Analysis I

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Start of lecture 1

1 Limits and Convergence [b]

Review from Numbers and Sets: sequences a_n, $(a_n)_{n=1}^{\infty}$, $a_n \in \mathbb{R}$.

Definition. We say that $a_n \to a$ as $n \to \infty$ if given $\varepsilon > 0$, $\exists$ N such that $|a_n - a| < \varepsilon$ for all $n \geq N$. Note $N = N(\varepsilon)$.

Definition (Monotonic sequence)**.** A sequence is *increasing* if $a_n \leq a_{n+1}$ for all n. Similarly, a sequence is *decreasing* if $a_n \geq a_{n+1}$ for all n. The sequence is *strictly increasing / decreasing* if equality never occurs. A sequence is *monotonic* if it is either increasing or decreasing.

Axiom (Fundamental Axiom of the Real Numbers)**.** Given an increasing sequence $(a_n)_{n=1}^{\infty}$ and some $A \in \mathbb{R}$ such that $a_n \leq A$ for all n, there exists $a \in \mathbb{R}$ such that $a_n \to a$ as $n \to \infty$. So an increasing sequence of real numbers bounded above *converges*. Equivalently a decreasing sequence of real numbers bounded below converges. Equivalent also to: "Every non-empty of real numbers bounded above has a *supremum*". (LUBA = Least Upper Bound Axiom).

Definition (supremum)**.** Given $S \subset \mathbb{R}$, $S \neq \emptyset$ we say that $\sup S = K$ if

(i) $x \leq K$ $\forall$ $x \in S$

(ii) given $\epsilon > 0$, $\exists x \in S$ such that $x > K - \varepsilon$.

Note. Supremum is unique. We also can define a similar notion of infimum.

Lemma 1.1. (i) The limit is unique. That is, if $a_n \to a$ and $a_n \to b$, then $a = b$.

(ii) If $a_n \to a$ as $n \to \infty$ and $n_1 < n_2 < n_3 < \cdots$, then $a_{n_j} \to a$ as $j \to \infty$ (subsequences converge to the same limit).

(iii) If $a_n = c \;\forall\; n$, then $a_n \to c$ as $n \to \infty$.

(iv) If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.

(v) If $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$.

(vi) If $a_n \to a$, $a_n \neq 0 \;\forall n$ and $a \neq 0$, then $\frac{1}{a_n} \to \frac{1}{a}$.

(vii) If $a_n \leq A \;\forall n$ and $a_n \to a$, then $a \leq A$.

Proof. We only do (i), (ii) and (v) and leave the others as exercise.

(i) given $\varepsilon > 0$, $\exists n_1$ such that $|a_n - a| < \varepsilon \;\forall\; n \geq n_1$, and $\exists n_2$ such that $|a_n - b| < \varepsilon \;\forall\; n \geq n_2$. Then let $N = \max\{n_1, n_2\}$. Then if $n \geq N$,

$$|a - b| \leq |a_n - a| + |a_n - b| < 2\varepsilon.$$

If $a \neq b$, take $\varepsilon = \frac{|a-b|}{3}$, then by triangle inequality

$$|a - b| < \frac{2}{3}|a - b|$$

which is a contradiction if $a \neq b$, hence $a = b$.

(ii) given $\varepsilon > 0$, $\exists N$ such that $|a_n - a| < \varepsilon$, $\forall\; n \geq N$ since $n_j \geq j$ by induction, we have $|a_{n_j} - a| < \varepsilon \;\forall\; j \geq N$, i.e. $a_{n_j} \to a$ as $j \to \infty$.

(v) $|a_n b_n - ab| \leq |a_n b_n - a_n b| + |a_n b - ab| = |a_n||b_n - b| + |b||a_n - a|$. Since $a_n \to a$, given $\varepsilon > 0$, $\exists n_1$ such that $|a_n - a| < \varepsilon \;\forall n \geq n_1$, and similarly since $b_n \to b$ $\exists n_2$ such that $|b_n - b| < \varepsilon \;\forall\; n \geq n_2$. If $n \geq n_1(1)$, $|a_n - a| < 1$, so $|a_n| \leq |a| + 1$. Hence

$$|a_n b_n - ab| \leq \epsilon(|a| + 1 + |b|)$$

for all $n \geq n_3(\epsilon) = \max\{n_1(1), n_1(\epsilon), n_2(\epsilon)\}$.

□

Lemma 1.2. $\frac{1}{n} \to 0$ as $n \to \infty$.

Proof. $\frac{1}{n}$ is a decreasing sequence bounded by below, so by the Fundamental Axiom it has a limit a. We claim that $a = 0$. Note that

$$\frac{1}{2n} = \frac{1}{2} \times \frac{1}{n} \to \frac{a}{2}$$

by Lemma 1.1(v). But $\frac{1}{2}$ is a subsequence, so by Lemma 1.1(ii), $\frac{1}{2n} \to a$. By uniqueness of limits (Lemma 1.1(i)), we have $a = \frac{a}{2} \implies a = 0$. □

Remark. The definition of limit of a sequence makes perfect sense for $a_n \in \mathbb{C}$.

Definition. $a_n \to a$ if given $\varepsilon > 0$, $\exists N$ such that $\forall\ n \geq N$, $|a_n - a| < \varepsilon$.

The first six parts of Lemma 1.1 are the same over $\mathbb{C}$. The last one does not make sense (over $\mathbb{C}$) since it uses the *order* of $\mathbb{R}$.

Start of lecture 2

The Bolzano-Weierstrass Theorem

Theorem 1.3. If $x_n \in \mathbb{R}$ and there exists K such that $|x_n| \leq K\ \forall\ n$, then we can find $n_1 < n_2 < n_3 < \cdots$ and $x \in \mathbb{R}$ such that $x_{n_j} \to x$ as $j \to \infty$.

In other words every *bounded* sequence has a convergent subsequence.

Remark. We say nothing about uniqueness of x, for example $x_n = (-1)^n$, then $x_{2n+1} \to -1$ and $x_{2n} \to 1$.

Proof. Set $[a_1, b_1] = [-K, K]$. Let $c_n = \frac{a_n + b_n}{2}$ for all n. Consider the following possibilities:

(1) $x \in [a_1, c_1]$ for infinitely many values of n.

(2) $x_n \in [c_1, b_1]$ for infinitely many values of n.

(1) and (2) could hold at the same time. But if (1) holds, we set $a_2 = a_1$ and $b_2 = c_1$. If (1) fails, we have that (2) must hold and we set $a_2 = c_1$ and $b_2 = b_1$. Proceed inductively to construct sequences a_n, b_n such that $x_m \in [a_n, b_n]$ for infinitely many values of m.

$$a_{n-1} \leq a_n \leq b_n \leq b_{n-1}$$

$$b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2} \qquad (*)$$

(bisection method). Note that a_n is an increasing sequence and bounded, and b_n is a decreasing sequence and bounded, so by the Fundamental Axiom, $a_n \to a \in [a_1, b_1]$ and $b_n \to b \in [a_1, b_1]$. Using $(*)$,

$$b - a = \frac{b - a}{2} \implies a = b.$$

Since $x_m \in [a_n, b_n]$ for infinitely many values of m, having chosen n_j such that $x_{n_j} \in [a_j, b_j]$, there is $n_{j+1} > n_j$ such that $x_{n_{j+1}} \in [a_{j+1}, b_{j+1}]$ (I have an "unlimited supply"!) Since $a_j \leq x_{n_j} \leq b_j$, we have $x_{n_j} \to a$. $\square$

Cauchy Sequences

Definition (Cauchy Sequence)**.** $a_n \in \mathbb{R}$ is called a *Cauchy sequence* if given $\varepsilon > 0$, $\exists\ N > 0$ such that $|a_n - a_m| < \varepsilon\ \forall\ n, m \leq N$. (Note: $N = N(\varepsilon)$.)

Lemma 1.4. A convergent sequence is a Cauchy sequence.

Proof. If $a_n \to a$, given $\varepsilon > 0$, $\exists\ N$ such that $\forall\ n \geq N$, $|a_n - a| < \varepsilon$. Take $m, n \geq N$, then

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < 2\varepsilon.$$

□

Theorem 1.5. Every Cauchy sequence is convergent.

Proof. First we note that if a_n is Cauchy, then it is *bounded*. Take $\varepsilon = 1$, $N = N(1)$ in the Cauchy property, then

$$|a_n - a_m| < 1, \quad \forall\ n, m \geq N(1)$$

$$|a_m| \leq |a_m - a_N| + |a_N| < 1 + |a_N| \quad \forall\ m \geq N.$$

Let $K = \max\{1 + |a_N|, |a_n|, n = 1, 2, \ldots, N-1\}$. Then $|a_n| \leq K\ \forall\ n$. So by the Bolzano-Weierstrass theorem, $a_{n_j \to a}$.

Claim: $a_n \to a$.
We now prove the claim: given $\varepsilon > 0$, $\exists\ j_0$ such that $\forall\ j \geq j_0$

$$|a_{n_j} - a| < \varepsilon.$$

Also, $\exists\ N(\varepsilon)$ such that $|a_m - a_n| < \varepsilon\ \forall\ m, n \geq N(\varepsilon)$. Take j such that $n_j \geq \max\{N(\varepsilon), n_{j_0}\}$.
Then if $n \geq N(\varepsilon)$

$$|a_n - a| \leq \underbrace{|a_n - a_{n_j}|}_{<\varepsilon} + \underbrace{|a_{n_j} - a|}_{<\varepsilon} < 2\varepsilon.$$

□

Summary: in $\mathbb{R}$ a sequence is convergent is and only if it is Cauchy.
"old fashioned name": the "general principle of convergence".
Useful property: since we do not need to know what the limit is.

Series

Definition. $a_n \in \mathbb{R}, \mathbb{C}$. We say that $\sum_{j=1}^{\infty} a_j$ converges to S if the sequence of partial sums

$$S_N = \sum_{j=1}^{N} a_j \to S$$

as $N \to \infty$. We write

$$\sum_{j=1}^{\infty} a_j = S.$$

If S_N does not converge, we say that $\sum_{j=1}^{\infty} a_j$ *diverges*.

Remark. Nay problem in series is really a problem about the sequence of partial sums.

Lemma 1.6. (i) If $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ converge, then so does

$$\sum_{j=1}^{\infty} (\lambda a_j + \mu b_j)$$

where $\lambda, \mu \in \mathbb{C}$.

(ii) Suppose $\exists\ N$ such that $a_j = b_j\ \forall\ j \geq N$ then either $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ both converge or they both diverge (initial terms do not matter).

Proof.

(i) Exercise

(ii) For $n \geq N$,

$$s_n = \sum_{j=1}^{n} a_j = \sum_{j=1}^{N-1} a_j + \sum_{j=N}^{n} a_j$$

$$d_n = \sum_{j=1}^{n} b_j = \sum_{j=1}^{N-1} b_j + \sum_{j=N}^{n} b_j$$

$$\implies s_n - d_n = \sum_{j=1}^{N-1} a_j - \sum_{j=1}^{N-1} b_j = \text{constant}$$

So s_n converges if and only if d_n does.

□

Start of lecture 3

Example (Geometric Series). $x \in \mathbb{R}$, set $a_n = x^{n-1}$ for $n \geq 1$. Now

$$s_n = \sum_{j=1}^{n} a_j = 1 + x + x^2 + \cdots + x^{n-1}$$

Then

$$s_n = \begin{cases} \frac{1-x^n}{1-x} & \text{for } x \neq 1 \\ n & \text{for } x = 1 \end{cases}$$

$$xs_n = x + x^2 + \cdots + x^n = s_n - 1 + x^n$$

$$s_n(1-x) = 1 - x^n$$

if $|x| < 1$, $x^n \to 0$ and $s_n \to \frac{1}{1-x}$. If $x > 1$, $x^n \to \infty$ and $s_n \to \infty$. (Note $s_n \to \infty$ if given A, there exists N such that $s_n > A$ such that $s_n > A \ \forall \ n \geq N$, and $s_n \to -\infty$ if given A there exists N such that $s_n < -A$ for all $n \geq N$.) If $x < -1$ then s_n does not converge (oscillates). If $x = -1$ then

$$s_n = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Thus the geometric series converges if and only if $|x| < 1$.

To see for example that $x^n \to 0$ if $|x| < 1$, consider first the case $0 < x < 1$. Write $\frac{1}{x} = 1 + \delta$, $\delta > 0$. So

$$x^n = \frac{1}{(1+\delta)^n} \leq \frac{1}{1+\delta n} \to 0.$$

because $(1+\delta)^n \geq 1 + n\delta$ from binomial expansion. An easy observation from this is that:

Lemma 1.7. If $\sum_{j=1}^{\infty} a_n$ converges, then $\lim_{j\to\infty} a_j = 0$.

Proof.

$$s_n = \sum_{j=1}^{n} a_j$$

Then

$$a_n = s_n - s_{n-1}.$$

If $s_n \to a$, then $a_n \to 0$ (since $s_{n-1} \to a$ as well). □

Remark. The converse of lemma 1.7 is false! For example, $\sum_{j=1}^{\infty} \frac{1}{j}$ diverges (harmonic series).

$$s_n = \sum_{j=1}^{n} \frac{1}{j}$$

$$s_{2n} = s_n + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} > s_n + \frac{1}{2}$$

since $\frac{1}{n+k} \geq \frac{1}{2n}$ for $k = 1, 2, \ldots, n$. So if $s_n \to a$, then $s_{2n} \to a$ also, and thus $a \geq a + \frac{1}{2}$ ※

Series of Non-negative Terms

$a_n \geq 0$. Basic result:

Theorem 1.8 (The comparison test)**.** Suppose that $0 \leq b_n \leq a_n \; \forall \; n$. Then if $\sum_{j=1}^{\infty} a_j$ converges, then so does $\sum_{j=1}^{\infty} b_j$.

Proof. Let $s_n = \sum_{j=1}^{N} a_j$ and let $d_N = \sum_{j=1}^{N} b_j$. Since $b_n \leq a_n$ we have that $d_N \leq s_N$. But $s_N \to s$, so $d_N \leq s_N \leq s \; \forall \; N$. Also, d_N is an increasing sequence bounded above, hence d_N converges. □

Example.

$$\sum_{j=1}^{\infty} \frac{1}{n^2}$$

$$\underbrace{\frac{1}{n^2} < \frac{1}{n(n-1)}}_{n \geq 2} = \frac{1}{n-1} - \frac{1}{n} = a_n$$

So

$$\sum_{j=2}^{N} a_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{N-1} - \frac{1}{N} = 1 - \frac{1}{N} \to 1$$

So by comparison, $\sum_{j=1}^{N} \frac{1}{n^2}$ converges. In fact we get that

$$\sum_{j=1}^{\infty} \frac{1}{n^2} \leq 1 + 1 = 2.$$

Theorem 1.9 (Root test / Cauchy's test for convergence)**.** Assume $a_n \geq 0$ and $a_n^{1/n} \to a$ as $n \to \infty$. Then if $a < 1$, $\sum a_n$ converges; if $a > 1$, $\sum a_n$ diverges.

Remark. Nothing can be said if $a = 1$ (examples coming up).

Proof. If $a < 1$, choose $a < r < 1$. By definition of limit and hypothesis, there exists N such that for all $n \geq N$,

$$a_n^{1/n} < r \implies a_n < r^n$$

But since $r < 1$, the geometric series $\sum r^n$ converges, so by theorem 1.8, $\sum a_n$ converges. If $a > 1$, then for $n \geq N$, then $a_n^{1/n} > 1 \implies a_n > 1$, thus $\sum a_n$ diverges (since a_n does *not* tend to zero). □

Theorem 1.10 (Ratio test / D'Alembert's test)**.** Suppose $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to \ell$. If $\ell < 1$, $\sum a_n$ converges. If $\ell > 1$, $\sum a_n$ converges.

Note. As before, nothing can be said for $\ell = 1$.

Proof. Suppose $\ell < 1$ and choose r with $\ell < r < 1$. Then there exists N such that for all $n \geq N$,

$$\frac{a_{n+1}}{a_n} < r$$

Therefore

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N < a_N r^{n-N}$$
$$\implies a_n < K r^n$$

with K *independent* of n. Since $\sum r^n$ converges, so does $\sum a_n$ by theorem 1.8. If $\ell > 1$, choose $1 < r < \ell$, then $\frac{a_{n+1}}{a_n} > r$ for all $n \geq N$, and as before

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N > a_N r^{n-N} > a_N$$

so $\sum a_n$ diverges. □

Start of lecture 4

Examples

- $\sum_{j=1}^{\infty} \frac{j}{2^j}$. Then

  $$\frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} \to \frac{1}{2} < 1.$$

  So we have convergence by ratio test.

