 - \gamma^2(\Delta x' + v\Delta t')^2 \qquad \text{inverse Lorentz transforms} \\
&= \gamma^2(c^2 - v^2)\Delta t'^2 - \gamma^2 \left(1 - \frac{v^2}{c^2} \right) \Delta x'^2 \\
&= c^2 \Delta t'^2 - \Delta x'^2
\end{aligned}$$

In three spatial dimensions, the invariant intervals is

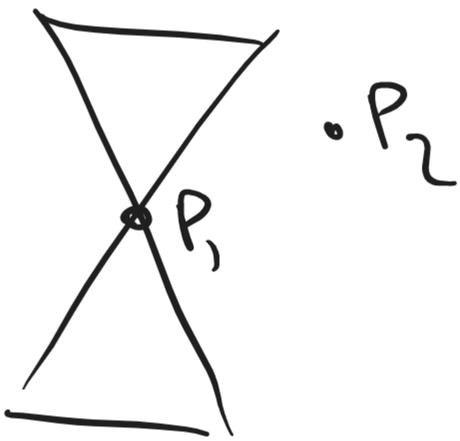
$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

Spacetime is parametrised by 4 numbers, i.e. $\mathbb{R}^4$. When endowed with the (non-positive definite) measure of distance Δs^2, it is called *Minkowski space*. We say it has dimension $1 + 3$, to stress the difference with Euclidean space.

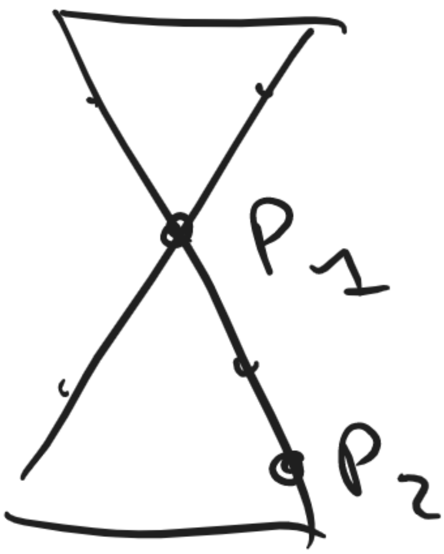
- Events with $\Delta s^2 > 0$ are *timlike separated*:



- Events with $\Delta s^2 < 0$ are *spacelike separated*:



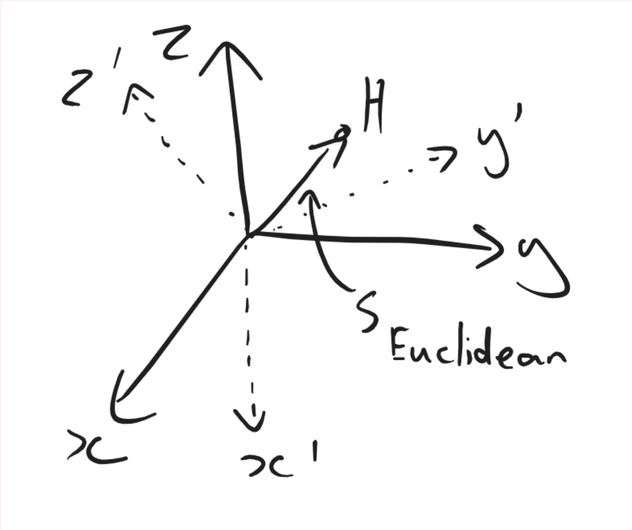
- Events with $\Delta s^2 = 0$ are *lightlike*- or *null*- separated:



Note. $\Delta s^2 = 0$ doesn't mean same point.

The Lorentz Group

Note. 3D rotations



distance of H to origin

$$S^2_{\text{Euclidean}} = x^2 + y^2 + z^2$$

In a rotated frame,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

3×3 rotation matrix such that $R^\top R = I_{3\times 3}$.

$$S^2_{\text{Euclidean}} = x'^2 + y'^2 + z'^2$$

It's best to think of Δs^2 as primary object in Minkowsky space. We'll use this now to redefine Lorentz transformation.

The coordinates of an event P in frame S can be written as a *four-vector* X:

$$X^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

(we label the entries with $\mu = 0, 1, 2, 3$ respectively.) The invariant distance from the origin and P can be written as an inner product

$$X \cdot X := X^\top \eta X$$

where

$$\eta := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is the *Minkowski metric.*

Note. $X \cdot X := X^\top \eta X := X^\mu \eta_{\mu\nu} X^\nu$ by Einstein summation convention.

Start of lecture 21

We see that

$$X \cdot X = ct^2 - x^2 - y^2 - z^2$$

is the invariant interval Δs^2 between the origin and point P. 4-vectors with $X \cdot X > 0$ are called timelike, $X \cdot X < 0$ are called spacelike and $X \cdot X = 0$ are called null.

We redefine a Lorentz transformation as a 4×4 matrix Λ, rotating the coords in frame S to those in S' such that

$$\underbrace{X'}_{\text{4-vector in } S'} = \underbrace{\Lambda}_{4\times 4 \text{ matrix}} \underbrace{X}_{\text{4-vector in } S}$$

Lorentz transformations are now defined to be any matrix which leaves the inner product invariant,

$$X' \cdot X' = X \cdot X \qquad \forall\ X$$

i.e.

$$\Lambda^\top \eta \Lambda = \eta \tag{‡}$$

There are 2 classes of solution to this equation:

- If R is a 3×3 matrix obeying $R^\top R = I_{3\times3}$ i.e. R is a rotation matrix (+ reflection). (one can parametrise R by three angles of rotation - one around each axis).

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}$$

- A boost in the x-direction:

$$\Lambda = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(Two further ones in y- and z-directions).

The set of all matrices solving (‡) form the *Lorentz group* (a group under matrix multiplication) denoted $O(1,3)$.

Taking det(‡), $+(\det\Lambda)^2 = +1$ hence $\det\Lambda = \pm 1$.
The subgroup with $\det\Lambda = +1$ is called the *proper Lorentz group*. Additionally requiring $\lambda^0_0 > 0$ leads to another subgroup, the *proper orthochronous Lorentz group*, denoted $SO(1,3)^{\uparrow}$ or $SO(1,3)^{+}$. (the S means special, and means that $\det\Lambda = +1$)

Rapidity

Focus on upper 2×2 block of Λ, boost in x-direction.

$$\Lambda[v] = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c}\gamma & \end{pmatrix}$$

Combining two successive boosts in the x-direction,

$$\Lambda[v_1]\Lambda[v_2] = \begin{pmatrix} \gamma_1 & -\frac{\gamma_1 v_1}{c} \\ -\frac{\gamma_1 v_1}{c} & \gamma_1 \end{pmatrix}\begin{pmatrix} \gamma_2 & -\frac{\gamma_2 v_2}{c} \\ -\frac{\gamma_2 v_2}{c} & \gamma_2 \end{pmatrix} = \cdots = \Lambda\left[\frac{v_1+v_2}{1+\frac{v_1 v_2}{c}}\right]$$

i.e. compatible with our velocity addition formula from last lecture.

There's a much nicer way of doing this.
Recall that, for 2D rotations,

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \implies R(\theta_1)R(\theta_2) = R(\theta_1+\theta_2)$$

- do an analogous thing for Lorentz boosts.

For Lorentz boosts, we define the *rapidity*, ϕ, defined by $\gamma = \cosh\phi$, so

$$\sinh\phi = \sqrt{\cosh^2\phi - 1} = \sqrt{\gamma^2 - 1} = \frac{\gamma v}{c}$$

$$\implies \Lambda[\phi] = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$$

so that rapidities just add, like angles of rotation:

$$\Lambda[\phi_1]\Lambda[\phi_2] = \Lambda[\phi_1+\phi_2]$$

- shows the relationships between rotations and boosts.

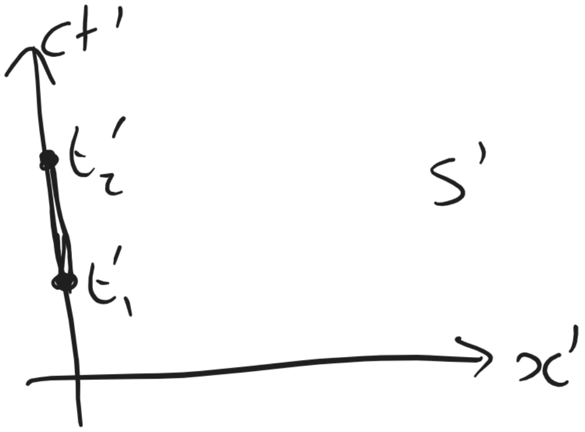
Remark. $c \approx 3.00\times 10^8 \mathsf{ms}^{-1}$. In fact $c = 299792458\mathsf{ms}^{-1}$ exactly because this is how the metre is defined:

$$1\mathsf{m} = \text{distance travelled by light in } \frac{1}{299792458} \text{ seconds.}$$