- $\sum_{j=1}^{\infty} \frac{1}{n}$ diverges, and $\sum_{j=1}^{\infty} \frac{1}{n^2}$ converges. Note ratio test gives limit 1 in both cases, so *inconclusive* if limit is 1. Since $n^{1/n} \to 1$ as $n \to \infty$, the root test is also inconclusive when limit is 1. To see this limit, write

$$n^{1/n} = 1 + \delta_n, \qquad \delta_n > 0.$$

$$n = (1 + \delta_n)^n > \frac{n(n-1)}{2}\delta_n^2$$

(binomial expansion)

$$\implies \delta_n^2 < \frac{2}{n-1} \implies \delta_n \to 0$$

- $\sum_{j=1}^{\infty} \left[\frac{n+1}{3n+5}\right]^n$ converges by root test since

$$\frac{n+1}{3n+5} \to \frac{1}{3} < 1.$$

Another useful test:

Theorem 1.11 (Cauchy's Condensation Test)**.** Let a_n be a decreasing sequence of positive terms. Then $\sum_{j=1}^{\infty} a_n$ converges if and only if

$$\sum_{j=1}^{\infty} 2^n a_{2^n}$$

converges.

Proof. First we observe that if a_n is decreasing, then

$$a_{2^k} \le a - 2^{k-1} + i \le a_{2^{k-1}}, \qquad 1 \le i \le 2^{k-1}$$

(for any $k \ge 1$.) Assume now that $\sum_{j=1}^{\infty} a_j$ converges with sum A. Then

$$2^{n-1} a_{2^n} \le a_{2^{n-1}+1} + a_{2^{n-1}+2} + \cdots + a_{2^n} = \sum_{m=2^{n-1}}^{2^n} a_m.$$

Thus

$$\sum_{n=1}^{N} 2^{n-1} a_{2^n} \le \sum_{n=1}^{N} \sum_{m=2^{n-1}+1}^{2^n} a_m = \sum_{m=2}^{2^N} a_m.$$

$$\implies \sum_{n=1}^{N} 2^n a_{2^n} \le 2 \sum_{m=2}^{2^N} a_m \le 2(A - a_1)$$

Thus $\sum_{n=1}^{N} 2^n a_{2^n}$ being increasing and bounded above, *converges*. Conversely, assume that $\sum_{j=1}^{\infty} 2^j a_{2^j}$ converges. Then

$$\sum_{m=2^{n-1}+1}^{2^n} a_m \leq \sum_{m=2^{n-1}+1}^{2^n} a_{2^{n-1}} = 2^{n-1} a_{2^{n-1}}.$$

$$\implies \sum_{m=2}^{2^N} a_m = \sum_{n=1}^{N} \sum_{m=2^{n-1}+1} a_m \leq \sum_{n=1}^{N} 2^{n-1} a_{2^{n-1}} \leq B.$$

So $\sum_{m=1}^{N} a_m$ is a bounded increasing sequence and thus it converges. □

Examples / Applications

Claim. $\sum_{j=1}^{\infty} \frac{1}{n^k}$ converges if and only if $k > 1$.

Proof. Note that it is a decreasing sequence of positive terms.

$$\frac{1}{(n+1)^k} < \frac{1}{n^k}, \qquad \left(\frac{n}{n+1}\right)^k < 1$$

Now:

$$2^n a_{2^n} = 2^n \left[\frac{1}{2^n}\right]^k = 2^{n-nk} = (2^{1-k})^n$$

so it is a geometric series with ratio 2^{1-k}, and it converges if and only if $2^{1-k} < 1$, so if and only if $k > 1$. □

Alternating Series

Theorem 1.12 (Alternating Series Test)**.** If a_n decreases and tends to zero as $n \to \infty$, then the series $\sum_{j=1}^{\infty} (-1)^{n+1} a_n$ converges.

Example. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Proof. Let $s_n = a_1 - a_2 + \cdots + (-1)^{n+1} a_n$. Note

$$s_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + \underbrace{(a_{2n-1} - a_{2n})}_{\geq 0} \geq s_{2n-2}$$

$$s_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1$$

So s_{2n} is increasing and bounded above, so $s_{2n} \to s$. Also note $s_{2n+1} = s_{2n} + a_{2n+1} \to s + 0 = s$. This implies that s_n converges to s:
Given $\varepsilon > 0$, there exists N_1 such that for all $n \geq N_1$, $|s_{2n} - s| < \varepsilon$ and there exists N_2 such that for all $n \geq N_2$, $|s_{2n+1} - s| < \varepsilon$. Take $N = 2\max\{N_1, N_2\} + 1$. Then if $k \geq N$, we have $|s_k - s| < \varepsilon$, so $s_k \to s$. □

Start of lecture 5

Absolute Convergence

Definition. Take $a_n \in \mathbb{C}$. If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then the series is called *absolutely convergent.*

Note. Since $|a_N| \geq 0$, we can use the previous tests to check absolute convergence. This is particularly useful for $a_n \in \mathbb{C}$.

Theorem 1.13. If $\sum a_n$ is absolutely convergent, then it is convergent.

Proof. Suppose first $a_n \in \mathbb{R}$. Let

$$v_N = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases}$$

$$w_n = \begin{cases} 0 & \text{if } a_n \geq 0 \\ -a_n & \text{if } a_n < 0 \end{cases}$$

$$v_n = \frac{|a_n| + a_n}{2}, \qquad w_n = \frac{|a_n| - a_n}{2}$$

Clearly, $v_n, w_n \geq 0$. Note $a_n = v_n - w_n$, and $|a_n| = v_n + w_n \geq v_n, w_n$. So if $\sum |a_n|$ converges, by comparison $\sum v_n$, $\sum w_n$ also converge, hence $\sum a_n$ converges. If $a_n \in \mathbb{C}$, then $a_n = x_n + iy_n$. Now $|x_n|, |y_n| \leq |a_n|$, so $\sum x_n$ and $\sum y_n$ are absolutely convergent, hence $\sum x_n$ and $\sum y_n$ converge. Since $a_n = x_n + iy_n$ we have that $\sum a_n$ converges as well. □

Examples

(1) $\sum \frac{(-1)^{n+1}}{n}$ converges but is *not* absolutely convergent.

(2) $\sum_{n=1}^{\infty} \frac{z^n}{2^n}$ for $z \in \mathbb{C}$, then if $|z| < 2$ we have absolute convergence. If $|z| \geq 2$, $\left|\frac{z}{2}\right|^n \geq 1$, so a_n does not tend to 0, hence the series diverges.

Definition. If $\sum a_n$ converges, but $\sum |a_n|$ does not, it is said sometimes, that $\sum a_n$ is *conditionally* convergent.

"conditional": because the sum to which the series converge is conditional on the order in which elements of the sequence are taken. If *rearranged*, the sum is altered.

Example. (Example Sheet 1, Q7)

(i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

(ii) $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \cdots$.

Let s_n be the partial sum of (i) and t_n the partial sum of (ii). Then $s_n \to s > 0$, and $t_n \to \frac{3s}{2}$.

Rearrangement:

Definition. Let σ be a bijection of the positive integers,

$$a_n' = a_{\sigma(n)}$$

is a rearrangement.

Theorem 1.14. If $\sum a_n$ is absolutely convergent, every series consisting of the same terms in any order (i.e. a rearrangement) has the *same sum*.

Proof. We do the proof first for $a_n \in \mathbb{R}$. Let $\sum a_n'$ be a rearrangement of $\sum a_n$. Let $s_n = \sum_{j=1}^n a_j$ and $t_n = \sum_{j=1}^n a_j'$, $s = \sum_{j=1}^\infty a_j$. Suppose first that $a_n \ge 0$. Given n, we can find q such that sq satisfies

$$t_n \le sq \le s$$

Now since t_n is an increasing sequence bounded above, $t_n \to t$. Clearly $t \le s$. But by symmetry, $s \le t$, hence $t = s$.
If a_n has any sign, consider v_n and w_n from theorem 1.13. Consider $\sum a_n'$, $\sum v_n'$ and $\sum w_n'$. Since $\sum |a_n|$ converges, both $\sum v_n$ and $\sum w_n$ converge. Use that $v_n, w_n \ge 0$ to deduce that $\sum v_n' = \sum v_n$ and $\sum w_n' = \sum w_n$. But $a_n = v_n - w_n$ hence $\sum a_n = \sum a_n'$.
For the case $a_n \in \mathbb{C}$, write $a_n = x_n + iy_n$. Since $|x_n|, |y_n| \le |a_n|$, we have that $\sum x_n$ and $\sum y_n$ are absolutely convergent. By the previous case, $\sum x_n' = \sum x_n$ and $\sum y_n' = \sum y_n$ since $a_n' = x_n' + iy_n' \implies \sum a_n = \sum a_n'$. □

Start of lecture 6

2 Continuity [3]

Let $E \subseteq \mathbb{C}$ non-empty, $f : E \to \mathbb{C}$ any function, and let $a \in E$. (This includes the case in which f Is real-valued and $E \subseteq \mathbb{R}$).

Definition 1. f is continuous at $a \in E$ if for every sequence $z_n \in E$ with $z_n \to a$, we have $f(z_n) \to f(a)$.

Definition 2. f is continuous at $a \in E$, if given $\varepsilon > 0$, $\exists \delta > 0$ such that if $|z - a| < \delta$, $z \in E$, then

$$|f(z) - f(a)| < \varepsilon$$

(ε-δ definition).

We will prove that these two definitions are equivalent.
Proof. We know that given $\varepsilon > 0$, $\exists \delta > 0$ such that $|z-a| < \delta$, $z \in E$, then $|f(z)-f(a)| < \varepsilon$. Let $z_n \to a$. Then $\exists n_0$ such that $\forall\ n \geq n_0$ we have $|z_n - a| < \delta$ hence $|f(z_n) - f(a)| < \varepsilon$ so $f(z_n) \to f(a)$. For the other direction, assume that $f(z_n) \to f(a)$ whenever $z_n \to a$ ($z_n \in E$). Suppose f is *not* continuous at a according to definition 2. Then:

$\exists \varepsilon > 0$ such that $\forall\ \delta > 0$, there exists $z \in E$ such that $|z - a| < \delta$ and $|f(z) - f(a)| \geq \varepsilon$.

Let $\delta = \frac{1}{n}$, from the above we get z_n such that $|z_n - a| < \frac{1}{n}$ and $|f(z_n) - f(a)| \geq \varepsilon$. Clearly $z_n \to a$, but $f(z_n)$ does not tend to $f(a)$ because $f(z_n) - f(a)| \geq \varepsilon$, contradiction. □

Proposition 2.1. $a \in E$, $g, f : E \to \mathbb{C}$ continuous at a¿ Then so are the functions $f(z) + g(z)$, $f(z)g(z)$ and $\lambda f(z)$ for any constant λ. In addition if $f(z) \neq 0\ \forall\ z \in E$ then $\frac{1}{f}$ is continuous at a.

Proof. Using definition 1, this is obvious. Using the analogous results for sequences (lemma 1.1), for example if $z_n \to a$ then $f(z_n) \to f(a)$ and $g(z_n) \to g(a)$ so by lemma 1.1 $f(z_n) + g(z_n) \to f(a) + g(a)$ etc. □

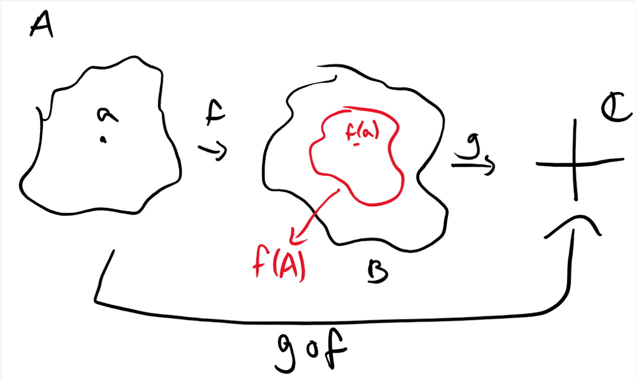
The function $f(z) = z$ is continuous, so by using the proposition, we get that every polynomial is continuous at every point in $\mathbb{C}$.

Note. We say that f is continuous on E if it is continuous at every $a \in E$.

Remark. Still it is *instructive* to prove proposition 2.1 directly from the ε-δ definition.

Next we look at compositions.

Theorem 2.2. Let $f : A \to \mathbb{C}$ and $g : B \to \mathbb{C}$ and $g : B \to \mathbb{C}$ be two functions such that $f(A) \subset B$. Suppose f is continuous at $a \in A$ and g is continuous at $f(a)$. Then $g \circ f : A \to \mathbb{C}$ is continuous at a.



Proof. Take any sequence $z_n \to a$. By assumption $f(z_n) \to f(a)$. Set $w_n = f(z_n) \in B$, $w_n \to f(a)$; thus $g(w_n) = g(f(z_n)) \to g(f(a))$. □

Examples

(1) $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\sin x$ is continuous (to be proved later!)
if $x \neq 0$, then 2.1 and 2.2 imply that $f(x)$ is continuous at every $x \neq 0$. Discontinuous at 0 because let $x_n = \frac{1}{(2n+\frac{1}{2})\pi}$, then $f(x_n) = 1$, $x_n \to 0$ but $f(0) = 0$.

(2)
$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

f is continuous at 0, take $x_n \to 0$ then

$$|f(x_n)| \leq |x_n|$$

so $f(x_n) \to 0 = f(0)$.

(3)
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Discontinuous at every point: if $x \in \mathbb{Q}$, take a sequence $x_n \to x$ with $x_n \notin \mathbb{Q}$, then $f(x_n) = 0$ which doesn't tend to $f(x) = 1$. Similarly if $x \notin \mathbb{Q}$, take $x_n \to x$ with $x_n \in \mathbb{Q}$. Then $f(x_n) = 1$ so doesn't tend to $f(x) = 0$.

Start of lecture 7

Limit of a function

$f : E \subseteq \mathbb{C} \to \mathbb{C}$. We wish to define what is meant by $\lim_{z\to a} f(z)$, even when a might *not* be in E. For example $\lim_{z\to 0} \frac{\sin z}{z}$, with $E = \mathbb{C} \setminus \{0\}$. Also, if $E = \{0\} \cup [1, 2]$ it does not make sense to speak about $z \in E$, $z \neq 0$, $z \to 0$.

Definition. $E \subseteq \mathbb{C}$, $a \in \mathbb{C}$. We say that a is a *limit point* of E if for any $\delta > 0$, $\exists \ \in E$ such that $0 < |z - a| < \delta$.

Remark. a is a limit point if and only if $\exists$ a sequence $z_n \in E$ such that $z_n \to a$ and $z_n \neq a \ \forall \ n$. (Check the equivalence!)

Definition. $f : E \subseteq \mathbb{C} \to \mathbb{C}$ and let $a \in \mathbb{C}$ be a limit point of E. We say that $\lim_{z\to a} f(z) = l$ ("f tends to l as z tends to a") if given $\varepsilon > 0$, $\exists\delta > 0$ such that whenever $0 < |z - a| < \delta$ and $z \in E$, then $|f(z) - l| < \varepsilon$.

Equivalently: $f(z_n) \to l$ for every sequence $z_i \in E$, $z_n \neq a$ and $z_n \to a$ (proved exactly as last time with definition 1 $\iff$ definition 2).

Remark. Straight from the definitions we have that if $a \in E$ is a limit point, then $\lim_{z\to a} f(z) = f(a)$ if and only if f is continuous at a.

If $a \in E$ is *isolated* (i.e. $a \in E$ and is not a limit point) then continuity of f at a always holds.
The limit of functions has very similar properties to limit of sequences.

(1) It is unique, $f(z) \to A$ and $f(z) \to B$ as $z \to a$

$$|A - B| \leq |A - f(z)| + |f(z) - B|$$

if $z \in E$ is such that $|z - a| < \delta_1, \delta_2$ then $|A - B| < 2\varepsilon$ so $A = B$. (the $\exists$ of such z is consequence of the condition that a is a limit point of E).

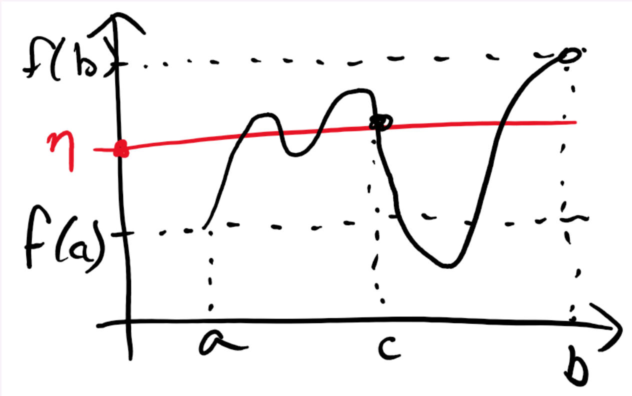
(2) $f(z) + g(z) \to A + B$ ($f(z) \to A$, $g(z) \to B$ as $z \to a$).

(3) $f(z)g(z) \to AB$

(4) if $B \neq 0$, $\frac{f(z)}{g(z)} \to \frac{A}{B}$ all proved in the same way as before.

2.1 The Intermediate Value Theorem

Theorem 2.3. $f : [a,b] \to \mathbb{R}$ continuous and $f(a) \neq f(b)$. Then f takes every value which lies between $f(a)$ and $f(b)$.



(for all $f(a) < \eta < f(b)$, $\exists c \in [a,b]$ such that $f(c) = \eta$)

Proof. Without loss of generality we may suppose that $f(a) < f(b)$. Take $f(a) < \eta < f(b)$. Let

$$S = \{x \in [a,b] : f(x) < \eta\}$$

$a \in S$, so $S \neq \emptyset$. Clearly S is bounded above by b. Then there is a supremum c where $c \leq b$. By definintion of supremum, given n, there exists $x_n \in S$ such that

$$c - \frac{1}{n} < x_n \leq c$$

so, $x_n \to c$ since $x_n \in S$, $f(x_n) < \eta$. By continuity of f, $f(x_n) \to f(c)$. Thus $f(c) \leq \eta$. Now observe that $c \neq b$. Then for n large, we can consider $c + \frac{1}{n} \in [a,b]$ and $c + \frac{1}{n} \to c$. Again by continuity

$$f\left(c + \frac{1}{n}\right) \to f(c)$$

but since $c + \frac{1}{n} > c$, $f\left(c + \frac{1}{n}\right) \geq \eta$ (by definition of supremum). Hence $f(c) \geq \eta$ and therefore $f(c) = \eta$. $\square$

Remark. The theorem is very useful for finding zeros or fixed points.

Example. Existence of the n-th root of a positive real number.

$$f(x) = x^n, \qquad x \geq 0$$

Let y be a positive real number. f is continuous on $[0, 1+y]$ and

$$0 = f(0) < y < (1+y)^n = f(1+y)$$

so by the Intermediate Value Theorem, $\exists c \in (0, 1+y)$ such that $f(c) = y$, i.e. $c^n = y$ so c is a positive n-root of y. We also have uniqueness! (check)

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Bounds of a continuous function

Theorem 2.4. Let $f : [a,b] \to \mathbb{R}$ be continuous. Then there exists K such that $|f(x)| \leq K \; \forall \; x \in [a,b]$.

Proof. We argue by contradiction. Suppose statement is false. Then given any integer $n \geq 1$, there exists $x_n \in [a,b]$ such that $|f(x_n)| > n$. By Bolzano-Weierstrass, x_n has a convergent subsequence $x_{n_j} \to x$. Since $a \leq x_{n_j} \leq b$, we must have $x \in [a,b]$. By the continuity of f, $f(x_{n_j}) \to f(x)$ but $|f(x_{n_j}| > n_j \to \infty$ (as $j \to \infty$). ※ □

Theorem 2.5. $f : [a,b] \in \mathbb{R}$ continuous. Then $\exists x_1, x_2 \in [a,b]$ such that

$$f(x_1) \leq f(x) \leq f(x_2)$$

for all $x \in [a,b]$. ("A continuous function on a closed bounded interval is bounded and attains its bounds").

Proof. Let $A = \{f(x) : x \in [a,b]\} = f([a,b])$. By Theorem 2.4, A is bounded. Since it is clearly non-empty, it has supremum, M. By definition of supremum, given an integer $n \geq 1$, $\exists x_n \in [a,b]$ such that

$$M - \frac{1}{n} < f(x_n) \leq M \qquad (*)$$

By Bolzano-Weierstrass, $\exists \; x_{n_j} \to x \in [a,b]$. Since $f(x_{n_j}) \to M$ (by $(*)$) and f is continuous, we deduce that $f(x) = M$. So $x_2 := x$. Similarly for the minimum. □

Proof (alternative proof). $A = f([a,b])$, $M = \sup A$ as before. Suppose $\nexists \; x_2$ such that $f(x_2) = M$. Let

$$g(x) = \frac{1}{M - f(x)}$$

for $x \in [a,b]$. It is defined and continuous on $[a,b]$. By Theorem 2.4 applied to g, $\exists k > 0$ such that

$$g(x) \leq K \qquad \forall \; x \in [a,b]$$

This means that $f(x) \leq M - \frac{1}{k}$ for all $x \in [a, b]$. This is absurd since it contradicts that M is the supremum. □

Note. Theorems 2.4 and 2.5 are *false* if the interval is not *closed* and bounded. For example, consider

$$(0, 1], \qquad f(x) = \frac{1}{x}$$

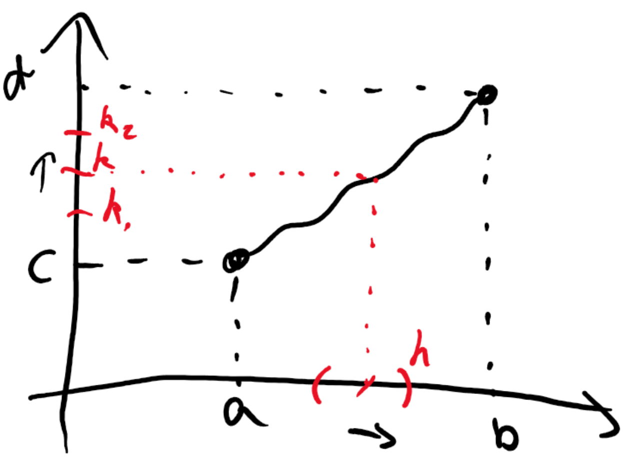
2.2 Inverse Functions

Definition. f is *increasing* for $x \in [a, b]$ if $f(x_1) \leq f(x_2)$ for all x_1, x_2 such that $a \leq x_1 < x_2 \leq b$. If $f(x_1) < f(x_2)$ we say that f is *strictly increasing*. Similarly for decreasing and strictly decreasing.

Theorem 2.6. $f : [a, b] \to \mathbb{R}$ continuous and strictly increasing function $x \in [a, b]$. Let $c = f(a)$ and $d = f(b)$. Then $f : [a, b] \to [c, d]$ is bijective and the inverse $g := f^{-1} : [c, d] \to [a, b]$ is continuous and strictly increasing.

Remark. A similar theorem holds for strictly *decreasing* functions.

Proof.



Take $c < k < d$. From the Intermediate Value Theorem, $\exists h$ such that $f(h) = k$. Since f is strictly increasing h is unique. Define $g(k) := h$ and this gives an inverse

$$g : [c, d] \to [a, b]$$

for f.

- g is strictly increasing because $y_1 < y_2$, $y_1 = f(x_1)$, $y_2 = f(x_2)$. If $x_2 \leq x_1$ then since f is increasing $f(x_2) \leq f(x_1)$ and so $y_2 \leq y_1$, contradiction.
- g is continuous because let $\varepsilon > 0$ be given, then let $k_1 = f(h - \varepsilon)$ and $k_2 = f(h + \varepsilon)$. f is strictly increasing so $k_1 < k < k_2$. If $k_1 < y < k_2$ then $h - \varepsilon < g(y) < h + \varepsilon$ so g is continuous at k. Here we took $k \in (c, d)$ but a very similar argument establishes continuity at the endpoints (check!)

□

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3 Differentiability [5]

Let $f : E \subseteq \mathbb{C} \to \mathbb{C}$, most of the time $E =$ interval $\subseteq \mathbb{R}$.

Definition. Let $x \in E$ be a point such that $\exists x_n \in E$ with $x_n \neq x \ \forall\ n$ and $x_n \to x$ (i.e. a limit point). f is said to be *differentiable* at x with derivative $f'(x)$ if

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = f'(x)$$

If f is differentiable at each $x \in E$ then we say that f is differentiable on E. (Think of E as an interval or a disc in the case of $\mathbb{C}$).

Important Remarks

(1) Other common notations:

$$\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \frac{\mathrm{d}f}{\mathrm{d}x}$$

(2) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $(y = x + h)$

(3) Another look at the definition: Let

$$\varepsilon(h) := f(x + h) - f(x) - hf'(x)$$

then

$$\lim_{h \to 0} \frac{\varepsilon(h)}{h} = 0$$

$$f(x + h) = f(x) + hf'(x) + \varepsilon(h)$$

Alternative definition of differentiability:

Definition. f is differentiable at x if $\exists A$ and ε such that

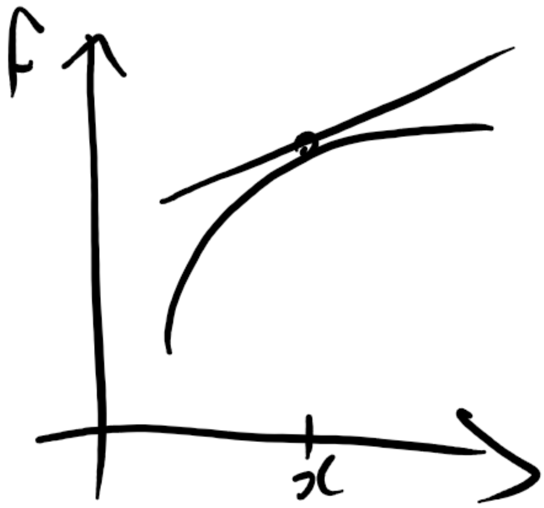
$$f(x + h) = f(x) + hA + \varepsilon(h)$$

where

$$\lim_{h \to 0} \frac{\varepsilon(h)}{h} = 0$$

If such an A exists, then it is *unique* since

$$A = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$



(4) If f is differentiable at x then f is continuous at x since $\varepsilon(h) \to 0$, so $f(x+h) \to f(x)$ as $h \to 0$.

(5) Alternative ways of writing things:

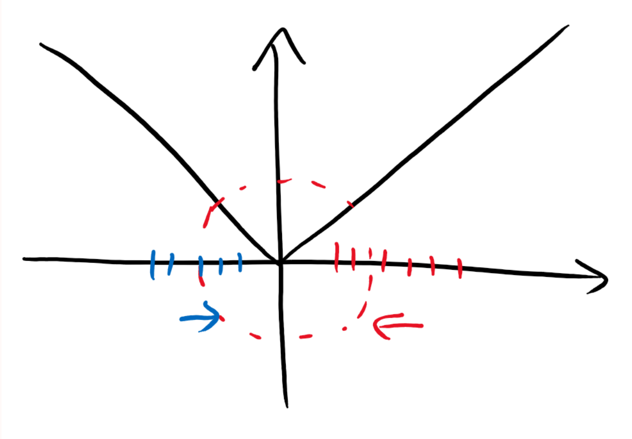
$$f(x+h) = f(x) + hf'(x) + h\varepsilon_f(h)$$

with $\varepsilon_f(h) \to 0$ as $h \to 0$. Or

$$f(x) = f(a) + (x-a)f'(a) + (x-a)\varepsilon_f(x)$$

where $\lim_{x\to a} \varepsilon_f(x) = 0$ as $x \to a$.

Example. $f(x) = |x|$, $f : \mathbb{R} \to \mathbb{R}$.



Clearly $f'(x) = 1$ for $x > 0$ and $f'(x) = -1$ for $x < 0$. Now for $x = 0$:
Take $h_n > 0$:

$$\lim_{n\to\infty} \frac{f(h_n) - f(0)}{h_n} = \lim \frac{h_n}{h_n} = 1$$

Take $h_n < 0$:

$$\lim_{n\to\infty} \frac{f(h_n) - f(0)}{h_n} = \lim -\frac{h_n}{h_n} = -1$$

so *not* differentiable at $x = 0$.

Differentiation of Sums, Products, etc

Proposition 3.1. (i) If $f(x) = c$ for all $x \in E$ then f is differentiable with $f'(x) = 0$.

(ii) f, g differentiable at x, then so is $f + g$ and

$$(f + g)'(x) = f'(x) + g'(x)$$

(iii) f, g differentiable at x, then so is fg and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

(iv) f differentiable at x and $f(x) \neq 0 \;\forall\; x \in E$, then $\frac{1}{f}$ is differentiable at x and

$$\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{[f(x)]^2}$$

Proof.

(i) $\lim_{h\to 0} \frac{c-c}{h} = 0$

(ii)

$$\lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$
$$= f'(x)+g'(x)$$

using properties of limits

(iii) $\phi(x) = f(x)g(x)$.

$$\lim_{h\to 0} \frac{\phi(x+h)-\phi(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0} f(x+h)\left[\frac{g(x+h)-g(x)}{h}\right] + g(x)\left[\frac{f(x+h)-f(x)}{h}\right] = f(x)g'(x)+f'(x)g(x)$$

using standard properties of limits and the fact that f is continuous at x.

(iv) $\phi(x) = \frac{1}{f(x)}$

$$\lim_{h\to 0} \frac{\phi(x+h)-\phi(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h}$$

$$= \lim_{h\to 0} \frac{f(x) = f(x)+h}{h} \times \frac{1}{f(x)f(x+h)} = -\frac{f'(x)}{[f(x)]^2}$$

□

Remark. from (iii) and (iv) we get

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

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Example. $f(x) = x^n$, $n \in \mathbb{Z}$, $n > 0$. $n = 1$, clearly $f(x) = x$ and $f'(x) = 1$.

Claim. $f'(x) = nx^{n-1}$

Proof. Induction ($n = 1$ is clear). $f(x) = xx^n = x^{n+1}$. So

$$f'(x) = x^n + x(nx^{n-1}) = (n+1)x^n$$

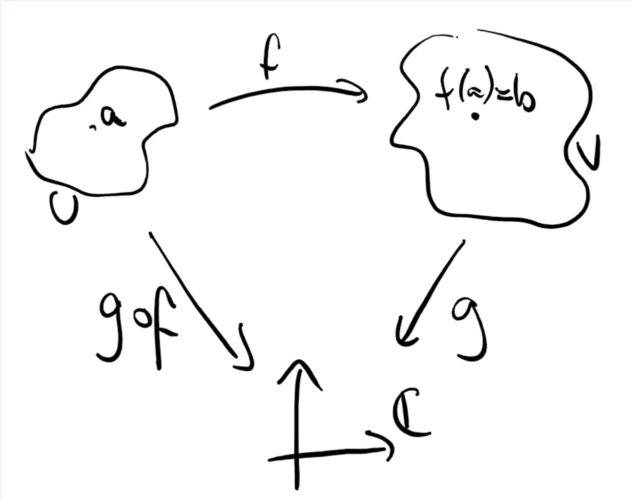
□ $n = 0$ can be done separately, and negative n can be done using Proposition 3.1 (iv):

$$f'(x) = -\frac{(x^n)'}{x^{2n}} = -\frac{nx^{n-1}}{x^{2n}} = -nx^{n-1}$$

Here is another useful result:

Theorem 3.2 (Chain rule)**.** $f : U \to \mathbb{C}$ is such that $f(x) \in V \; \forall \, x \in U$. If f is differentiable at $a \in U$ and $g : V \to \mathbb{C}$ is differentiable at $f(a)$, then $g \circ f$ is differentiable at a with

$$(g \circ f)'(a) = f'(a)g'(f(a))$$



Proof. We know:

$$f(x) = f(a) + (x-a)f'(a) + \varepsilon_f(x)(x-a)$$

(where $\lim_{x \to a} \varepsilon_f(x) = 0$). Also

$$g(y) = g(b) + (y-b)g'(b) + \varepsilon_g(y)(y-b)$$

(where $\lim_{y \to b} \varepsilon_g(y) = 0$).
Let $b = f(a)$. Set $\varepsilon_f(a) = 0$ and $\varepsilon_g(b) = 0$ to make them continuous at $x = a$ and $y = b$.
Now $y = f(x)$ gives

$$g(f(x)) = g(b) + (f(x) - b)g'(b) + \varepsilon_g(f(x))(f(x) - b)$$

$$= g(f(a)) + [(x-a)f'(a) + \varepsilon_f(x)(x-a)][g'(B) + \varepsilon_g(f(x))]$$

$$= g(f(a)) + (x-a)f'(a)g'(b) + (x-a)[\varepsilon_f(x)g'(b) + \varepsilon_g(f(x))(f'(a) + \varepsilon_f(x))]$$

$$\sigma(x) = \underbrace{\varepsilon_f(x)g'(b)}_{\to 0} + \underbrace{\varepsilon_g(f(x))}_{\to 0}\underbrace{(f'(a) + \varepsilon_f(x))}_{\to f'(a)} \to 0$$

□

Examples

(1) $f(x) = \sin(x^2)$, $(\sin x)' = \cos x$

$$f'(x) = 2x\cos(x^2)$$

(2)

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

From previous lectures f is continuous. It is differentiable at every $x \neq 0$ by the previous theorems. At $x = 0$,

$$\frac{f(t) - f(0)}{t} = \sin\left(\frac{1}{t}\right)$$

so the limit does not exist, so f is not differentiable at $x = 0$.