Could have defined c to be 1 (light second) s^{-1}. (later on in tripos, you'll redefine units such that $c = 1$)

4-velocity

$$\Delta s^2 = c^2 \Delta t'^2$$



We define *propertime* τ as

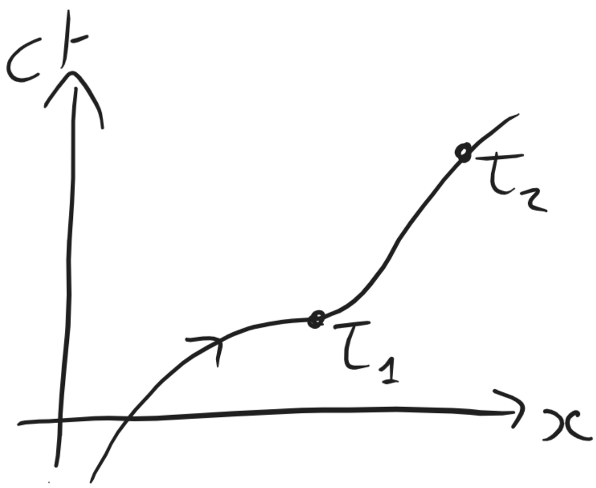
$$\Delta\tau = \frac{\Delta s}{c}$$

the time experienced by the particle, but $\Delta\tau = \frac{\Delta s}{c}$ holds in all inertial frames, since Δs is invariant under Lorentz transformations.

Start of lecture 22

τ provides a way to parametrise the world-line of a particle in a way that all inertial observers agree on.
In frame S, the wordline of any particle can be parametrised

$$\mathbf{x}(\tau), \qquad t(\tau)$$



Along a small segment of worldline

$$\begin{aligned} d\tau &= \sqrt{dt^2 - \frac{(d\mathbf{x})^2}{c^2}} \\ &= dt\sqrt{1 - \frac{\mathbf{u}^2}{c^2}} \end{aligned}$$

where $\mathbf{u} := \frac{d\mathbf{x}}{dt}$

$$\implies \frac{dt}{d\tau} = \gamma_u \tag{$*$}$$

with

$$\gamma_u := \frac{1}{\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}}.$$

The total time experienced by the particle along its worldline

$$T = \int d\tau \overset{(*)}{=} \int \frac{dt}{\gamma_u}$$

4-velocity

The general trajectory of a particle traces out a 4-vector

$$X(\tau) = \begin{pmatrix} ct(\tau) \\ \mathbf{x}^\top(\tau) \end{pmatrix}.$$

The *4-velocity* is defined as

$$U = \frac{dX}{d\tau} = \begin{pmatrix} c\frac{dt(\tau)}{d\tau} \\ \left(\frac{d\mathbf{x}(\tau)}{d\tau}\right)^\top \end{pmatrix} = \frac{dt}{d\tau}\begin{pmatrix} c \\ \mathbf{u}^\top \end{pmatrix} \overset{(*)}{=} \gamma_u \begin{pmatrix} c \\ \mathbf{u}^\top \end{pmatrix}$$

Because $d\tau$ is invariant for inertial observers, we easily write down the 4-velocity in other frames: If S' is related to S by $X' = \Lambda X$ then the 4-velocity of the particle in S' is

$$U' = \Lambda U$$

(take $\frac{d}{d\tau}$).

Note. Wouldn't work for $\frac{dX}{dt}$ since both X *and* t transform under Lorentz boosts.

We know that

$$U \cdot U = U' \cdot U'$$

i.e. is the same for all observers. In the rest-frame of the particle,

$$U = \begin{pmatrix} c \\ \mathbf{0}^\top \end{pmatrix} \implies U \cdot U = c^2.$$

i.e. 4-velocities have 3 independent parameters.

Note. In general, a *4-vector* is a 4-component vector that transforms as $A \to \Lambda A$ under Lorentz transformations.

Addition of velocities at an angle

In frame S, a particle travels with 4-velocity

$$U = \gamma_u \begin{pmatrix} c \\ u\cos\alpha \\ u\sin\alpha \\ 0 \end{pmatrix}$$

Frame S' moves with velocity v in the x-direction (with respect to S).

$$\begin{aligned}
\implies U' &= \Lambda[v]U \\
&= \begin{pmatrix} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_u c \\ U\gamma_u \cos\alpha \\ U\gamma_u \sin\alpha \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} c\gamma\gamma_u \left(1 - \frac{uv}{c^2}\cos\alpha\right) \\ \gamma\gamma_u(u\cos\alpha - v) \\ u\gamma_u \sin\alpha \\ 0 \end{pmatrix} \\
&:= \begin{pmatrix} \gamma_{u'} c \\ u'\gamma_{u'}\cos\alpha' \\ u'\gamma_{u'}\sin\alpha' \\ 0 \end{pmatrix}
\end{aligned}$$

Divide first component by zeroth on each side:

$$u'\cos\alpha' = \frac{u\cos\alpha - v}{1 - \frac{uv}{c^2}\cos\alpha}$$

Divide second component by first:

$$\tan\alpha' = \frac{u\sin\alpha}{\gamma(u\cos\alpha - v)}$$

4-momentum

The *4-momentum* of a particle of mass m is

$$\mathbf{P} = mU = \begin{pmatrix} mc\gamma \\ m\gamma\mathbf{u}^\top \end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

where m is the *rest-mass*. In relativity, 4-momentum is conserved. Spatial components give the 3-momenta $\mathbf{p} = m\gamma\mathbf{u}$.

Note. As $|\mathbf{u}| \to c$, $|\mathbf{p}| \to \infty$.

What's the interpretation of $\mathbf{P}^0 = \gamma mc$? Taylor expand for $|\mathbf{u}| \ll c$:

$$\mathbf{P}^0 = \frac{mc}{\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}} = \frac{1}{c}\left(mc^2 + \frac{1}{2}m\mathbf{u}^2 + \cdots\right)$$

suggesting we should interpret $\mathbf{P}$ as

$$\mathbf{P} = \begin{pmatrix} \frac{E}{c} \\ \mathbf{p}^\top \end{pmatrix}$$

where $E =$ energy. (Noether's theorem).
We learn that both mass and kinetic energy contribute to the energy $\boxed{E = \gamma mc^2}$.

Note. As $|\mathbf{u}| \to c$, $E \to \infty$, i.e. can't get particles to exceed the speed of light.

For a stationary particle, $\gamma = 1$ so $E = mc^2 \to$ mass is a form of energy.

We can also write the energy in terms of momentum since

$$\mathbf{P} \cdot \mathbf{P} = \frac{E^2}{c^2} - \mathbf{p}^2,$$

since same in all frames, calculate in rest frame, $\mathbf{p} = 0$, $E = mc^2$.

$$\implies \mathbf{P} \cdot \mathbf{P} = \frac{m^2c^4}{c^2} \implies \boxed{E^2 = \mathbf{p}^2c^2 + m^2c^4}$$

In Newtonian physics, mass and energy are conserved separately. In relativity, mass is just another form of energy and we can convert kinetic energy $\leftrightarrow$ mass.

Note. E and $\mathbf{p}$ are frame-dependent! Lorentz transform:

$$U \to \Lambda U$$

$$U \cdot U = U^\top \eta U \overset{\text{LT}}{\to} U^\top \Lambda^\top \eta \Lambda U = u^\top \eta U = U \cdot U$$

Start of lecture 23

Massless Particles

Proper time τ is only defined for massive particles that move slowed than c. But from last lecture, have

$$\mathbf{P}\cdot\mathbf{P} = \frac{E^2}{c^2} - \mathbf{p}^2 = \mathbf{p}^2 - \mathbf{p}^2 = 0$$

i.e. the 4-momentum is null - and lies on a light ray i.e. $\exists$ a maximum speed for massless particles (and it happens that photons - particles of light - are massless). (Gravitons are the other ones).

For $m = 0$, $E^2 = \mathbf{p}^2 c^2$ so

$$\mathbf{P} = \frac{E}{c}\begin{pmatrix}1\\ \hat{\mathbf{p}}\end{pmatrix}$$

(ω is angular frequency, and λ is wavelength of light) To get an expression for the energy, we take it from quantum mechanics

$$E = \hbar\omega = \frac{2\phi\hbar c}{\lambda}$$

$h =$ Planck's constant $= 6.6\times 10^{-340}\mathsf{m}^2\mathsf{kgs}^{-1}$. ($\hbar := \frac{h}{2\pi}$)