The Mean Value Theorem

Theorem 3.3 (Rolle's Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then $\exists c \in (a, b)$ such that $f'(c) = 0$.

Proof. Let $M = \max_{x\in[a,b]} f(x)$ and $m = \min_{x\in[a,b]} f(x)$. Recall by Theorem 2.5 that these values are achieved. Let $k = f(a) = f(b)$. If $M = m = k$, then f is constant and $f'(c) = 0 \;\forall\; c \in (a, b)$. If f is not constant then $M > k$ or $m < k$. Suppose $M > k$. By Theorem 2.5 $\exists\; c \in (a, b)$ such that $f(c) = M$. If $f'(c) > 0$, then there are values to the right of c for which $f(x) > f(c)$. Why?

$$f(h + c) - f(c) = h(f'(c) + \varepsilon_f(h))$$

since $\varepsilon_f(h) \to 0$ as $h \to 0$, $f'(c) + \varepsilon_f(h) > 0$ for h small. This contradicts that M is the maximum. Similarly if $f'(c) < 0$ there exists x to the left of c for which $f(x) > f(c)$. Hence $f'(c) = 0$. □

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Theorem 3.4 (Mean Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Then $\exists\; c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a)$$

Proof. Write

$$\phi(x) = f(x) - kx$$

choose k such that $\phi(a) = \phi(b)$. Hence $f(b) - bk = f(a) - ak$

$$\implies k = \frac{f(b) - f(a)}{b - a}$$

By Rolle's theorem applied to ϕ, $\exists\; c \in (a, b)$ such that $\phi'(c) = 0$, i.e. $f'(c) = k$. □

Remark. We will often write

$$f(a+h) = f(a) + hf'(a+\theta h)$$

for $\theta \in (0,1)$. (Note that $\theta = \theta(h)$!)

Corollary 3.5. $f : [a,b] \to \mathbb{R}$ continuous and differentiable on (a,b).

(i) If $f'(x) > 0 \ \forall\ x \in (a,b)$ then f is strictly increasing. (i.e. if $b \geq y > x \geq a$, then $f(y) > f(x)$)

(ii) If $f'(x) \geq 0 \ \forall\ x \in (a,b)$ then f is increasing (i.e. if $b \geq y > x \geq a$ then $f(y) \geq f(x)$)

(iii) If $f'(x) = 0 \ \forall\ x \in (a,b)$ then f is constant on $[a,b]$.

Proof.

(i) MVT

$$\implies f(y) - f(x) = f'(c)(y-x)$$
$$f'(c) > 0 \implies f(y) > f(x)$$

(ii) Same but $f(c) \geq 0 \implies f(y) \geq f(x)$.

(iii) Take $x \in [a,b]$. Then use the MVT in $[a,x]$ to get $c \in (a,x)$ such that

$$f(x) - f(a) = f'(c)(x-a) = 0$$

$$\implies f(x) = f(a)$$

so f is continuous.

□

Inverse Rule / Inverse Function Theorem

Theorem 3.6. $f : [a,b] \to \mathbb{R}$ continuous and differentiable on (a,b) with $f'(x) > 0 \ \forall\ x \in (a,b)$.
Let $f(a) = c$ and $f(b) = d$. Then the function $f : [a,b] \to [c,d]$ is bijective and f^{-1} is differentiable with

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Proof. By Corollary 3.5, f is strictly increasing on $[a, b]$. By Theorem 2.6 $\exists\ y : [c, d] \to [a, b]$ which is a continuous strictly increasing inverse of f.
Need to prove that g is differentiable and that

$$g'(y) = \frac{1}{f'(x)}$$

where $y = f(x)$ and $x \in (a, b)$. If $k \neq 0$ is given, let h be given by

$$y + k = f(x + h)$$

That is, $g(y + k) = x + h$, $h \neq 0$. Then

$$\frac{g(y + k) - g(y)}{k} = \frac{x + h - x}{f(x + h) - f(x)}$$

Let $k \to 0$, then $h \to 0$ (since g is continuous), and then

$$g'(y) = \lim_{k\to 0} \frac{g(y + k) - g(y)}{k} = \frac{1}{f'(x)}.$$

□

Example. $g(x) = x^{\frac{1}{q}}$ ($x > 0$, q a positive integer).

$$f(x) = x^q, \qquad f'(x) = qx^{q-1}$$

f is differentiable, then so is g and by Theorem 3.6 (inverse rule)

$$g'(x) = \frac{1}{q(x^{\frac{1}{q}})^{q-1}} = \frac{1}{q} x^{\frac{1}{q}-1}$$

Remark. If $g(x) = x^r$, $r \in \mathbb{Q}$ then $g'(x) = rx^{r-1}$ (check!)

Suppose $f, g : [a, b] \to \mathbb{R}$ continuous and differentiable on (a, b) and $g(a) \neq g(b)$, then the MVT gives us $s, t \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{(b - a)f'(s)}{(b - a)g'(t)} = \frac{f'(s)}{g'(t)}$$

Cauchy showed that we can take $s = t$.

Theorem 3.7 (Cauchy's Mean Value Theorem)**.** Let $f, g : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Then $\exists\ t \in (a, b)$ such that

$$(f(b) - f(a))g'(t) = f'(t)(g(b) - g(a))$$

> **Note.** We recover the MVT if we take $g(x) = x$.

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Proof. Let

$$\phi(x) = \begin{vmatrix} 1 & 1 & 1 \\ f(a) & f(x) & f(b) \\ g(a) & g(x) & g(b) \end{vmatrix}$$

ϕ is continuous on $[a, b]$ and differentiable on (a, b). Also

$$\phi(a) = \phi(b) = 0$$

By Rolle's Theorem $\exists\ t \in (a, b)$ such that

$$\phi'(t) = 0$$

$$\begin{aligned} \phi'(x) &= f'(x)g(b) - g'(x)f(b) + f(a)g'(x) - g(a)f'(x) \\ &= f'(x)[g(b) - g(a)] + g'(x)[f(a) - f(b)] \end{aligned}$$

and $\phi'(t) = 0$ gives the desired result. □

"Lesson": good choice of auxiliary function + Rolle!

> **Example** (L'Hôpital's Rule)**.** The example:
>
> $$\lim_{x \to 0} \frac{e^x - 1}{\sin x} = \lim_{x \to 0} \frac{e^x - e^0}{\sin x - \sin 0}$$
>
> $$= \lim_{x \to 0} \frac{e^t}{\cos t} = 1$$
>
> where $t = t(x) \in (0, x)$ is chosen using Cauchy's Mean Value Theorem.

Goal: we want to extend the MVT to include higher order derivatives.

> **Theorem 3.8** (Taylor's Theorem With Lagrange's Remainder)**.** Suppose f and its derivatives up to order $h - 1$ are continuous in $[a, a + h]$ and $f^{(n)}$ exists for $x \in (a, a + h)$. Then
>
> $$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1} f^{(n-1)}(a)}{(n-1)!} + \frac{h^n}{n!} f^{(n)}(a + \theta h)$$
>
> where $\theta \in (0, 1)$.

Note. (1) For $n = 1$ we get back MVT, so this is a "n-th order MVT".

(2) $R_n = \frac{h^n}{n!} f^{(n)}(a + \theta h)$ is known as Lagrange's form of the remainder

Proof. Define $0 \leq t \leq h$

$$\phi(t) = f(a+t) - f(a) - tf'(a) - \cdots - \frac{t^{n-1}}{(n-1)!} f^{(n-1)}(a) - \frac{t^n}{n!} B$$

where we choose B such that $\phi(h) = 0$. (Note $\phi(0) = 0$.) (Recall that in the proof of the MVT we used $f(x) - kx$ and picked k so we could use Rolle). We see that

$$\phi(0) = \phi; (0) = \cdots = \phi^{(n-1)}(0) = 0$$

We use Rolle's Theorem then n-times. Since $\phi(0) = \phi(h) = 0$

$$\text{Rolle} \quad \Longrightarrow \quad \phi'(h_1) = 0 \qquad 0 < h_1 < h$$

Since $\phi'(0) = 0 = \phi'(h_1)$

$$\text{Rolle} \quad \Longrightarrow \quad \phi''(h_2) = 0 \qquad 0 < h_2 < h_1$$

Finally $\phi^{(n-1)}(0) = \phi^{(n-1)}(h_{n-1}) = 0$

$$\text{Rolle} \quad \Longrightarrow \quad \phi^{(n)}(h_n) = 0 \qquad 0 < h_n < h_{n-1} < \cdots < h$$

so $h_n = \theta h$ for $\theta \in (0, 1)$. Now

$$\phi^{(n)}(t) = f^{(n)}(a+t) - B$$

$$\Longrightarrow \quad B = f^{(n)}(a + \theta h)$$

Set $t = h$, $\theta(h) = 0$ and put this value of B in the second line in the proof. □

Theorem 3.9 (Taylor's Theorem with Cauchy's Form of Remainder)**.** With the same hypothesis as in Theorem 3.8 and $a = 0$ (to simplify) we have

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n$$

where

$$R_n = \frac{(1-\theta)^{n-1} f^{(n)}(\theta h) h^n}{(n-1)!}$$

for $\theta \in (0, 1)$.

Proof. Define

$$F(t) = f(h) - f(t) - (h-t)f'(t) - \cdots - \frac{(h-t)^{n-1}f^{(n-1)}(t)}{(n-1)!}$$

for $t \in [0, h]$.

$$F'(t) = -f'(t) + f'(t) - (h-t)f''(t) + (h-t)f''(t) - \frac{(h-t)^2}{2}f'''(t) + \cdots - \frac{(h-t)^{n-1}}{(n-1)!}f^{(n)}(t)$$

$$\implies F'(t) = -\frac{(h-t)^{n-1}}{(n-1)!}f^{(n)}(t)$$

Set

$$\phi(t) = F(t) - \left[\frac{h-t}{h}\right]^p F(0)$$

with $p \in \mathbb{Z}$, $1 \le p \le n$. Then $\phi(0) = \phi(h) = 0$. By Rolle's $\exists\ \theta \in (0,1)$ such that

$$\phi'(\theta h) = 0$$

but

$$\phi'(\theta h) = F'(\theta h) + \frac{p(1-\theta)^{p-1}}{h}F(0) = 0$$

$$\implies 0 = -\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\theta h) + \frac{p(1-\theta)^{p-1}}{h}\left[f(h) - f(0) - hf'(0) - \cdots - \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0)\right]$$

$$\implies f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{h^n(1-\theta)^{n-1}}{(n-1)!p(1-\theta)^{p-1}}f^{(n)}(\theta h)$$

If $p = n$ we get Lagrange's remainder. If $p = 1$ we get Cauchy's remainder. □

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To get a Taylor series for f one needs to show that $R_n \to 0$ as $n \to \infty$. This requires "estimates" and "effort".

Remark. Theorems 3.8 and 3.9 work equally well in an interval $[a+h, a]$ with $h < 0$.

Example. The binomial *series*:

$$f(x) = (1+x)^r, \qquad r \in \mathbb{Q}$$

Claim. If $|x| < 1$, then

$$(1+x)^r = 1 + \binom{r}{1}x + \cdots + \binom{r}{n}x^n + \cdots$$

where

$$\binom{r}{n} \stackrel{\text{def}}{=} \frac{r(r-1)\cdots(r-n+1)}{n!}$$

Proof. Clearly

$$f^{(n)}(x) = r(r-1)\cdots(r-n+1)(1+x)^{r-n}$$

If $r \in \mathbb{Z}$, $r \geq 0$, then

$$f^{(r+1)} \equiv 0$$

we have a polynomial of degree r. In general (Lagrange)

$$R_n = \frac{x^n}{n!}f^{(n)}(\theta x) = \binom{r}{n}\frac{x^n}{(1+\theta x)^{n-r}}$$

$(\theta \in (0,1))$

Note. In principle, θ depends on both x and n.

For $0 < x < 1$,

$$(1+\theta x)^{n-r} > 1$$

for $n > r$. Now observe that the series

$$\sum \binom{r}{n}x^n$$

is absolutely convergent for $|x| < 1$. Indeed by the ratio test

$$a_n = \binom{r}{n}x^n$$

$$\begin{aligned}\implies \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{r(r-1)\cdots(r-n+1)(r-n)x^{n+1}}{(n+1)!}\right| \cdots \left|\frac{n!}{r(r-1)\cdots(r-n+1)x^n}\right| \\ &= \left|\frac{(r-n)x}{n+1}\right| \to |x| < 1\end{aligned}$$

In particular $a_n \to 0$ so $\binom{r}{n}x^n \to 0$. Hence for $n > r$ and $0 < x < 1$, we have

$$|R_n| \leq |\binom{r}{n}x^n| = |a_n| \to 0$$

as $n \to \infty$. So the claim is proved in the range $0 \le x < 1$.
If $-1 < x < 0$ the argument above breaks, but Cauchy's form for R_n works:

$$\begin{aligned}
R_n &= \frac{(1-\theta)^{n-1} r(r-1)\cdots(r-n+1)(1+\theta x)^{r-n} x^n}{(n-1)!} \\
&= \underbrace{\frac{r(r-1)\cdots(r-n+1)}{(n-1)!}}_{r\binom{r-1}{n-1}} \frac{(1-\theta)^{n-1}}{(1+\theta x)^{n-r}} x^n \\
&= r\binom{r-1}{n-1} x^n (1+\theta x)^{r-1} \left(\underbrace{\frac{1-\theta}{1+\theta x}}_{\substack{<1 \\ \forall x \in (-1,1)}} \right)^{n-1} \\
\implies |R_n| &\le \left| r\binom{r-1}{n-1} x^n \right| (1+\theta x)^{r-1}
\end{aligned}$$

Check:

$$(1+\theta x)^{r-1} \le \max\{1, (1+x)^{r-1}\}$$

(do it!) Let

$$K_r = |r| \max\{1, (1+x)^{r-1}\}$$

independent of n.

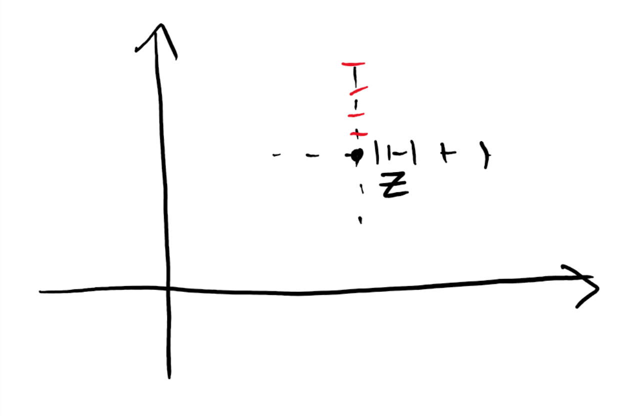
$$|R_n| \le K_r \left| \binom{r-1}{n-1} x^n \right| \to 0$$

because $a_n \to 0$, thus $R_n \to 0$. □

Remarks on Complex Differentiation

Formally for functions $f : E \subseteq \mathbb{C} \to \mathbb{C}$ we have properties for sums, products, chain rule etc. But it is *much more restrictive* than differentiability on the real line.

Example. $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \overline{z}$



If

$$z_n = z + \frac{1}{n} \to z$$

then

$$\frac{f(z_n) - f(z)}{z_n - z} = \frac{\overline{z} + \frac{1}{n} - \overline{z}}{z + \frac{1}{n} - z} = 1$$

but on the other hand if

$$z_n = z + \frac{i}{n} \to z$$

then

$$\frac{f(z_n) - f(z)}{z_n - z} = \frac{\overline{z} - \frac{i}{n} - \overline{z}}{z + \frac{i}{n} - z} = -1$$

so

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

does *not exist*, so it is nowhere $\mathbb{C}$-differentiable!