Newton's Laws of Motion

In special relativity, N2 becomes

$$\frac{\mathrm{d}\mathbf{P}^\mu}{\mathrm{d}\tau} = F^\mu$$

where F^μ is a 4-vector force, related to the 3-vector force $\mathbf{f}$ by

$$F^\mu = \begin{pmatrix}F^0\\ \gamma\mathbf{f}^\top\end{pmatrix}$$

so

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = \frac{1}{\gamma}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = \mathbf{f}$$

(agrees with N2). The time component is related to the power

$$F^0 = \frac{\mathrm{d}\mathbf{P}^0}{\mathrm{d}\tau} = \frac{\gamma}{c}\frac{\mathrm{d}E}{\mathrm{d}t}$$

8.4 Particle Physics

We'll discuss problems which just need conservation of 4-momentum

$$\mathbf{P} = \begin{pmatrix}\frac{E}{c}\\ \mathbf{p}\end{pmatrix}$$

Recall

$$\mathbf{P}\cdot\mathbf{P} = m^2c^2$$

and $E^2 = \mathbf{p}^2c^2 + m^2c^4$.

Often make problems easy by choosing a handy inertial frame - almost always this is the centre of mass (should be "centre of momentum", $\sum_i \mathbf{p}_i = 0$). Sometimes, $\exists$ a $\mathbf{P}^\mu$ we don't know. Can use conservation of 4-momentum, re-write equations as $\mathbf{P}^\mu = \cdots$ and then form $\mathbf{P}\cdot\mathbf{P} = \cdots$.

Particle Decay

A particle of mass m, decays to particles of mass m_2 and m_3.

$$\begin{aligned} \mathbf{P}_1 &= \mathbf{P}_2 + \mathbf{P}_3 && \text{(4-momentum conserved)} \\ \Longrightarrow E_1 &= E_2 + E_3 && \text{zeroth component} \\ \mathbf{p}_1 &= \mathbf{p}_2 + \mathbf{p}_3 && \text{first, second, third components} \end{aligned}$$

In the rest-frame of decaying particle,

$$E_1 = m_1c^2 = \sqrt{\mathbf{p}^2c^2 + m_2^2c^4} + \sqrt{\mathbf{p}_3^2c^2 + m_3^2c^4} \geq m_2c^2 + m_3c^2$$

i.e. decay only happens if $m_1 \geq m_2 + m_3$.

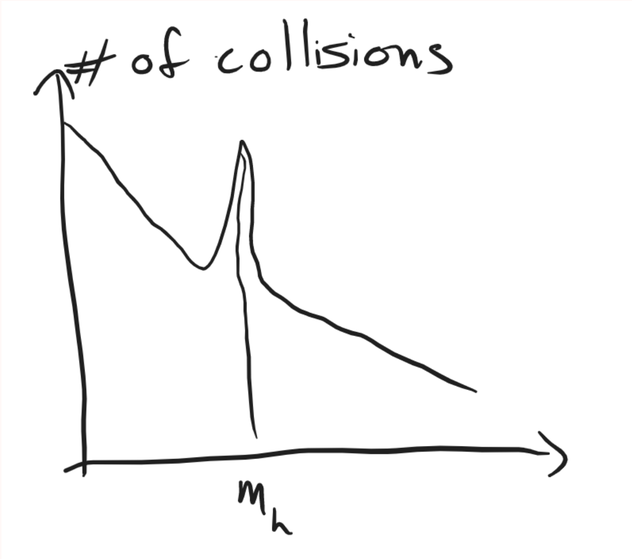
Example. $h \to \gamma\gamma$ (higgs boson decaying into 2 photons). In h rest-frame,

$$\mathbf{P}_h = \begin{pmatrix} m_h c \\ \mathbf{0}^\top \end{pmatrix}; \qquad \mathbf{P}_h = \mathbf{P}_\gamma + \mathbf{P}_{\gamma'}$$

4-momentum conserved (first, second, third components):

$$\mathbf{p}_\gamma = -\mathbf{p}_{\gamma'}$$

$$\implies E_\gamma = E_{\gamma'} = \frac{m_h c^2}{2}$$



$$(\mathbf{P}_\gamma + \mathbf{P}_{\gamma'}) \cdot (\mathbf{P}_\gamma + \mathbf{P}_{\gamma'}) = m_h^2$$

Colliders

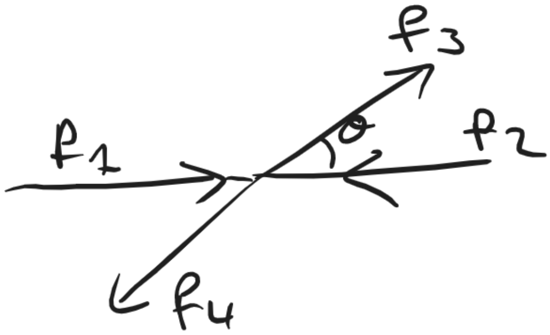
Collide 2 particles of mass m:

$$\underbrace{\mathbf{P}_1 + \mathbf{P}_2}_{\text{incoming}} = \underbrace{\mathbf{P}_3 + \mathbf{P}_4}_{\text{outgoing}}$$

In the centre of mass frame, $\mathbf{p}_1 + \mathbf{p}_2 = 0$. Pick axes so that

$$\mathbf{P}_1^\mu = (mc\gamma_v, mv\gamma_v, 0, 0)$$

$$\mathbf{P}_2^\mu = (mc\gamma_v, -mv\gamma_v, 0, 0)$$



After collision, particles must still have equal and opposite 3-momenta. Choose axes such that

$$\mathbf{P}_3^\mu = (mc\gamma_v, mv\gamma_v \cos\theta, mv\gamma_v \sin\theta, 0)$$
$$\mathbf{P}_4^\mu = (mc\gamma_v, -mv\gamma_v \cos\theta, -mv\gamma_v \sin\theta, 0)$$

Let's say that in the lab frame, one particle is at rest. Velocity addition formula implies that other particles has speed

$$u = \frac{2v}{1 + \frac{v^2}{c^2}}$$

Particle Creation

Collide 2 particles of mass m energetically enough to create an extra particle of mass M. 4-momentum conserved:

$$\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_3 + \mathbf{P}_4 + \mathbf{P}_5 \tag{†}$$

$$\mathbf{P}_1^2 = \mathbf{P}_2^2 = \mathbf{P}_3^2 = \mathbf{P}_4^2 = m^2c^2$$
$$\mathbf{P}_5^2 = M^2c^2$$

Question: what is v?
Centre of mass frame

$$\mathbf{P}_1 = (\gamma mc, \gamma m\mathbf{v})$$
$$\mathbf{P}_2 = (\gamma mc, -\gamma m\mathbf{v})$$

$(\dagger)\cdot(\dagger)$:

$$(\mathbf{P}_1 + \mathbf{P}_2)^2 := (\mathbf{P}_1 + \mathbf{P}_2)\cdot(\mathbf{P}_1 + \mathbf{P}_2) = (\mathbf{P}_3 + \mathbf{P}_4 + \mathbf{P}_5)^2 \tag{$*$}$$

Lemma. If $\mathbf{P}^2 = m_1^2c^2$ and $Q^2 = m_2^2c^2$ then $\mathbf{P}\cdot Q \geq m_1m_2c^2$.

Proof. In the rest frame of m,

$$\mathbf{P} \cdot Q = (m_1, c\mathbf{0}) \cdot \begin{pmatrix} \frac{E_2}{c} \\ |\mathbf{p}_2| \end{pmatrix}$$

$$= m_1 E_2$$

$$m_1 \sqrt{m_2^2 c^4 + \mathbf{p}^2 c^2}$$

$$\geq m_1 m_2 c^2$$

□

Expand (∗) and use lemma:

$$\mathbf{P}_1 + \mathbf{P}_2 = (\gamma mc, \mathbf{0})$$

so LHS is

$$\begin{aligned} 4\gamma^2 m^2 c^2 &= \mathbf{P}_3^2 + \mathbf{P}_4^2 + \mathbf{P}_5^2 + 2(\mathbf{P}_3 \cdot \mathbf{P}_4 + \mathbf{P}_3 \cdot \mathbf{P}_5 + \mathbf{P}_4 \cdot \mathbf{P}_5) \\ &= 2m^2c^2 + M^2c^2 + 2(\mathbf{P}_3 \cdot \mathbf{P}_4 + \mathbf{P}_3 \cdot \mathbf{P}_5 + \mathbf{P}_4 \cdot \mathbf{P}_5) \\ &\geq 2m^2c^2 + M^2c^2 + 2(m^2c^2 + mMc^2 + mMc^2) \\ \implies &\boxed{\gamma \geq 1 + \frac{M}{2m}} \end{aligned}$$

Note. Kinetic energy of incoming particle

$$T = \gamma mc^2 - mc^2 \geq \frac{1}{2}Mc^2$$