Note. IB Complex Analysis explores the consequences of $\mathbb{C}$-differentiability.

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4 Power Series [4-5]

We want to look at

$$\sum_{n=0}^{\infty} a_n z^n \tag{$*$}$$

$z \in \mathbb{C}$, $a_n \in \mathbb{C}$. (The case $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, z_0 fixed, can be reduced to $(*)$ by translation).

Lemma 4.1. If $\sum_{n=0}^{\infty} a_n z_1^n$ converges and $|z| < |z_1|$, then $\sum_{n=0}^{\infty} a_n z^n$ *converges absolutely.*

Proof. Since $\sum_{n=0}^{\infty} a_n z_1^n$ converges, $a_n z_1^n \to 0$. Thus $\exists\ K > 0$ such that $|a_n z_1^n| \leq K\ \forall\ n$. Then

$$\begin{aligned} |a_n z^n| &= |a_n z^n| \frac{|z_1^n|}{|z_1^n|} \\ &\leq K \underbrace{\left| \frac{z}{z_1} \right|}_{<1}^n \end{aligned}$$

Since the geometric series

$$\sum_{n=0}^{\infty} \left| \frac{z}{z_1} \right|^n$$

converges, the lemma follows by comparison. □

Using this lemma, we'll prove that every power series has a *radius of convergence.*

Theorem 4.2. A power series either

(1) Converges absolutely for all z, or

(2) Converges absolutely for all z inside a circle $|z| = R$ and diverges for all z outside it, or

(3) Converges for $z = 0$ only.

Definition. The circle $|z| = R$ is called the circle of convergence and R is the radius of convergence. In (1) we agree that $R = \infty$ and in (3) $R = 0$ (so $R \in [0, \infty]$).

Proof. Let

$$S = \{x \in \mathbb{R} : x \geq 0 \text{ and } \sum a_n x^n \text{ converges}\}$$

Clearly $0 \in S$. By Lemma 4.1 if $x_1 \in S$, then $[0, x_1] \subset S$. If $S = [0, \infty)$ we have case (1). If not, there exists a finite supremum for S. Let $R = \sup S < \infty$, $R \geq 0$. If $R > 0$, we'll prove that if $|z_1| < R$, then $\sum a_n z_1^n$ converges absolutely. Pick R_0 such that

$$|z_1| < R_0 < R$$

Then $R_0 \in S$ and the series converges for $z = R_0$. By Lemma 4.1, $\sum |a_n z_1^n|$ converges. Finally we show that if $|z_2| > R$, then the series does not converge for z_2. Pick $R < R_0 < |z_2|$. If $\sum a_n z_2^n$ converges then by Lemma 4.1 $\sum a_n R_0^n$ would be convergent, which contradicts that $R = \sup S$. □

The following lemma is useful for computing R:

Lemma 4.3. If $\left|\frac{a_{n+1}}{a_n}\right| \to l$ as $n \to \infty$, then $R = \frac{1}{l}$.

Proof. By the ratio test we have absolute convergence if

$$\lim \left| \frac{a_{n+1}}{a_n} \frac{z^{n+1}}{z^n} \right| < 1$$

so if $|z| < \frac{1}{l}$ we have absolute convergence. If $|z| > \frac{1}{l}$, the series diverges, again by ratio test. □

Remark. One can also use the root test to get that if $|a_n|^{1/n} \to l$, then $R = \frac{1}{l}$.

Examples

(1) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0 = l \implies R = \infty$$

(2) Geometric series, $\sum_{n=0}^{\infty} z^n$. $R = 1$. Note that at $|z| = 1$ we have divergence.

(3) $\sum_{n=0}^{\infty} n! z^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{n!} = n + 1 \to \infty \implies R = 0$$

(4) $\sum_{n=1}^{\infty} \frac{z^n}{n}$, $R = 1$. (for $z = 1$ it diverges (harmonic series)) What happens for $|z| = 1$

and $z \neq 1$? Consider $\sum_{n=1}^{\infty} \frac{z^n}{n}(1-z)$. Then

$$
\begin{aligned}
s_N &= \sum_{n=1}^{N} \left(\frac{z^n - z^{n+1}}{n} \right) \\
&= \sum_{n=1}^{N} \frac{z^n}{n} - \sum_{n=1}^{N} \frac{z^{n+1}}{n} \\
&= \sum_{n=1}^{N} \frac{z^n}{n} - \sum_{n=2}^{N+1} \frac{z^n}{n-1} \\
&= z - \frac{z^{N+1}}{N} + \sum_{n=2}^{N} z^n \left(-\frac{1}{n(n-1)} \right)
\end{aligned}
$$

if $|z| = 1$, then $\frac{z^{N+1}}{N} \to 0$ as $N \to \infty$ and $\sum \frac{1}{n(n-1)}$ converges, so s_N *converges.*

(5) $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$, $R = 1$ but converges for all z with $|z| = 1$.

Conclusion

In principle nothing can be said about $|z| = R$ and each case has to be discussed separately. Within the radius of convergence "life is great". Power series behave as if "they were polynomials".

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Theorem 4.4. $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R. Then f is differentiable at all points with $|z| < R$ with

$$
f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}
$$

Proof (non-examinable). We need two auxiliary lemmas:

Lemma 4.5. If $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R, so do

$$
\sum_{n=1}^{\infty} n a_n z^{n-1} \qquad \text{and} \qquad \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}
$$

Lemma 4.6. (i) $\binom{n}{r} \leq n(n-1)\binom{n-2}{r-2}$ for all $2 \leq r \leq n$

(ii) $|(z+h)^n - z^n - nhz^{n-1}| \leq n(n-1)(|z|+|h|)^{n-2}|h|^2$ for all $z \in \mathbb{C}, h \in \mathbb{C}$.

Proof of 4.4. (after which we prove the lemmas) By Lemma 4.5 we may define

$$f'(Z) := \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad |z| < R$$

Then we are required to prove that

$$\lim_{h \to 0} \frac{f(z+h) - f(z) - hf'(z)}{h} = 0$$

$$\begin{aligned}
I &:= \frac{f(z+h) - f(z) - hf'(z)}{h} \\
&= \frac{1}{h} \sum_{n=0}^{\infty} a_n((z+h)^n - z^n - hnz^{n-1}) \\
|I| &= \frac{1}{|h|} \left| \lim_{N \to \infty} \sum_{n=0}^{N} a_n((z+h)^n - z^n - nhz^{n-1}) \right| \\
&= \frac{1}{|h|} \lim_{N \to \infty} \left| \sum_{n=0}^{N} a_n((z+h)^n - z^n - nhz^{n-1}) \right| \\
&\leq \frac{1}{|h|} \sum_{n=0}^{N} |a_n||(z+h)^n - z^n - nhz^{n-1}| \\
&\leq \frac{1}{|h|} \sum_{n=2}^{\infty} |a_n| n(n-1)(|z| + |h|)^{n-2} |h|^2 \\
&= |h| \sum_{n=2}^{\infty} |a_n| n(n-1)(|z| + |h|)^{n-2}
\end{aligned}$$

By Lemma 4.5, for $|h|$ small enough,

$$\sum_{n=2}^{\infty} |a_n| n(n-1)(|z| + |h|)^{n-2}$$

converges to $A(h)$, but $A(h) \leq A(r)$ for $|h| < r$ and $|z| + r < R$. Hence

$$|I| \leq |h| A(h) \leq |h| A(r) \to 0$$

as $h \to 0$. $\square$ *Proof of Lemma 4.5.* Take z and R_0 such that $0 < |z| < R_0 < R$. Since $a_n R_0^n \to 0$, $\exists$ K such that $|a_n R_0^n| \leq K$, $\forall$ $n \geq 0$. Thus

$$\begin{aligned}
|n a_n z^{n-1}| &= \frac{n}{|z|} |a_n R_0^n| \left| \frac{z}{R_0} \right|^n \\
&\leq \frac{Kn}{|z|} \left| \frac{z}{R_0} \right|^n
\end{aligned}$$

But $\sum n \left|\frac{z}{R_0}\right|^n$ converges by the ratio test:

$$\frac{n+1}{n}\left|\frac{z}{R_0}\right|^{n+1}\left|\frac{R_0}{z}\right|^n = \frac{n+1}{n}\left|\frac{z}{R_0}\right| \to \left|\frac{z}{R_0}\right| < 1$$

if $|z| > R$, the series diverges since $|a_n z^n|$ is unbounded hence so is $n|a_n z^n|$. The same proof applies to $\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}$. □ *Proof of Lemma 4.6.*

(i)
$$\frac{\binom{n}{r}}{\binom{n-2}{r-2}} = \frac{n!}{r!\cancel{(n-r)!}}\frac{(r-2)!\cancel{(n-r)!}}{(n-2)!} = \frac{n(n-1)}{r(r-1)} \le n(n-1)$$

(ii)
$$(z+h)^n - z^n - nhz^{n-1} = \sum_{r=2}^{n}\binom{n}{r}z^{n-r}h^r$$

Thus

$$\begin{aligned}|(z+h)^n - z^n - nhz^{n-1}| &\le \sum_{r=2}^{n}\binom{n}{r}|z|^{n-r}|h|^r \\ &\le n(n-1)\left[\sum_{r=2}^{n}\binom{n-2}{r-2}|z|^{n-r}|h|^{r-2}\right]|h|^2 \\ &= n(n-1)(|z|+|h|)^{n-2}|h|^2\end{aligned}$$

□ □

4.1 The Standard Functions

(exponentials, logs, trigonometric, etc)

We have already seen that

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}$$

has $R = \infty$. Define $e : \mathbb{C} \to \mathbb{C}$ by

$$e(z) = \sum_{n=0}^{\infty}\frac{z^n}{n!}$$

Straight from Theorem 4.4, e is differentiable and

$$\boxed{e'(z) = e(z)}.$$

Lemma. If $F : \mathbb{C} \to \mathbb{C}$ has $F'(z) = 0$ for all $z \in \mathbb{C}$, then F is constant.

Proof. Consider $g(t) = F(tz)$. By chain rule:

$$g'(t) = F'(tz)z = 0$$

if $g(t) = u(t) + iv(t)$ then $g'(t) = u'(t) + iv'(t)$ so $u' = v' = 0$. Apply Corollary 3.5 to get the claim. $\square$

Now let $a, b \in \mathbb{C}$. Consider

$$F(z) = e(a + b - z)e(z)$$

$$F'(z) = -e(a + b - z)e(z) + e(a + b - z)e(z) = 0$$

so F is constant. Use $z = b$ and $z = 0$ to deduce that

$$\boxed{e(a)e(b) = e(a + b)}$$

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Now we restrict to $\mathbb{R}$:

Theorem 4.7.

(i) $e : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable and $e'(x) = e(x)$

(ii) $e(x + y) = e(x)e(y)$

(iii) $e(x) > 0$ for all $x \in \mathbb{R}$

(iv) e is strictly increasing

(v) $e(x) \to \infty$ as $x \to \infty$, $e(x) \to 0$ as $x \to -\infty$

(vi) $e : \mathbb{R} \to (0, \infty)$ is a *bijection*.

Proof.

(i) Already done.

(ii) Clearly

$$e(x) > 0 \qquad \forall x \geq 0$$

and $e(0) = 1$. Also

$$e(0) = e(x - x) = e(x)e(x) = 1 \implies e(-x) > 0$$

for all $x > 0$.

(iii) Already done.

(iv) $e'(x) = e(x) > 0$ so e is strictly increasing.

(v) $e(x) > 1 + x$ for $x > 0$ so if $x \to \infty$, $e(x) \to \infty$. For $x > 0$ since

$$e(-x) = \frac{1}{e(x)}$$

then $e(x) \to 0$ as $x \to -\infty$.

(vi) Injectivity follows right away from being strictly increasing. Surjectivity: Take $y \in (0 \in \infty)$. From (v) there exist $a, b \in \mathbb{R}$ such that

$$e(a) < y < e(b)$$

so by the Intermediate Value Theorem there exists $x \in \mathbb{R}$ such that $e(x) = y$.

□

Remark. $e : (\mathbb{R}, +) \to ((0, \infty), \times)$ is a *group isomorphism.*

Since e is a bijection we have an inverse:

$$l : (0, \infty) \to \mathbb{R}$$

Theorem 4.8. (i) $l : (0, \infty) \to \mathbb{R}$ is a bijection and $l(e(x)) = x$ for all $x \in \mathbb{R}$ and $r(l(t)) = t$ for all $t \in (0, \infty)$.

(ii) l is differentiable and $l'(t) = \frac{1}{t}$.

(iii) $l(xy) = l(x) + l(y)$ for all $x, y \in (0, \infty)$.

Proof.

(i) Obvious from the definition of l.

(ii) Inverse rule (Theorem 3.6) l is differentiable and

$$l'(t) = \frac{1}{e(l(t))} = \frac{1}{t}$$

(iii) From IA Groups if e is an isomorphism, so is its inverse.

□

Now define for $\alpha \in \mathbb{R}$ and $x > 0$:

$$r_\alpha(x) \overset{\text{def}}{=} e(\alpha l(x))$$

Theorem 4.9. Suppose $x, y > 0$ and $\alpha, \beta \in \mathbb{R}$. Then

(i) $r_\alpha(xy) = r_\alpha(x) r_\alpha(y)$

(ii) $r_{\alpha+\beta}(x) = r_\alpha(x) r_\beta(x)$

(iii) $r_\alpha(r_\beta(x)) = r_{\alpha\beta}(x)$

(iv) $r_1(x) = x$, $r_0(x) = 1$.

Proof.

(i)
$$\begin{aligned} r_\alpha(xy) &= e(\alpha l(xy)) \\ &= e(\alpha l(x) + \alpha l(y)) \\ &= e(\alpha l(x)) e(\alpha l(y)) \\ &= r_\alpha(x) r_\alpha(y) \end{aligned}$$

(ii)
$$\begin{aligned} r_{\alpha+\beta}(x) &= e((\alpha+\beta) l(x)) \\ &= e(\alpha l(x)) e(\beta l(x)) \\ &= r_\alpha(x) r_\beta(x) \end{aligned}$$

(iii)
$$\begin{aligned} r_\alpha(r_\beta(x)) &= r_\alpha(e(\beta l(x))) \\ &= e(\alpha l e(\beta l(x))) \\ &= e(\alpha\beta l(x)) \\ r_{\alpha\beta}(x) \end{aligned}$$

(iv) $r_1(x) = e(l(x)) = x$, $r_0(x) = e(0) = 1$.

□

For $n \geq 1$, $n \in \mathbb{Z}$
$$r_n(x) = r_{1+\cdots+1}(x) = x \cdots x = x^n$$
$$r_1(x) r_{-1}(x) = r_0(x) = 1$$
$$\implies r_{-1}(x) = \frac{1}{x}$$
$$r_{-n}(x) = \frac{1}{x^n}$$
$$(r_{\frac{1}{q}}(x))^q = r_1(x) = x$$
$(q \in \mathbb{Z}, q \geq 1)$
$$\implies r_{\frac{1}{q}}(x) = x^{\frac{1}{q}}$$

$$r_{\frac{p}{q}}(x) = (r_{\frac{1}{q}}(x))^p = x^{\frac{p}{q}}$$

Thus $r_\alpha(x)$ agrees with x^α when $a \in \mathbb{Q}$ as previously defined. Now we give them names:

$$\begin{aligned} \exp(x) &= e(x) & & x \in \mathbb{R} \\ \log x &= l(x) & & x \in (0, \infty) \\ x^\alpha &= r_\alpha(x) & & \alpha \in \mathbb{R}, x \in (0, \infty) \end{aligned}$$

$$e(x) = e(x \log e) = e_x(e) = e^x$$

where

$$e \overset{\text{def}}{=\!=\!=} \sum_{n=0}^{\infty} \frac{1}{n!}$$

$\exp(x)$ is also a power, which we may as well write as e^x. Finally we compute

$$\begin{aligned} (x^\alpha)' &= (e^{\alpha \log x})' \\ &= e^{\alpha \log x} \frac{\alpha}{x} \\ &= \alpha x^{\alpha - 1} \end{aligned}$$

$f(x) = a^x$, $a > 0$ then

$$f'(x) = (e^{x \log a})' = e^{x \log a} \log a = a^x \log a$$

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Remark. "Exponentials beat polynomials"

$$\lim_{x \to \infty} \frac{e^x}{x^k} = \infty$$

$(k > 0)$. This is easy to prove since

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} > \frac{x^n}{n!}$$

for $x > 0$. Now pick $n > k$ and then

$$\frac{e^x}{x^k} > \frac{x^{n-k}}{n!} \to \infty$$

as $x \to \infty$.

Trigonometric Functions

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

Both power series have infinite radius of convergence and by Theorem 4.4 we get

$$(\sin z)' = \cos z, \qquad (\cos z)' = -\sin z$$

$(e^z = e(z))$

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{n=0}^{\infty} \frac{(iz)^{2k}}{(2k)!} + \sum_{n=0}^{\infty} \frac{(iz)^{2k+1}}{(2k+1)!}$$

$$(iz)^{2k} = (-1)^k z^{2k}, \qquad (iz)^{2k+1} = i(-1)^k z^{2k+1}$$

$$\implies e^{iz} = \cos z + i \sin z$$

Similarly

$$e^{-iz} = \cos z - i \sin z$$

which gives

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

From this we get many trigonometric identities:

$$\cos z = \cos(-z), \qquad \sin(-z) = -\sin z$$

$$\cos(0) = 1, \qquad \sin(0) = 0$$

Addition formulas:

(1) $\sin(z + w) = \sin z \cos w + \cos z + \sin w$

(2) $\cos(z + w) = \cos z \cos w - \sin z \sin w, \ z, w \in \mathbb{C}$.

These follow from $e^{a+b} = e^a e^b$. To prove (2) write

$$\begin{aligned}
\cos(z + w) &= \frac{1}{2}(e^{i(z+w)} + e^{-i(z+w)}) \\
&= \frac{1}{2}(e^{iz}e^{iw} + e^{iz}e^{iw}) \\
\cos z \cos w - \sin z \sin w &= \frac{1}{4}(e^{iz} + e^{-iz})(e^{iw} + e^{-iw}) + \frac{1}{4}(e^{iz} - e^{-iz})(e^{iw} - e^{-iw})
\end{aligned}$$

operate to get the result. Also we can easily deduce that $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$. Now if $x \in \mathbb{R}$, then $\sin x, \cos x \in \mathbb{R}$ and so $|\sin x|, |\cos x| \leq 1$ for $x \in \mathbb{R}$.

Remark. They are not bounded over $\mathbb{C}$. For example take

$$\cos(iy) = \frac{1}{2}(e^{-y} + e^{y})$$

$(y \in \mathbb{R})$ then as $y \to \infty$, $\cos(iy) \to \infty$!

Periodicity of the Trigonmetric Functions

Proposition 4.10. There is a smallest positive number ω (where $\sqrt{2} < \frac{\omega}{2} < \sqrt{3}$) such that

$$\cos\frac{\omega}{2} = 0$$

Proof. If $0 < x < 2$ then

$$\sin x = \underbrace{\left(x - \frac{x^3}{3!}\right)}_{>0} + \underbrace{\left(\frac{x^5}{5!} - \frac{x^7}{7!}\right)}_{>0} + \cdots$$

(If $0 < x < 2$ then $\frac{x^{2n-1}}{(2n-1)!} > \frac{x^{2n+1}}{(2n+1)!}$) Hence $\sin x > 0$. Since $(\cos x)' = -\sin x < 0$ for $0 < x < 2$, $\cos x$ is strictly decreasing. We'll show that $\cos\sqrt{2} > 0$ and $\cos\sqrt{3} < 0$. Then by the intermediate value theorem the existence of ω follows.

$$\cos\sqrt{2} = \underbrace{\left(\frac{(\sqrt{2})^4}{4!} - \frac{(\sqrt{2})^6}{6!}\right)}_{>0} + \underbrace{(\cdots)}_{>0} + \underbrace{(\cdots)}_{>0} + \cdots$$

So $\cos\sqrt{2} > 0$. Now note that

$$\cos\sqrt{3} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \underbrace{\left(\frac{x^6}{6!} - \frac{x^8}{8!}\right)}_{>0} - \cdots$$

But

$$1 - \frac{3}{2} + \frac{9}{4 \times 3 \times 2} = 1 - \frac{3}{2} + \frac{3}{8} = -\frac{1}{8} < 0$$

so $\cos\sqrt{3} < 0$. □

Corollary 4.11. $\sin\frac{\omega}{2} = 1$.

Proof. Use $\sin^2\frac{\omega}{2} + \cos^2\frac{\omega}{2} = 1$ and $\sin\frac{\omega}{2} > 0$. □

Now define $\pi = \omega$.

Theorem 4.12. (1) $\sin\left(z + \frac{\pi}{2}\right) = \cos z$, $\cos\left(z + \frac{\pi}{2}\right) = -\sin z$.

(2) $\sin(z + \pi) = -\sin z$, $\cos(z + \pi) = -\cos z$.

(3) $\sin(z + 2\pi) = \sin z$, $\cos(z + 2\pi) = \cos z$.

Proof. Immediate from addition formulas and $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$. □

This implies

$$\begin{aligned} e^{iz+2\pi i} &= \cos(z + 2\pi) + i\sin(z + 2\pi) \\ &= \cos z + i \sin z \\ e^{iz} & \end{aligned}$$

so e^z is periodic with period $2\pi i$.

Remark. "Relation with geometry"
Given two vectors $x, y \in \mathbb{R}^2$ define $x \cdot y$ as in vectors and matrices:

$$x \cdot y = x_1 y_2 + x_2 y_2$$

$$x = (x_1, x_2) \qquad y = (y_1, y_2)$$

Cauchy-Schwarz:

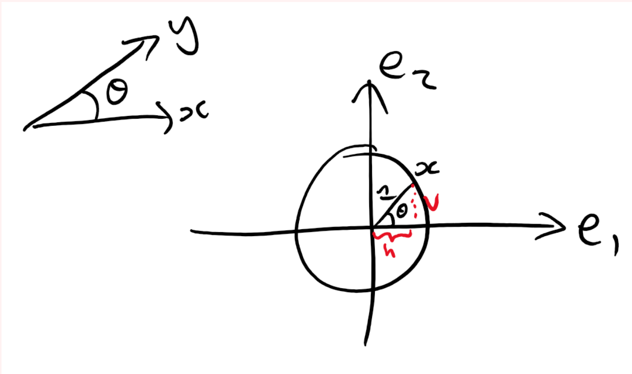
$$|x \cdot y| \le \|x\| \|y\|$$

where $\|x\|^2 = x_1^2 + x_2^2$. So, for $x \neq 0$, $y \neq 0$

$$-1 \le \frac{x \cdot y}{\|x\| \|y\|} \le 1$$

Define the angle between x and y as the unique $\theta \in [0, \pi]$ such that

$$\cos \theta = \frac{x \cdot y}{\|x\| \|y\|}$$



$x = (h, v)$, $\cos \theta = x \cdot e_1 = h$

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Hyperbolic Functions

(Hyperbolic sine and cosine)

Definition. $\cosh z = \frac{1}{2}(e^z + e^{-z})$, $\sinh z = \frac{1}{2}(e^z - e^{-z})$. Alternatively, $\cosh z = \cos(iz)$, $\sinh z = -i \sin(iz)$.

One can also prove that $(\cosh z)' = \sinh z$ and $(\sinh z)' = \cosh z$. (This is left as an exercise). We also have

$$\cosh^2 z - \sinh^2 z = 1$$

The rest of the trigonometric functions $(\tan, \cot, \sec, \text{cosec})$ are defined in the usual way.

5 Integration

$f : [a, b] \to \mathbb{R}$ *bounded.* (i.e. there exists K such that $|f(x)| \le K \ \forall \ x \in [a, b]$)

Definition. A dissection (or partition) $\mathcal{D}$ of $[a, b]$ is a finite subset of $[a, b]$ containing the endpoints a and b. We write

$$\mathcal{D} = \{x_0, x_1, \dots, x_4\}$$

with $a = x_0 < x_1 < \cdots < x_{n-1} < x_= b$.

Definition. We define the upper sum and lower sum associated with $\mathcal{D}$ by

$$S(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{x \in [x_{j-1}, x_j]} f(x) \qquad \text{(upper)}$$

$$s(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{x \in [x_{j-1}, x_j]} f(x) \qquad \text{(lower)}$$

Clearly $s(f, \mathcal{D}) \le S(f, \mathcal{D})$ for all $\mathcal{D}$.

Lemma 5.1. If $\mathcal{D}$ and $\mathcal{D}'$ are dissections with $\mathcal{D}' \supseteq \mathcal{D}$, then

$$S(f, \mathcal{D}) \ge S(f, \mathcal{D}') \ge s(f, \mathcal{D}') \ge s(f, \mathcal{D})$$

Proof.

$$S(f, \mathcal{D}') \ge s(f, \mathcal{D}')$$

is obvious. Suppose $\mathcal{D}'$ contains an extra point than $\mathcal{D}$, let's say $y \in (x_{r-1}, x_r)$. Then clearly

$$\sup_{x \in [x_{r-1}, y]} f(x), \ \sup_{x \in [y, x_r]} \le \sup_{x \in [x_{r-1}, x_r]} f(x)$$

$$\implies (x_r - x_{r-1}) \sup_{x \in [x_{r-1}, x_r]} f(x) \ge (y - x_{r-1}) \sup_{x \in [x_{r-1}, y]} f(x) + (x_r - y) \sup_{x \in [y, x_r]} f(x)$$

$$\implies S(f, \mathcal{D}) \ge S(f, \mathcal{D}')$$

The same for s and the same if $\mathcal{D}'$ has more extra points than $\mathcal{D}$. □

Lemma 5.2. $\mathcal{D}_1, \mathcal{D}_2$ two arbitrary dissections. Then

$$S(f, \mathcal{D}_1) \geq S(f, \mathcal{D}_1 \cup \mathcal{D}_2) \geq s(f, \mathcal{D}_1 \cup \mathcal{D}_2) \geq s(f, \mathcal{D}_2)$$

and in particular

$$S(f, \mathcal{D}_1) \geq s(f, \mathcal{D}_2)$$

Proof. Take $\mathcal{D}' = \mathcal{D}_1 \cup \mathcal{D}_2 \supseteq \mathcal{D}_1, \mathcal{D}_2$ in the previous lemma. □

Definition. The *upper integral* of f is

$$I^*(f) = \inf_{\mathcal{D}} S(f, \mathcal{D})$$

(always exists!) The *lower integral* of f is

$$I_*(f) = \sup_{\mathcal{D}} s(f, \mathcal{D})$$

By Lemma 5.2,

$$I^*(f) \geq I_*(f)$$

because

$$S(f, \mathcal{D}_1) \geq s(f, \mathcal{D}_2)$$
$$I^*(f) = \inf_{\mathcal{D}_1} S(f, \mathcal{D}_1) \geq s(f, \mathcal{D}_2)$$
$$I^*(f) \geq \sup_{\mathcal{D}_2} s(f, \mathcal{D}_2) = I_*(f)$$

Definition. A bounded function $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* (or just integrable) if

$$I^*(f) = I_*(f)$$

and we set

$$\int_a^b f(x)\mathrm{d}x = I^*(f) = I_*(f) = \int_a^b f$$

Example.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & x \notin \mathbb{Q} \cap [0,1] \end{cases}$$

$f : [0,1] \to \mathbb{R}$; f is *not* Riemann integrable:

$$\sup_{x \in [x_{j-1}, x_j]} f(x) = 1, \qquad \inf_{x \in [x_{j-1}, x_j]} f(x) = 0$$

Hence $s(f, \mathcal{D}) = 1$ and $s(f, \mathcal{D}) = 0$ for all $\mathcal{D}$. Hence $I^*(f) = 1$, but $I_*(f) = 0$.

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Theorem 5.3. A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if given $\varepsilon > 0$, $\exists \mathcal{D}$ such that

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) < \varepsilon$$

Proof. For every dissection $\mathcal{D}$ we have

$$0 \le I^*(f) - I_*(f) \le S(f, \mathcal{D}) - s(f, \mathcal{D})$$

If the given condition holds, then

$$0 \le I^*(f) - I_*(f) \le S(f, \mathcal{D}) - s(f, \mathcal{D}) < \varepsilon$$

for all $\varepsilon > 0$ hence $I^*(f) = I_*(f)$.
Conversely, if f is integrable, by definition of sup and inf there are partitions $\mathcal{D}_1$ and $\mathcal{D}_2$ such that

$$\int_a^b f - \frac{\varepsilon}{2} = I_*(f) + \frac{\varepsilon}{2} = \int_a^b f + \frac{\varepsilon}{2}$$

By Lemma 5.1 ($\mathcal{D}_1 \cup \mathcal{D}_2 \supseteq \mathcal{D}_1, \mathcal{D}_2$)

$$\begin{aligned} S(f, \mathcal{D}_1 \cup \mathcal{D}_2) - s(f, \mathcal{D}_1 \cup \mathcal{D}_2) &\le S(f, \mathcal{D}_2) - s(f, \mathcal{D}_1 \\ &< \int_a^b f + \frac{\varepsilon}{2} - \int_a^b f + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

□

We now use this criterion to show that monotone and continuous functions are *integrable*.

Remark. Monotone and continuous functions are bounded (theorem 2.6 for the case of continuous functions).

Theorem 5.4. Let $f : [a, b] \to \mathbb{R}$ be monotone. Then f is integrable.

Proof. Suppose f is *increasing* (same proof for f decreasing). Then

$$\sup_{x \in [x_{j-1}, x_j]} f(x) = f(x_j)$$

$$\inf_{x \in [x_{j-1}, x_j]} f(x) = f(x_{j-1})$$

Thus

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) = \sum_{j-1}^{n} (x_j - x_{j-1})[f(x_j) - f(x_{j-1})]$$

Now choose

$$\mathcal{D} = \left\{ a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \dots, b \right\}$$

$$x_j = a + \frac{(b-a)j}{n} \qquad 0 \le j \le n$$

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) = \frac{(b-a)}{n}(f(b) - f(a))$$

Take n large enough such that

$$\frac{(b-a)}{n}(f(b) - f(a)) < \varepsilon$$

and use Theorem 5.3. □

Continuous Functions

First we need an auxiliary lemma.

Lemma 5.5. $f : [a, b] \to \mathbb{R}$ continuous. Then given $\varepsilon > 0$, $\exists\, \delta > 0$ such that if $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$ (uniform continuity). The point is that δ works $\forall\, x, y$ as long as $|x - y| < \delta$. (in the definition of continuity of f at, $\delta = f(x)$).

Proof. Suppose the claim is false. Then $\exists\, \varepsilon > 0$ such that $\forall\, \delta > 0$, we can find $x, y \in [a, b]$ such that $|x - y| < \delta$, but $|f(x) - f(y)| \ge \varepsilon$. Take $\delta = \frac{1}{n}$, to get $x_n, y_n \in [a, b]$ with $|x_n - y_n| < \frac{1}{n}$, but

$$|f(x_n) - f(y_n)| \ge \varepsilon$$

By Bolzano-Weierstrass, $\exists\, x_{n_k} \to c \in [a, b]$

$$|y_{n_k} - c| \le |y_{n_k} - x_{n_k}| + |x_{n_k} - c| \to 0$$

so $y_{n_k} \to c$. But

$$|f(x_{n_k} - f(y_{n_k}) \ge \varepsilon.$$

Let $k \to \infty$, then by continuity of f

$$|f(c) - f(c)| \geq \varepsilon \implies 0 \geq \varepsilon$$

Absurd. $\square$

Theorem 5.6. Let $f : [a, b] \to \mathbb{R}$ continuous. Then f is Riemann integral.

Proof. By 5.5, given $\varepsilon > 0$, $\exists\, \delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. Let

$$\mathcal{D} = \left\{a + \frac{(b-a)j}{n} : 0 \leq j \leq n\right\}$$

Choose n large enough such that $\frac{b-a}{n} < \delta$. Then for $x, y \in [x_{j-1}, x_j]$

$$|f(x) - f(y)| < \varepsilon,$$

since

$$|x - y| \leq |x_j - x_{j-1}| = \frac{b-a}{n} < \delta$$

Observe that

$$\max_{x \in [x_{j-1}, x_j]} f(x) - \min_{x \in [x_{j-1}, x_j]} f(x) = f(p_j) - f(q_j)$$

$o_j, q_j \in [x_{j-1}, x_j]$ (max and min are achieved due to continuity). Hence

$$\begin{aligned} S(f, \mathcal{D}) - s(f, \mathcal{D}) &= \sum_{j=1}^{n} (x_j - x_{j-1})[f(p_j) - f(q_j)] \\ &< \varepsilon(b - a) \end{aligned}$$

$\square$

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Remark. We have shown that monotone functions and continuous functions are Riemann integrable, but there do exist more complicated functions that are Riemann integrable.s

Example. $f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \in (0,1] \text{in its lowest form} \\ 0 & \text{otherwise} \end{cases}$$

Clearly $s(f, \mathcal{D}) = 0 \ \forall \ \mathcal{D}$. We'll show that given $\varepsilon > 0$, $\exists \mathcal{D}$ such that $S(f, \mathcal{D}) < \varepsilon$. This would imply that f is integrable with $\int_0^1 f = 0$. Consider the set

$$\left\{ x \in [0,1] : f(x) \geq \frac{1}{N} \right\} = \left\{ \frac{p}{q} : 1 \leq q \leq N, 1 \leq p \leq q \right\}$$

Take $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. This is a finite set

$$0 < t_1 < t_2 < \cdots < t_R = 1$$

Consider a dissection $\mathcal{D}$ of $[a, b]$ such that

(1) Each t_k, $1 \leq k < R$ is some (x_{j-1}, x_j)

(2) $\forall \ k$, the unique interval containing t_k has length at most $\frac{\varepsilon}{2R}$.

Note $f \leq 1$ everywhere so

$$S(f, \mathcal{D}) \leq \frac{1}{N} + \frac{\varepsilon}{2} < \varepsilon$$

Elementary Properties of the Integral

Let f, g bounded and integrable on $[a, b]$.

(1) If $f \leq g$ on $[a, b]$ then

$$\int_a^b f \leq \int_a^b g$$

(2) $f + g$ is integrable on $[a, b]$ and

$$\int_a^b f + g = \int_a^b f + \int_a^b g$$

(3) For any constant k, kf is integrable and

$$\int_a^b kf = k \int_a^b f$$

(4) $|f|$ is integrable and

$$|\int_a^b f| \leq \int_a^b |f|$$

(5) The product fg is integrable.

Proof.

(1) If $f \leq g$, then

$$\begin{aligned}\int_a^b f &= I^*(f) \\ &\leq S(f, \mathcal{D}) \\ &\leq S(g, \mathcal{D}) \\ \implies \int_a^b f &= I^*(f) \\ &\leq I^*(g) \\ &= \int_a^b g\end{aligned}$$

(2)

$$\begin{aligned}\sup_{[x_{j-1},x_j]} (f+g) &\leq \sup_{[x_{j-1},x_j]} f + \sup_{[x_{j-1},x_j]} g \\ \implies S(f+g, \mathcal{D}) &\leq S(f, \mathcal{D}) + S(g, \mathcal{D})\end{aligned}$$

Now take dissections $\mathcal{D}_1$ and $\mathcal{D}_2$

$$\begin{aligned}I^*(f+g) &\leq S(f+g, \mathcal{D}_1 \cup \mathcal{D}_2) \\ &\leq S(f, \mathcal{D}_1 \cup \mathcal{D}_2) + S(g, \mathcal{D}_1 \cup \mathcal{D}_2) \\ &\leq S(f, \mathcal{D}_1) + S(g, \mathcal{D}_2)\end{aligned}$$

Fix $\mathcal{D}_1$ and take inf over $\mathcal{D}_2$ to get

$$I^*(f+g) \leq S(f, \mathcal{D}_1) + I^*(g)$$

now take inf over all $\mathcal{D}_1$ to get

$$I^*(f+g) \leq I^*(f) + I^*(g) = \int_a^b f + \int_a^b g$$

Similarly

$$\int_a^b f + \int_a^b g \leq I_*(f+g)$$

so $f + g$ is integrable with integral equal to the sum of integrals.

(3) Exercise!

(4) Consider

$$f_+(x) = \max(f(x), 0)$$

$$\sup_{[x_{j-1},x_j]} f_+ - \inf_{[x_{j-1},x_j]} f_+ \leq \sup_{[x_{j-1},x_j]} f - \inf_{[x_{j-1},x_j]} f$$

We know that given $\varepsilon > 0$ there exists $\mathcal{D}$ such that

$$S(f,\mathcal{D}) - s(f,\mathcal{D}) < \varepsilon$$

(criterion from last time)

$$S(f,\mathcal{D}) - s(f,\mathcal{D}) = \sum_{j=1}^{n} (\sup_{[x_{j-1},x_j]} f - \inf_{[x_{j-1},x_j]} f)(x_j - x_{j-1})$$

$$\implies S(f_+,\mathcal{D}) - s(f_+,\mathcal{D}) \leq S(f,\mathcal{D}) - s(f,\mathcal{D}) < \varepsilon$$

$$\implies f_+ \text{ is integrable}$$

But $|f| = 2f_+ - f$, so by (2) and (3), $|f|$ is integrable. Since

$$-|f| \leq f \leq |f|$$

property (1) gives

$$|\int_a^b f| \leq \int_a^b |f|$$

(5) Take f integrable and ≥ 0. Then

$$\sup_{[x_{j-1},x_j]} f^2 = \left(\sup_{[x_{j-1},x_j]} f\right)^2 = M_j^2$$

$$\inf_{[x_{j-1},x_j]} f^2 = \left(\inf_{[x_{j-1},x_j]} f\right)^2 = m_j^2$$

Thus

$$S(f^2,\mathcal{D}) - s(f^2,\mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1})(M_j^2 - m_j^2)$$

$$= \sum_{j=1}^{n} (x_j - x_{j-1})(M_j + m_j)(M_j - m_j) \leq 2K(S(f,\mathcal{D}) - s(f,\mathcal{D}))$$

($|f(X)| \leq K \ \forall\ x \in [a,b]$) Using the criterion in Theorem 5.3 we deduce that f^2 is integrable. Now take any f, then $|f| \leq 0$. Since $f^2 = |f|^2$ we deduce that f^2 is integrable for any f. Finally for fg note that

$$4fg = (f+g)^2 - (f-g)^2$$

hence fg is integrable given what we proved before.

□

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Here is another property of Riemann integrals:

(6) f is integrable on $[a,b]$. If $a < c < b$, then f is integrable over $[a,c]$ and $[c,b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Conversely, if f is integrable over $[a,c]$ and $[c,b]$, then f is integrable over $[a,b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof of (6). We first make two observations:

- If $\mathcal{D}_1$ is a dissection of $[a,c]$ and $\mathcal{D}_2$ is a dissection of $[c,b]$, then $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ is a dissection of $[a,b]$ and

$$S(f, \mathcal{D}_1 \cup \mathcal{D}_2) = S(f_{[a,c]}, \mathcal{D}_1) + S(f_{[c,b]}, \mathcal{D}_2 \qquad (*_1)$$

- Also if $\mathcal{D}$ is a dissection of $[a,b]$, then

$$\begin{aligned} S(f,\mathcal{D}) &\geq S(f, \mathcal{D} \cup \{c\}) \\ &= S(f_{[a,c]}, \mathcal{D}_1) + S(f_{[c,b]}, \mathcal{D}_2) \end{aligned} \qquad (*_2)$$

  where $\mathcal{D}_1$ dissects $[a,c]$ and $\mathcal{D}_2$ dissects $[c,b]$.

Then $(*_1)$ gives

$$I^*(f) \leq I^*(f_{[a,c]}) + I^*(f_{[c,b]})$$

and $(*_2)$ gives

$$\begin{aligned} &I^*(f) \geq I^*(f_{[a,c]}) + I^*(f_{[c,b]}) \\ \implies &I^*(f) = I^*(f_{[a,c]}) + I^*(f_{[c,b]}) \end{aligned}$$

Similarly

$$I_*(f) = I_*(f_{[a,c]} + I_*(f_{[c,b]})$$

Thus

$$\begin{aligned} 0 &\leq I^*(F) - I_*(f) \\ &= [I^*(f_{[a,c]}) - I_*(f_{[a,c]})] + [I^*(f_{[c,b]}) - I_*(f_{[c,b]})] \end{aligned}$$

From this (6) follows right away. □

Notation. It is a convention that if $a > b$, then

$$\int_a^b f = -\int_b^a f$$

and if $a = b$ we agree that its value is zero. With this convention if $|f| \leq K$, then

$$\left| \int_a^b f \right| \leq K|b-a|$$

Fundamental Theorem of Calculus (FTC)

$f : [a, b] \to \mathbb{R}$ bounded and integrable. Write:

$$F(x) = \int_a^x f(t)\mathrm{d}t$$

$x \in [a, b]$.

Theorem 5.7. F is continuous.

Proof.

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\mathrm{d}t$$

so

$$\begin{aligned}|F(x+h) - F(x)| &= \left|\int_x^{x+h} f(t)\mathrm{d}t\right| \\ &\leq K|h|\end{aligned}$$

if $|f| \leq K \ \forall \ t \in [a, b]$. Now let $h \to 0$ and we're done. □

Theorem 5.8 (FTC)**.** If in addition f is continuous at x, then F is differentiable at x and

$$F'(x) = f(x).$$

Proof. We need to consider

$$\left|\frac{F(x+h) - F(x)}{h} - f(x)\right|$$

(for $x + h \in [a, b]$ and $h \neq 0$).

$$\begin{aligned}\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| &= \frac{1}{|h|}\left|\int_x^{x+h} f(t)\mathrm{d}t - hf(x)\right| \\ &= \frac{1}{|h|}\left|\int_x^{x+h} [f(t) - f(x)]\mathrm{d}t\right|\end{aligned}$$

f is continuous at x, means that given $\varepsilon > 0$, $\exists \ \delta > 0$ such that if $|t - x| < \delta$ then

$$|f(t) - f(x)| < \varepsilon$$

If $|h| < \delta$, we can write

$$\leq \frac{1}{|h|}\varepsilon|h| = \varepsilon$$

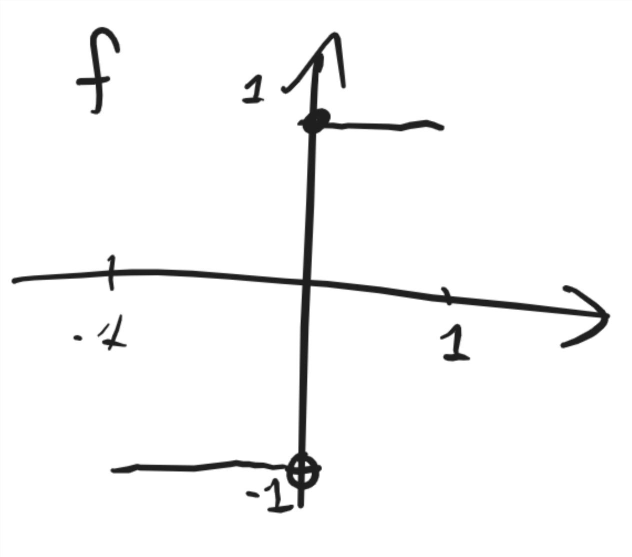
This means

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

□

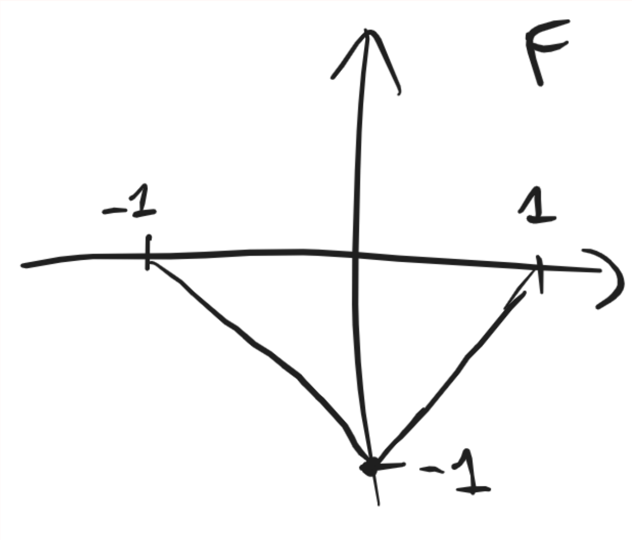
Example.

$$f(x) = \begin{cases} -1 & x \in [-1, 0] \\ 1 & x \in (0, 1] \end{cases}$$



Since monotone, it's integrable. One can check that

$$F(x) = \begin{cases} -x - 1 & x \leq 0 \\ x - 1 & x > 0 \end{cases} = -1 + |x|$$



Corollary 5.9 (integration is the inverse of differentiation)**.** If $f = g'$ is continuous on $[a, b]$, then

$$\int_a^x f(t)\mathrm{d}t = g(x) - g(a) \qquad \forall\ x \in [a, b]$$

Proof. From Theorem 5.8 $F - g$ has zero derivative in $[a, b]$. Hence $F - g$ is constant and since $F(a) = 0$ this implies that $F(x) = g(x) - g(a)$. □

Every continuous has an *indefinite integral* or anti-derivative written $\int f(x)\mathrm{d}x$ which is determined up to a constant.

Remark. We have solved the ODE:

$$\begin{cases} y'(x) = f(x) \\ y(a) = y_0 \end{cases}$$

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Corollary 5.10 (integration by parts)**.** Suppose f' and g' exist and are continuous on $[a, b]$. Then

$$\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'$$

Proof. By the product rule

$$(fg)' = f'g + fg'$$

By 5.9

$$f(b)g(b) - f(a)g(a) = \int_a^b f'g + \int_a^b fg'$$

□

Corollary 5.11 (integration by substitution)**.** Let $g : [\alpha, \beta] \to [a, b]$ with $g(\alpha) = a$, $g(\beta) = b$ and g' exists and is continuous on $[\alpha, \beta]$. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then

$$\int_a^b f(x)\mathrm{d}x = \int_\alpha^\beta f(g(t))g'(t)\mathrm{d}t$$

Proof. Set $F(x) = \int_a^x f(t)\mathrm{d}t$ as before. Let $h(t) = F(g(t))$ (defined since g takes values in $[a, b]$). Then

$$\begin{aligned}
\int_\alpha^\beta f(g(t))g'(t)\mathrm{d}t &= \int_\alpha^\beta F'(g(t))g'(t)\mathrm{d}t && \text{(FTC)} \\
&= \int_\alpha^\beta h'(t)\mathrm{d}t && \text{(Chain rule)} \\
&= h(\beta) - h(\alpha) \\
&= F(b) - F(a) \\
&= \int_a^b f(x)\mathrm{d}x
\end{aligned}$$

□

Theorem 5.12 (Taylor's Theorem with remainder an integral)**.** Let $f^{(n)}(x)$ be continuous for $x \in [0, h]$. Then

$$f(h) = f(0) + \cdots + \frac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} + R_n$$

where

$$R_n = \frac{h^n}{(n-1)!} \int_0^1 (1-t)^{n-1} f^{(n)}(th)\mathrm{d}t$$

Proof. Substitution $u = th$.

$$R_n = \frac{1}{(n-1)!} \int_0^h (h-u)^{n-1} f^{(n)}(u)\mathrm{d}u$$

Integrating by parts now, we get:

$$R_n = -\frac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} + \underbrace{\frac{1}{(n-2)!} \int_0^h (h-u)^{n-2} f^{(n-1)}(u)\mathrm{d}u}_{R_{n-1}}$$

If we integrate by parts $n-1$ times we arrive at:

$$R_n = -\frac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} - \cdots - hf'(0) + \underbrace{\int_0^h f'(u)\mathrm{d}u}_{f(h)-f(0)}$$

□

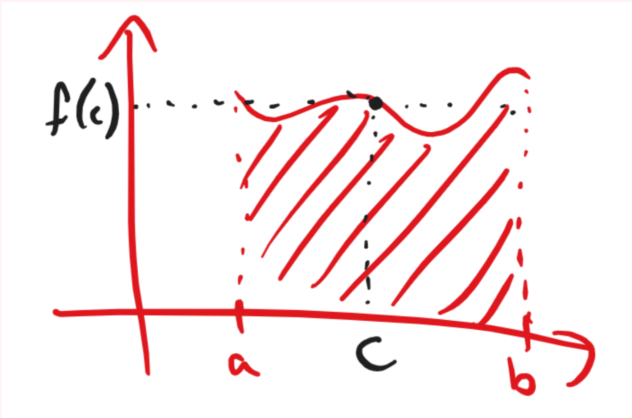
Now we can get the Cauchy & Lagrange form of the remainder. However note that the proof above uses continuity of $f^{(n)}$ not just mere existence as in section 3. But first we need to prove:

Theorem 5.13. $f, g : [a, b] \to \mathbb{R}$ continuous with $g(x) \neq 0 \; \forall \; x \in (a, b)$. Then $\exists \; c \in (a, b)$ such that

$$\int_a^b f(x)g(x)\mathrm{d}x = f(c) \int_a^b g(x)\mathrm{d}x$$

Note. If we take $g(x) = 1$ we get

$$\int_a^b f(x)\mathrm{d}x = f(c)(b - a)$$



Proof. We're going to use Cauchy's MVT (Theorem 3.7).

$$F(x) = \int_a^x fg, \qquad G(x) = \int_a^x g$$

Theorem 3.7 implies $\exists \; c \in (a, b)$ such that

$$(F(b) - F(a))G'(c) = F'(c)(G(b) - G(a))$$
$$\left(\int_a^b fg\right) g(c) = f(c)g(c) \int_a^b g$$

Since $g(c) \neq 0$ we simplify and we're done. □

Now we want to apply this to

$$R_n = \frac{h^n}{(n-1)!} \int_0^1 (1-t)^{n-1} f^{(n)}(th)\mathrm{d}t$$

First we use Theorem 5.13 with $g \equiv 1$, to get

$$R_n \frac{h^n}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(\theta h)$$

($\theta \in (0,1)$), which is Cauchy's form of remainder!

To get Lagrange, we use Theorem 5.13 with $g(t) = (1-t)^{n-1}$ which is > 0 for $t \in (0,1)$. Therefore $\exists\ \theta \in (0,1)$ such that

$$R_n = \frac{h^n}{(n-1)!} f^{(n)}(\theta h) \left[\int_0^1 (1-t)^{n-1} \mathrm{d}t\right]$$

and

$$\int_0^1 (1-t)^{n-1}\mathrm{d}t = -\left.\frac{(1-t)^n}{n}\right|_0^1 = \frac{1}{n}$$

$$\implies R_n = \frac{h^n}{n!} f^{(n)}(\theta h), \qquad \theta \in (0,1)$$

which is Lagrange's form of the remainder!

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5.1 Improper Integrals (infinite integrals)

Definition. Suppose $f : [a,\infty) \to \mathbb{R}$ integrable (and bounded) on every interval $[a,R]$ and that as $R \to \infty$

$$\int_a^R f(x)\mathrm{d}x \to l$$

Then we say that $\int_a^\infty f(x)\mathrm{d}x$ *exists* or *converges* and that its value is l. If $\int_a^R f(x)\mathrm{d}x$ does not tend to a limit, we say that $\int_a^\infty f(x)\mathrm{d}x$ *diverges*. A similar definition applies to $\int_{-\infty}^a f(x)\mathrm{d}x$. If

$$\int_a^\infty f = l_1 \qquad \text{and} \qquad \int_{-\infty}^a f = l_2$$

we write

$$\int_{-\infty}^\infty f = l_1 + l_2$$

(independent of the particular value of a).

Note. This last bit is *not* the same as saying that

$$\lim_{R\to\infty} \int_{-R}^R f(x)\mathrm{d}x$$

exists. It is stronger: for example

$$\int_{-R}^R x\mathrm{d}x = 0$$

Example. $\int_1^\infty \frac{\mathrm{d}x}{x^k}$ converges if and only if $k > 1$. Indeed, if $k \neq 1$ then

$$\int_1^R \frac{\mathrm{d}x}{x^k} = \frac{x^{1-k}}{1-k}\bigg|_1^R = \frac{R^{1-k}-1}{1-k}$$

and as $R \to \infty$ this limit is finite if and only if $k > 1$. If $k = 1$,

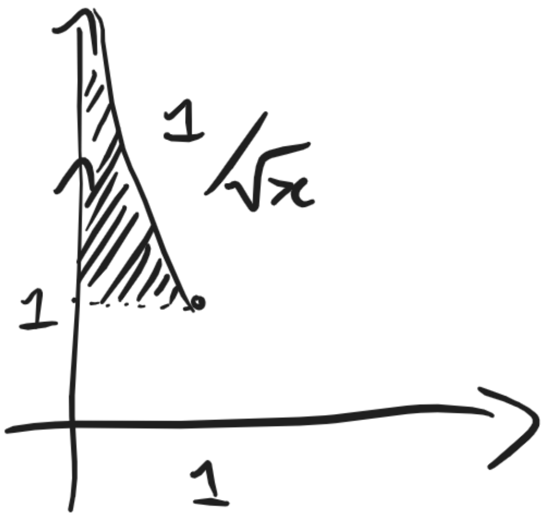
$$\int_1^R \frac{\mathrm{d}x}{x} = \log R \to \infty$$

Remarks

(1) $\frac{1}{\sqrt{x}}$ continuous on $[\delta, 1]$ for any $\delta > 0$, and

$$\int_\delta^1 \frac{1}{\sqrt{x}}\mathrm{d}x = 2\sqrt{x}\big|_\delta^1 = 2 - 2\sqrt{\delta} \to 2$$

as $\delta \to 0$.



$\frac{1}{\sqrt{x}}$ is unbounded on $(0, 1]$.

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}} = \lim_{\delta\to 0}\int_\delta^1 \frac{\mathrm{d}x}{\sqrt{x}} = 2$$

Exercise: give a general definition for cases like this.

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{x} &= \lim_{\delta\to 0}\int_\delta^1 \frac{\mathrm{d}x}{x} \\ &= \lim_{\delta\to 0}\left(\log x|_\delta^1\right) \\ &= \lim_{\delta\to 0}(\log 1 - \log\delta)\end{aligned}$$

does *not exist.*

(2) If $f \geq 0$ and $g \geq 0$, for $x \geq a$ and

$$f(x) \leq Kg(x) \qquad \forall\, x \geq a$$

with K a constant, then

$$\int_a^\infty g \text{ converges} \implies \int_a^\infty f \text{ converges}$$

and

$$\int_a^\infty f \leq K\int_a^\infty g$$

Just note that

$$\int_a^R f \leq K\int_a^R g$$

The function $R \to \int_a^R f$ is increasing ($f \geq 0$) and bounded above (since $\int_a^\infty g$ converges). Take $l = \sup_{R\geq a}\int_a^R f < \infty$, and check that

$$\lim_{R\to\infty}\int_a^R f = l.$$

Given $\varepsilon > 0$, $\exists\ R_0$ such that

$$\int_a^{R_0} f \geq l - \varepsilon$$

Thus if $R \geq R_0$,

$$\int_a^R f \geq \int_a^{R_0} \geq l - \varepsilon$$

$$\implies 0 \leq l - \int_a^R f \leq \varepsilon$$

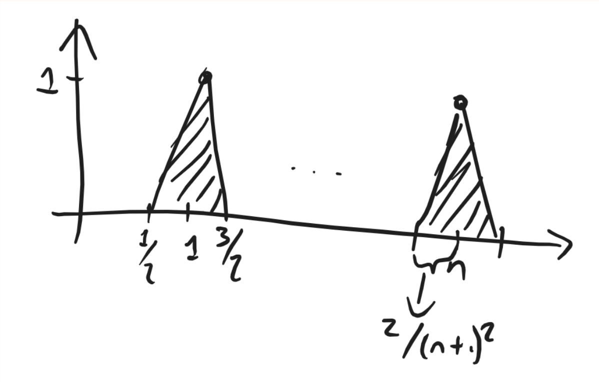
Example. $\int_0^\infty e^{-\frac{x^2}{2}}\mathrm{d}x$. Note $e^{-\frac{x^2}{2}} \leq e^{-\frac{x}{2}}$ for $x \geq 1$. Note that

$$\int_1^R e^{-\frac{x}{2}}\mathrm{d}x = \frac{1}{2}[e^{-\frac{1}{2}} - e^{-\frac{R}{2}}] \to \frac{e^{-\frac{1}{2}}}{2}$$

hence $\int_0^\infty e^{-\frac{x^2}{2}}$ converges.

(3) We know that if $\sum a_n$ converges, then $a_n \to 0$. $\int_a^\infty f$ converges may *not* imply that $f \to 0$.

Example.



$$\text{Area}(\triangle) = \frac{2}{(n+1)^2}$$

so since $\sum \frac{2}{(n+1)^2}$ converges, $\int_0^\infty f$ converges. But $f(n) = 1$, so $f \not\to 0$.

5.2 The Integral Test

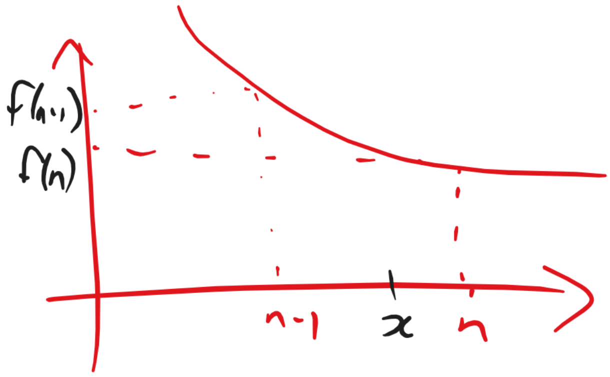
Theorem 5.14 (integral test)**.** Let $f(x)$ be a positive *decreasing* function for $x \geq 1$. Then

(1) The integral $\int_1^\infty f(x)\mathrm{d}x$ and the series $\sum_1^\infty f(n)$ both converge or diverge.

(2) As $n \to \infty$,

$$\sum_{r=1}^n f(r) - \int_1^n f(X)\mathrm{d}x$$

tends to a limit l such that $0 \leq l \leq f(1)$.



Note. f decreasing $\implies$ f integrable on every bounded subinterval by Theorem 5.4.

Proof. If $n-1 \leq x \leq n$, then

$$f(n-1) \geq f(x) \geq f(n)$$

hence

$$f(n-1) \geq \int_{n-1}^{n} f(x)\mathrm{d}x \geq f(n) \tag{$*$}$$

Adding

$$\sum_{1}^{n-1} f(r) \geq \int_{1}^{n} f(x)\mathrm{d}x \geq \sum_{2}^{n} f(r) \tag{$**$}$$

From this claim (1) is *clear*. For the proof of (2) set

$$\phi(n) = \sum_{1}^{n} f(r) - \int_{1}^{n} f(x)\mathrm{d}x$$

Then

$$\phi(n) - \phi(n-1) = f(n) - \int_{n-1}^{n} f(x)\mathrm{d}x \leq 0$$

(using ($*$)) From ($**$)

$$0 \leq \phi(n) \leq f(1)$$

Thus $\phi(n)$ is decreasing and tends to a limit l such that

$$0 \leq l \leq f(1).$$

□

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Examples

(1) $\sum_1^\infty \frac{1}{n^k}$ converges if and only if $k > 1$ $(*)$. We saw that $\int_1^\infty \frac{1}{x^k}$ converges if and only if $k > 1$, so from the integral test we get $(*)$.

(2) $\sum_2^\infty \frac{1}{n \log n}$, $f(x) = \frac{1}{x \log x}$, $x \geq 2$.

$$\int_2^R \frac{\mathrm{d}x}{x \log x} = \log(\log x)|_2^R = \log(\log R) - \log(\log 2) \to \infty$$

as $R \to \infty$. Integral test implies

$$\sum_2^\infty \frac{1}{n \log n}$$

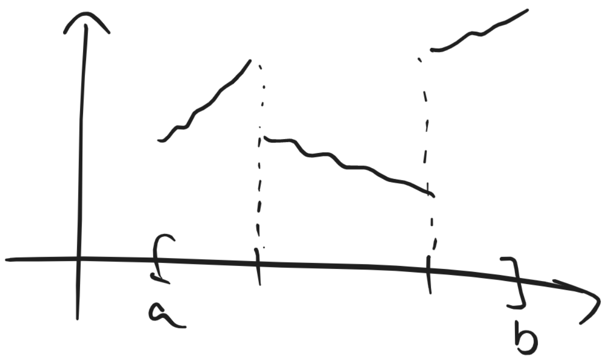
diverges.

Corollary 5.15 (Euler's constant)**.** As $n \to \infty$, $1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \to \gamma$ with $0 \leq \gamma \leq 1$.

Proof. Set $f(x) = \frac{1}{x}$ and use Theorem 5.14. $\square$

Note. An open problem asks "Is γ irrational? ($\gamma \approx 0.577$)"

We have seen: monotone functions and continuous functions are Riemann integrable. We can generalize this a bit and say that *piece-wise continuous* functions are integrable.



Definition. A function $f : [a, b] \to \mathbb{R}$ is said to be piece-wise continuous if there is a dissection

$$\mathcal{D} = \{a = x_0, x_1, \ldots, x_n = b\}$$

such that

(1) f is continuous on (x_{j-1}, x_j) $\forall\, j$

(2) The one-sided limits

$$\lim_{x \to x_{j-1}^+} f(x), \qquad \lim_{x \to x_{j-1}^-} f(x)$$

exist.

It is now an *exercise* to check that f is Riemann integrable: just check that $f|_{[x_{j-1}, x_j]}$ is integrable for each j. (the values of f and the endpoints won't really matter) and use additivity of domain (property (6)).

Question: How large can the discontinuity set of f be while f is still Riemann integrable?

Recall the example:

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \\ 0 & \text{otherwise} \end{cases}$$

on $[0, 1]$.

Note. What follows is non-examinable.

Answer: Henri Lebesgue characterization of Riemann integrability: $f : [a, b] \to \mathbb{R}$ bounded. Then f is Riemann integrable if and only if the set of discontinuity points has *measure zero*.

Definition. Let $l(I)$ be the length of an interval I. A subset $A \subset \mathbb{R}$ is said to have measure zero if for each $\varepsilon > 0$, $\exists$ a countable collection of intervals I_j such that

$$A \subset \bigcup_{j=1}^{\infty} I_j$$

and

$$\sum_j l(I_j) < \varepsilon$$

Lemma. (1) Every countable set has measure zero.

(2) If B has measure zero and $A \subset B$, then A has measure zero.

(3) If A_k has measure zero $\forall\ k \in \mathbb{N}$, then $\bigcup_{k\in\mathbb{N}} A_k$ also has measure zero.

Oscillation of f

I interval:

$$\omega_f(I) = \sup_I f - \inf_I f$$

oscillation of f at a point:

$$\omega_f(x) = \lim_{\varepsilon\to 0} \omega_f(x-\varepsilon, x+\varepsilon)$$

Lemma. f is continuous at x if and only if $\omega_f(x) = 0$.

Proof. Exercise. □

Brief Sketch of Lebesgue's criteria

$$D = \{x \in [a,b] : f \text{ discontinuous at } x\} = \{x : \omega_f(x) > 0\}$$
$$N(\alpha) = \{x : \omega_f(x) \geq \alpha\}$$
$$D = \bigcup_1^\infty N\left(\frac{1}{k}\right)$$

Required to prove: D has measure zero. Let $\varepsilon > 0$ be given, $\exists\ \mathcal{D}$ such that

$$\sum_{j=1}^n \omega_f([x_{j-1}, x_j])(x_j - x_{j-1}) S(f,\mathcal{D}) - s(f,\mathcal{D}) < \frac{\varepsilon\alpha}{2}$$

$$F = \{j : (x_{j-1}, x_j) \cap N(\alpha) \neq \emptyset\}$$

then for each $j \in F$,

$$\omega_f([x_{j-1}, x_j]) \geq \alpha$$
$$\implies \alpha \sum_{j\in F}(x_j - x_{j-1}) \leq \sum_{j\in F} \omega_f([x_{j-1}, x_j])(x_j - x_{j-1}) < \frac{\varepsilon\alpha}{2}$$
$$\implies \sum_{j\in F}(x_j - x_{j-1}) < \frac{\varepsilon}{2}$$

These cover $N(\alpha)$ except perhaps for $\{x_0, x_1, \ldots, x_n\}$. But these can be covered by intervals of total length $< \frac{\varepsilon}{2}$ hence $N(\alpha)$ can be covered by total length $< \varepsilon$.

For the other direction, let $\varepsilon > 0$ be given. $N(\varepsilon) \subset D$, so $N(\varepsilon)$ has measure zero. $N(\varepsilon)$ is closed and bounded hence it can be covered by finitely many open intervals of total length $< \varepsilon$.

$$N(\varepsilon) = \bigcup_{i=1}^{m} U_i$$

$$K = [a, b] \setminus \bigcup_{i=1}^{m} U_i$$

compact so it can be covered by finitely many intervals J_j such that

$$\omega_f(J_j) < \varepsilon.$$